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Transient Analysis of Markovian Queueing System with Balking and Reneging Subject to Catastrophes and Server Failures

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Abstract: In this paper, we present an analysis of the transient and steady states of the infinite Markovian queueing system where both reneging and balking are defined and the system may be entered under catastrophes and server failures repair. Moreover, some other special cases are shown as special case of our new result.

Keywords: Markovian queue, Transient analysis Infinite buffer queue Balk, Renege, Catastrophes, Server failures.

1 Introduction

Queueing systems with reneging have attracted many researchers due to their applications in real life congestion problems such impatient telephone switchboard customers, perishable goods storage inventory systems and communication networks, for examples. Reneging is common phenomen in queues ; as consequence, customers depart after joining the queue without getting service due to impatience. These models with impatient customers have been extensively considered due to their versatility and applicability. An M/M/c queueing system with reneging has been discussed in Haight ([8],[9]). Ancker and Gafarian [3] considered an M/M/1/N queue with balking and reneging . For other examples of articles that address queueing systems which use balking and reneging (see, Ke [13], Shawky [18] , Haghighi, Medhi, and Mohanty [6], Abou-El-Ata and Hariri [1] and Wang and Chang[21])).

Another salient feature, which has been widely studied in the literature, is queueing systems subject to disasters (Gelenbe and Pujolle [11]). The catastrophes arrive as negative customers to the system and their characteristic is to remove some or all of the regular customers in the system. The catastrophes may come either from outside the system or from another service station. For example, in computer networks, if a job infected with a virus arrives, it transmits virus to other processors inactivating them Chao, Miyazawa and Pinedo [5]. Hence, computer networks with a virus infection may be modeled by queueing networks with catastrophes. Other interesting articles in the area include (Harrison and Pitle [7], Henderson [10] and Jain and Sigman [12]).

Queueing systems with repairable servers often arise in practice (Avi-Itzhak and Naor [4],,Neuts and Lucantoni [16] and Vinod [21]). Such repairable server queueing models are interesting, either from the point of view of queueing theory or of reliability theory. These phenomena occur in the area of computer and communication systems where failure and repair of processors have a major impact on the flow of jobs that have to be handled by those processors (Towsley and Tripathi [19] and Wartenhorst [22]). Our motivation is to extend the work done by Kumar and Madheswari [14] and obtain a general case.

The rest of this paper is organized as follows. In Section 2, the new model is described and the governed equations are formulated under pursue the given assumptions. In Section 3, the transient solution of the given model is derived and in Section 4 the transient probability of $p_{c-1}(t)$ is also obtained. Moreover, the steady state probabilities are easily shown for

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completeness .The main conclusion and summarized are finally given.

2 Model Description

We consider the M/M/c queuing system as well as balking and reneging with the probability of catastrophes. We consider the following assumptions:

Customers arrive at the system one by one according to a Poisson stream with rate λ. On arrival a customer either decides to join the queue with probability β, where $\beta = Pr\{a\ unit\ joints\ the\ queue\}$ or balk with probability $(1-\beta)$, where $0 \leq \beta < 1$ if $n = c(1)\infty$ and $\beta = 1$ if $n = 0(1)\overline{c-1}$.

After joining the queue, each customer will wait a certain length of time T for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting served. This time T is a random variable with the following density function: $f(t) = \alpha\exp\{-\alpha t\}$, $\alpha \geq 0$, $t \geq 0$, where α is the rate of time T. Since the arrival and the departure of the impatient customers without service are independent, the average reneging rate of the customer can be given by$(n-c)\alpha$. Hence, the used function of customers average reneging rate is given by: :

$$r(n) = \begin{cases} 0, & \text{at} \quad 0 \leq n \leq c; \\ (n-c)\alpha, & \text{at} \quad n \geq c+1. \end{cases}$$

The service order is assumed on first-come first-served (FCFS) basis and the inter-arrival times, service times, and vacations are mutually independent. The service times are assumed to be independent and identically distributed (i.i.d) exponential random variables with mean $1/\mu$.

Apart from arrival and service processes, the catastrophe occurs at the service facility as a Poisson process with rate ϑ when server is operational (or up). During operational periods, the system under consideration behaves as a standard $M/M/c$ queue. Whenever a catastrophe occurs at the operational server, all the customers in the system are flushed out immediately and the sever gets inactivated. The repair times of failed server are i.i.d, according to an exponential distribution with mean $1/\eta$.

After a repair on the server is completed, the server immediately returns to its working position for service when a new customer arrives. Further, it is assumed that the newly arriving customers will be lost forever during the repair time of failed server.

3 Analysis of the model

Using the assumptions given above, the forward Kolmogorov equations can be written for the state probabilities $p_n(t)$, as follows:

$$\frac{dQ(t)}{dt} = -\eta Q(t) + \vartheta[1-Q(t)], \tag{1}$$

$$\frac{dp_0(t)}{dt} = -(\vartheta+\lambda)p_0(t) + \mu p_1(t) + \eta Q(t), \tag{2}$$

$$\frac{dp_n(t)}{dt} = -(\vartheta+\lambda+n\mu)p_n(t) + \lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t); \qquad 1 \leq n < c, \tag{3}$$

$$\frac{dp_c(t)}{dt} = -(\vartheta+\lambda\beta+c\mu)p_c(t) + \lambda p_{c-1}(t) + (c\mu+\alpha)p_{c+1}(t), \tag{4}$$

$$\frac{dp_n(t)}{dt} = -[\vartheta+\lambda\beta+c\mu+(n-c)\alpha]p_n(t) + \lambda\beta p_{n-1}(t) + [c\mu+(n-c+1)\alpha]p_{n+1}(t); \quad n \geq c+1, \tag{5}$$

where $p_n(t)$ is the probability that there are n customers in the system at time t given that there customers i initially and the failure probability $Q(t)$ (with $p_n(0) = \delta_{ni}$).

Let $P(s,t)$ be the probability generating function for the number of customers a waiting commencement of service as

$$P(s,t) = Q(t) + q_{c-1}(t) + \sum_{n=c}^{\infty} p_n(t)s^{n-c+1} \tag{6}$$

and $P(s,0) = s^{\tau(i)}$. With

$$q_{c-1}(t) = \sum_{n=0}^{c-1} p_n(t), \text{and } \tau(i) = (i-c+1)[1-\sum_{k=0}^{c-1}\delta_{ik}]. \tag{7}$$

It is easily seen that the probability generating function $P(s,t)$ satisfies the following partial differential equation,

$$\frac{\partial P(s,t)}{\partial t} - \alpha(1-s)\frac{\partial P(s,t)}{\partial s} = [P(s,t) - Q(t) - q_{c-1}(t)] \times\{\lambda\beta(s-1) + (c\mu-\alpha)(\frac{1}{s}-1)\} - \vartheta P(s,t) + \lambda\beta p_{c-1}(t)(s-1) + \vartheta. \tag{8}$$

With the initial condition $Q(0) = 0$, the solution of equation (8) is obtained as

$$\begin{aligned} P(s,t) &= \exp\{[\lambda\beta(s-1) + (c\mu-\alpha)(\frac{1}{s}-1) - \vartheta]t\} \\ &\times \sum_{\zeta=0}^{\infty}\binom{\tau(i)}{\zeta}\exp\{-\alpha(\tau(i)-\zeta)t\}s^{(\tau(i)-\zeta)} \\ &\times (1-\exp\{-\alpha t\})^{\zeta} + \int_0^t \exp\{[\lambda\beta(s-1) \\ &+ (c\mu-\alpha)(\frac{1}{s}-1) - \vartheta](t-u)\}\{\lambda\beta p_{c-1}(u) \\ &\times (s-1) - [\lambda\beta(s-1) + (c\mu-\alpha)(\frac{1}{s}-1)] \\ &\times [Q(u) + q_{c-1}(u)]\}du + \vartheta\int_0^t \exp\{[\lambda\beta(s-1) \\ &+ (c\mu-\alpha)(\frac{1}{s}-1) - \vartheta]u\}du. \end{aligned} \tag{9}$$

Using the Bessel function identity (Watson [23]), Let $\theta = 2\sqrt{\lambda\beta(c\mu-\alpha)}$, and $\gamma = \sqrt{\lambda\beta/(c\mu-\alpha)}$, then

$$\exp\{(\lambda\beta s + \frac{c\mu-\alpha}{s})t\} = \sum_{m=-\infty}^{m=\infty} (\gamma s)^m I_m(\theta t), \quad (10)$$

where $I_m(.)$ is the modified Bessel function of order m. Substituting this in equation (9), we get

$$\begin{aligned} P(s,t) = & \sum_{n=-\infty}^{n=\infty} \sum_{\zeta=0}^{\infty} \binom{\tau(i)}{\zeta} \exp\{-\alpha(\tau(i)-\zeta)t\} s^{n+\tau(i)-\zeta} \\ & \times (1-\exp\{-\alpha t\})^{\zeta} \exp\{-\omega t\} I_n(\theta t) \gamma^{n-\tau(i)+\zeta} \\ & + \vartheta \int_0^t \sum_{n=-\infty}^{n=\infty} I_n(\theta u) \gamma^n s^n \exp\{-\omega u\} du \\ & + \int_0^t \sum_{n=-\infty}^{n=\infty} I_n(\theta(t-u)) \gamma^n s^n \exp\{-\omega(t-u)\} \\ & \times \{\lambda\beta p_{c-1}(u)(s-1) - [\lambda\beta(s-1) \\ & + (c\mu-\alpha)(\frac{1}{s}-1)][Q(u)+q_{c-1}(u)]\} du, \quad (11) \end{aligned}$$

where $\omega = \lambda\beta + c\mu - \alpha + \vartheta$ and comparing the coefficient of s^n on right and left hand side, we have for $n = 1,2,3,\ldots$

$$\begin{aligned} p_{n+c-1}(t) = & \sum_{\zeta=0}^{\infty} \binom{\tau(i)}{\zeta} \exp\{-\alpha(\tau(i)-\zeta)t\} I_{n-\tau(i)+\zeta}(\theta t) \\ & \times (1-\exp\{-\alpha t\})^{\zeta} \exp\{-\omega t\} \gamma^{n-\tau(i)+\zeta} \\ & + (c\mu-\alpha)\gamma^n \int_0^t \exp\{-\omega(t-u)\} I_n(\theta(t-u)) \\ & \times [Q(u)+q_{c-1}(u)] du + \lambda\beta\gamma^{n-1} \int_0^t p_{c-1}(u) \\ & \times (I_{n-1}(\theta(t-u)) - \gamma I_n(\theta(t-u))) \\ & \times \exp\{-\omega(t-u)\} du + \lambda\beta\gamma^{n-1} \int_0^t [q_{c-1}(u) \\ & + Q(u)] \exp\{-\omega(t-u)\}[\gamma I_n(\theta(t-u)) \\ & - I_{n-1}(\theta(t-u)) - I_{n+1}(\theta(t-u))] du \\ & + \vartheta\gamma^n \int_0^t \exp\{-\omega u\} I_n(\theta u) du, \quad (12) \end{aligned}$$

and, for $n = 0$,

$$\begin{aligned} q_{c-1}(t) = & \sum_{\zeta=0}^{\infty} \binom{\tau(i)}{\zeta} \exp\{-\alpha(\tau(i)-\zeta)t\} I_{\tau(i)-\zeta}(\theta t) \gamma^{\zeta-\tau(i)} \\ & \times \exp\{-\omega t\}(1-\exp\{-\alpha t\})^{\zeta} + \lambda\beta \int_0^t p_{c-1}(u) \\ & \times (\frac{I_1(\theta(t-u))}{\gamma} - I_0(\theta(t-u))) \exp\{-\omega(t-u)\} du \\ & + \lambda\beta \int_0^t \exp\{-\omega(t-u)\}[Q(u)+q_{c-1}(u)] \\ & \times (I_0(\theta(t-u)) - \frac{2I_1(\theta(t-u))}{\gamma}) du + (c\mu-\alpha) \\ & \times \int_0^t \exp\{-\omega(t-u)\} I_0(\theta(t-u))[q_{c-1}(u) \\ & + Q(u)] du + \vartheta \int_0^t \exp\{-\omega u\} I_0(\theta u) du - Q(t) \quad (13) \end{aligned}$$

Where $P(s,t)$ does not contain terms with negative powers of s, the right-hand side of equation (12) with n replaced by $-n$, must be zero. Thus,

$$\begin{aligned} & \sum_{\zeta=0}^{\infty} \binom{\tau(i)}{\zeta} \exp\{-\alpha(\tau(i)-\zeta)t\}(1-\exp\{-\alpha t\})^{\zeta} \gamma^{n-\tau(i)+\zeta} \\ & \times \exp\{-\omega t\} I_{n+\tau(i)-\zeta}(\theta t) + (c\mu-\alpha)\gamma^n \int_0^t I_n(\theta(t-u)) \\ & \times \exp\{-\omega(t-u)\}[Q(u)+q_{c-1}(u)] du + \lambda\beta\gamma^{n-1} \\ & \times \int_0^t (I_{n+1}(\theta(t-u)) - \gamma I_n(\theta(t-u))) \exp\{-\omega(t-u)\} \\ & \times p_{c-1}(u) du + \lambda\beta\gamma^{n-1} \int_0^t \exp\{-\omega(t-u)\}[\gamma I_n(\theta(t-u)) \\ & - I_{n+1}(\theta(t-u)) - I_{n-1}(\theta(t-u))][q_{c-1}(u) \\ & + Q(u)] du + \vartheta\gamma^n \int_0^t \exp\{-\omega u\} I_n(\theta u) du = 0. \quad (14) \end{aligned}$$

Where we have used $I_{-m}(.) = I_m(.)$. Usage of equation (14) in equation (12) considerably simplifies the work and results in a simple expression for $p_n(t)$. This yields, for $n = 1,2,3,\ldots,$

$$\begin{aligned} p_{n+c-1}(t) = & \sum_{\zeta=0}^{\infty} \binom{\tau(i)}{\zeta} \exp\{-\alpha t(\tau(i)-\zeta)\} \exp\{-\omega t\} \\ & \times (I_{n-\tau(i)+\zeta}(\theta t) - I_{n+\tau(i)-\zeta}(\theta t)) \gamma^{n-\tau(i)+\zeta} \\ & \times (1-\exp\{-\alpha t\})^{\zeta} + n\gamma^n \int_0^t \frac{I_n(\theta(t-u))}{t-u} \\ & \times \exp\{-\omega(t-u)\} p_{c-1}(u) du. \quad (15) \end{aligned}$$

Here, we have obtained $p_{n+c-1}(t)$ for $n = 1,2,3,\ldots$. However, this expression depends upon $p_{c-1}(t)$. In order to determine $p_{n+c-1}(t)$ in next section.

Put $\vartheta = 0$ and $\eta = 0$,

$$\begin{aligned} p_{n+c-1}(t) = & \sum_{\zeta=0}^{\infty} \binom{\tau(i)}{\zeta} \exp\{-\alpha t(\tau(i)-\zeta)\} \gamma^{n-\tau(i)+\zeta} \\ & \times (I_{n-\tau(i)+\zeta}(\theta t) - I_{n+\tau(i)-\zeta}(\theta t)) \\ & \times (1-\exp\{-\alpha t\})^{\zeta} \exp\{-(\lambda\beta+c\mu-\alpha)t\} \\ & + n\gamma^n \int_0^t \frac{I_n(\theta(t-u))}{t-u} p_{c-1}(u) \\ & \times \exp\{-(\lambda\beta+c\mu-\alpha)(t-u)\} du. \quad (16) \end{aligned}$$

This result have obtained by AL-seedy, El-Sherbiny, El-Shehawy and Ammar [2].

4 The transient probability $p_{c-1}(t)$

To determine the probabilities $p_n(t), n = 0,1,2,\ldots,c-1$, consider the system of equations (1)-(3) subject to the condition (13), the system (3) together with (2) can be expressed in the form:

$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{AP}(t) + (c-1)\mu p_{c-1}(t) e_{c-1} + \eta Q(t) e_1, \quad (17)$$

where
$\mathbf{P}(t) = (p_0(t), p_1(t), p_2(t), ..., p_{c-2}(t))^T$, and

$$\mathbf{A} = (a_{kj})_{(c-1)\times(c-1)}, \tag{18}$$

with

$$a_{kj} = \begin{cases} \lambda, & \text{j=k-1, k=1,2,...,c-2;} \\ -(\lambda+\vartheta+k\mu), & \text{j=k, k=0,1,2,...,c-2;} \\ (k+1)\mu, & \text{j=k+1, k=0,1,2,...,c-3,} \end{cases} \tag{19}$$

and $e_{c-1} = (0,0,0,...,1)^T, e_1 = (1,0,0,...,0)^T$.

Let $\hat{f}(s)$ denote the Laplace transform for the function $f(t)$, by transforming of equation (17) and solving, we obtain

$$\hat{\mathbf{P}}(s) = (s\mathbf{I}\text{-}\mathbf{A})^{-1}\{(c-1)\mu\hat{p}_{c-1}(s)e_{c-1} + \eta\hat{Q}(s)e_1 + \mathbf{P}(0)\}, \tag{20}$$

with $\mathbf{P}(0) = (\delta_{i0}, \delta_{i1}, \delta_{i2}, ..., \delta_{ic-2})^T$. Thus, only $\hat{p}_{c-1}(s)$ remains to be found. We observe that if $e = (1,1,1,...,1)^T$ and

$$\hat{q}_{c-1}(s) = e^T\hat{\mathbf{P}}(s) + \hat{p}_{c-1}(s), \tag{21}$$

using equation (13) and simplifying, we get

$$\begin{aligned} \hat{p}_{c-1}(s) = &\{\sum_{\zeta=0}^{\infty}\sum_{g=0}^{\zeta}(-1)^g\binom{\zeta}{g}\binom{\tau(i)}{\zeta}\left(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta\gamma}\right)^{\tau(i)-\zeta} \\ &\times \left(\frac{\sqrt{p^2-\theta^2}}{\sqrt{\Gamma^2-\theta^2}}\right) + \frac{\vartheta\eta}{s(s+\vartheta+\eta)}[1-(s+\vartheta)e^T \\ &\times (s\mathbf{I}\text{-}\mathbf{A})^{-1}e_1] - (s+\vartheta)e^T(s\mathbf{I}\text{-}\mathbf{A})^{-1}P(0)\} \\ &\times \{s+\vartheta+\lambda\beta - \left(\frac{p-\sqrt{p^2-\theta^2}}{2}\right) \\ &+ (c-1)\mu(s+\vartheta)e^T(s\mathbf{I}\text{-}\mathbf{A})^{-1}e_{c-1}\}^{-1}, \end{aligned} \tag{22}$$

where $p = s+\omega$ and $\Gamma = p+\alpha(\tau(i)-\zeta+g)$.

In (20) and (22), $(s\mathbf{I}\text{-}\mathbf{A})^{-1}$ has to be found. For smaller order matrices the usual procedure can be employed. For higher order matrices, we can follow the procedure given in Raju and Bhat [17] to get the element of the matrix $(sI-A)^{-1}$. To this end, let

$$(s\mathbf{I}\text{-}\mathbf{A})^{-1} = (\hat{a}_{kj}(s))_{(c-1)\times(c-1)}, \tag{23}$$

we note that $(s\mathbf{I}\text{-}\mathbf{A})$ is almost lower triangular. Following Raju and Bhat [17], we obtain

$$\hat{a}_{kj}(s) = \begin{cases} \frac{(u_{c-1,0}(s))^{-1}}{(j+1)\mu}[u_{c-1,j+1}(s)u_{k,0}(s) \\ \quad -u_{k,j+1}(s)u_{c-1,0}(s)], & \text{j=0,1,2,...c-3;} \\ u_{k,0}(s)(u_{c-1,0}(s))^{-1}, & \text{j=c-2.} \end{cases} \tag{24}$$

For $k = 0,1,2,...,c-2$, where $u_{k,j}(s)$ are defined by

$$u_{k,k}(s) = 1, \qquad k = 0,1,2,...,c-2,$$

$$u_{k+1,k}(s) = \frac{s+\lambda+\vartheta+k\mu}{(k+1)\mu}, \qquad k = 0,1,2,...,c-3,$$

$$\begin{aligned} u_{k+1,k-j}(s) = &\frac{1}{(k+1)\mu}[(s+\lambda+\vartheta+k\mu)u_{k,k-j}(s) \\ &-\lambda u_{k-1,k-j}(s)], \qquad k = 1,2,...,c-3,, j \le k \end{aligned}$$

$$u_{c-1,j}(s) = \begin{cases} [s+\lambda+\vartheta+(c-2)\mu] \\ \quad \times u_{c-2,j}(s) - \lambda u_{c-3,j}(s), & \text{j=0,1,2,...,c-3;} \\ s+\lambda+\vartheta+(c-2)\mu, & \text{j=c-2,} \end{cases}$$

and $u_{k,j}(s) = 0$ for k and j.

To facilitate computation, we have suppressed the argument s. The advantage in using these relations is that the authors do not evaluate any determinant. Using these in (22), we have

$$\begin{aligned} \hat{p}_{c-1}(s) = &\{\sum_{\zeta=0}^{\infty}\sum_{g=0}^{\zeta}(-1)^g\binom{\zeta}{g}\binom{\tau(i)}{\zeta}\left(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta\gamma}\right)^{\tau(i)-\zeta} \\ &\times \left(\frac{\sqrt{p^2-\theta^2}}{\sqrt{\Gamma^2-\theta^2}}\right) + \frac{\vartheta\eta}{s(s+\vartheta+\eta)}[1-(s+\vartheta) \\ &\times \sum_{k=0}^{c-2}\hat{a}_{k0}(s)] - (s+\vartheta)\sum_{j=0}^{c-2}\sum_{k=0}^{c-2}\delta_{ij}\hat{a}_{kj}(s)\} \\ &\times \{s+\vartheta+\lambda\beta - \left(\frac{p-\sqrt{p^2-\theta^2}}{2}\right) \\ &+ (c-1)\mu(s+\vartheta)\sum_{k=0}^{c-2}\hat{a}_{kc-2}(s)\}^{-1}, \end{aligned} \tag{25}$$

and $k = 0,1,2,...,c-2$,

$$\begin{aligned} \hat{p}_k(s) = &\sum_{j=0}^{c-2}\delta_{ij}\hat{a}_{kj}(s) + (c-1)\mu\hat{a}_{kc-2}(s)\hat{p}_{c-1}(s) \\ &+\eta\hat{a}_{k0}(s)\hat{Q}(s). \end{aligned} \tag{26}$$

It is clear that $\hat{a}_{kj}(s)$ are all rational algebraic function in s. The cofactor of the $(i,j)th$ element of $(s\mathbf{I}\text{-}\mathbf{A})$ is polynomial of degree $c-2-|i-j|$. In particular, the cofactor of the diagonal elements are polynomials in s of degree $c-2$ with the leading coefficient equal to 1. In fact $u_{c-1,0}(s) = 0$ is the characteristic equation of $\mathbf{A}$. Since $a_{00} = -(\lambda+\vartheta), (\lambda+\vartheta \neq 0)$ it is also known that the characteristic roots of $\mathbf{A}$ are all distinct and negative Lederman and Reuter [15]. Hence the inverse transform $\tilde{a}_{kj}(t)$ of $\hat{a}_{kj}(s)$ can be obtained by partial fraction decomposition. Let s_k, $k = 0,1,2,...,c-2$, be the characteristic roots of matrix $\mathbf{A}$. Then

$$\hat{a}_{kj}(s) = \sum_{m=0}^{c-2}\frac{\rho_{kj}^{(m)}}{(s-s_m)}, \tag{27}$$

where the constants are defined by $\rho_{kj}^{(m)}$

$$\rho_{kj}^{(m)} = \lim_{s\to s_m}\{(s-s_m)\hat{a}_{kj}(s)\}. \tag{28}$$

Thus, we can write $\tilde{a}_{kj}(t)$ in the following form:-

$$\tilde{a}_{kj}(t) = \sum_{m=0}^{c-2}\rho_{kj}^{(m)}\exp\{s_m t\}, \tag{29}$$

similarly

$$\sum_{k=0}^{c-2}(c-1)s\hat{a}_{kj}(s)=(c-1)+\hat{b}_j(s),\, j=0,1,2,...,c-2, \tag{30}$$

where $\hat{b}_j(s)$ can be resolved as

$$\hat{b}_j(s)=\sum_{m=0}^{c-2}\frac{B_j^{(m)}}{s-s_m}, \tag{31}$$

with

$$B_j^{(m)}=\lim_{s\to s_m}\{(s-s_m)(c-1)\sum_{k=0}^{c-2}s\hat{a}_{kj}(s)\}. \tag{32}$$

Then the inverse transform $b_j(t)$ of $\hat{b}_j(s)$ is given by

$$b_j(t)=\sum_{m=0}^{c-2}B_j^{(m)}\exp\{s_m t\},\, j=0,1,2,...,c-2. \tag{33}$$

Using (30) in (25), we obtained

$$\begin{aligned}\hat{p}_{c-1}(s)=&\{\sum_{\zeta=0}^{\infty}\sum_{g=0}^{\zeta}(-1)^g\tbinom{\zeta}{g}\tbinom{\tau(i)}{\zeta}\Big(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta\gamma}\Big)^{\tau(i)-\zeta}\\&\times\Big(\frac{\sqrt{p^2-\theta^2}}{\sqrt{\Gamma^2-\theta^2}}\Big)-\frac{\vartheta\eta}{s(s+\vartheta+\eta)}\Big(\frac{\hat{b}_0(s)}{c-1}\Big)\\&-\sum_{j=0}^{c-2}\delta_{ij}\{\frac{(c-1)+\hat{b}_j(s)}{c-1}\}\}\\&\times\{\frac{p-\sqrt{p^2-\theta^2}}{2}+\alpha-\mu+\mu\hat{b}_{c-2}(s)\}^{-1},\end{aligned} \tag{34}$$

hence (34) simplifies to

$$\begin{aligned}p_{c-1}(t)=&\sum_{m=0}^{\infty}\Big(\frac{2}{\theta}\Big)^{m+1}\sum_{k=0}^{m}(-1)^k(m+1)\tbinom{m}{k}(\mu-\alpha)^{m-k}\mu^k\\&\times\int_0^t b_{c-2}^{*k}(t-u)\{\int_0^u\sum_{\zeta=0}^{\infty}\sum_{g=0}^{\zeta}(-1)^g\tbinom{\zeta}{g}\tbinom{\tau(i)}{\zeta}\\&\times\gamma^{\zeta-\tau(i)}\exp\{-[\omega+\alpha(\tau(i)-\zeta+g)](u-r)\}\\&\times I_{\tau(i)-\zeta}(\theta[u-r])[\int_0^r(\delta(r-y)+\frac{I_0(\theta[r-y])}{\theta(r-y)})\\&\times\exp\{-\omega y\}\Big(\frac{I_{m+1}(\theta y)}{y}\Big)dy]dr-\frac{(m+1)\vartheta\eta}{(\vartheta+\eta)}\\&\times\int_0^u\{\exp\{-\omega(u-r)\}\exp\{-r(\vartheta+\eta)\}\\&\times\Big(\frac{I_{m+1}(\theta[u-r])}{u-r}\Big)\int_0^r(\delta(r-x)+\frac{b_0(r-x)}{(r-x)})dx\}dr\\&+\frac{(m+1)\vartheta\eta}{(\vartheta+\eta)}\int_0^u(\delta(u-x)+\frac{b_0(u-x)}{c-1})\frac{I_{m+1}(\theta x)}{x}\\&\times\exp\{-\omega x\}dx-(m+1)\sum_{j=0}^{c-2}\delta_{ij}\int_0^u\{\delta(u-x)\\&+\frac{b_j(u-x)}{u-x}\}\frac{I_{m+1}(\theta x)}{x}\exp\{-\omega x\}dx\}du,\end{aligned} \tag{35}$$

where $b_{c-2}^{*k}(t)$ is $k-$fold convolution of $b_{c-2}(t)$ wit itself, we note that $b_{c-2}^{*0}(t)=\delta(t)$. Finally for $k=0,1,2,...,c-2$,

$$\begin{aligned}p_k(t)=&\sum_{j=0}^{c-2}\delta_{ij}\tilde{a}_{kj}(t)+(c-1)\mu\int_0^t\tilde{a}_{kc-2}(u)p_{c-1}(t-u)du\\&+\Big(\frac{\vartheta\eta}{\vartheta+\eta}\Big)\int_0^t[1-\exp\{-u(\vartheta+\eta)\}]\\&\times\tilde{a}_{k0}(t-u)du.\end{aligned} \tag{36}$$

Thus (15), (35) and (36) completely determine all the state probabilities of the queue size.

Remark at $\vartheta=0$ and $\eta=0$,

$$\begin{aligned}p_k(t)=&\sum_{j=0}^{c-2}\delta_{ij}\tilde{a}_{kj}(t)+(c-1)\mu\\&\times\int_0^t\tilde{a}_{kc-2}(u)p_{c-1}(t-u)du.\end{aligned} \tag{37}$$

This result have obtained by AL-seedy, El-Sherbiny, El-Shehawy and Ammar [2].

5 Steady state probabilities

Let $\frac{\lambda\beta}{(c\mu-\alpha)}<1$ and

$$\lim_{t\to\infty}p_n(t)=\pi_n=\lim_{s\to 0}s\hat{p}_n(s), \tag{38}$$

$$\lim_{t\to\infty}Q(t)=Q=\lim_{s\to 0}s\hat{Q}(s). \tag{39}$$

The Laplace transform of equations (13) and (15) are

$$\begin{aligned}\hat{q}_{c-1}(s)=&\{\sum_{\zeta=0}^{\infty}\sum_{g=0}^{\zeta}(-1)^g\tbinom{\zeta}{g}\tbinom{\tau(i)}{\zeta}\Big(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta\gamma}\Big)^{\tau(i)-\zeta}\\&\times\frac{\sqrt{p^2-\theta^2}}{\sqrt{\Gamma^2-\theta^2}}+\Big(\frac{p-\sqrt{p^2-\theta^2}}{2}-\lambda\beta\Big)\hat{p}_{c-1}(s)\\&+\frac{\vartheta\eta}{s(s+\vartheta+\eta)}\}(s+\vartheta)^{-1},\end{aligned} \tag{40}$$

and

$$\begin{aligned}\hat{p}_{n+c-1}(s)=&\sum_{\zeta=0}^{\infty}\sum_{g=0}^{\zeta}(-1)^g\tbinom{\zeta}{g}\tbinom{\tau(i)}{\zeta}[\Big(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta}\Big)^{\zeta-\tau(i)}\\&-\Big(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta}\Big)^{\tau(i)-\zeta}]\Big(\frac{\Gamma-\sqrt{\Gamma^2-\theta^2}}{\theta}\Big)^n\\&\times\frac{\gamma^{n-\tau(i)+\zeta}}{\sqrt{\Gamma^2-\theta^2}}+\Big(\gamma\frac{p-\sqrt{p^2-\theta^2}}{\theta}\Big)^n\hat{p}_{c-1}(s)\end{aligned} \tag{41}$$

from (20), we have

$$\begin{aligned}\pi=\lim_{s\to 0}s\hat{\mathbf{P}}(s)=&(c-1)\mu\pi_{c-1}[\lim_{s\to 0}(s\mathbf{I}\text{-}\mathbf{A})^{-1}]e_{c-1}\\&+\eta Q[\lim_{s\to 0}(s\mathbf{I}\text{-}\mathbf{A})^{-1}]e_1.\end{aligned} \tag{42}$$

Also from (40) and (41), we get

$$\mathbb{Q}_{c-1} = \lim_{s\to 0} s\hat{q}_{c-1}(s) = \frac{\eta}{\vartheta+\eta} + \frac{1}{\vartheta} \times \left(\frac{\omega - \sqrt{\omega^2-\theta^2}}{2} - \lambda\beta\right)\pi_{c-1}, \quad (43)$$

$$\pi_{n+c-1} = \lim_{s\to 0} s\hat{p}_{n+c-1}(s) = \left(\gamma\frac{\omega - \sqrt{\omega^2-\theta^2}}{\theta}\right)^n \pi_{c-1}, \quad (44)$$

and (21) and (38), we have

$$\mathbb{Q}_{c-1} = e^T\pi + \pi_{c-1} = \frac{\eta}{\vartheta+\eta} + \frac{1}{\vartheta} \times \left(\frac{\omega - \sqrt{\omega^2-\theta^2}}{2} - \lambda\beta\right)\pi_{c-1}, \quad (45)$$

This, together with equation (42), yields

$$\pi_{c-1} = \frac{\vartheta\eta}{\vartheta+\eta}\left(1 - \vartheta e^T[\lim_{s\to 0}(s\mathbf{I}\text{-}\mathbf{A})^{-1}]e_1\right) \times \left\{\vartheta + \lambda\beta - \frac{\omega - \sqrt{\omega^2-\theta^2}}{2} + \mu\vartheta(c-1)e^T[\lim_{s\to 0}(s\mathbf{I}\text{-}\mathbf{A})^{-1}]e_{c-1}\right\}^{-1}, \quad (46)$$

where

$$e^T[\lim_{s\to 0}(s\mathbf{I}\text{-}\mathbf{A})^{-1}]e_{c-1} = \lim_{s\to 0}\sum_{k=0}^{c-2}\hat{a}_{kc-2}(s) = \sum_{k=0}^{c-2}\lim_{s\to 0}\frac{\hat{u}_{k,0}(s)}{\hat{u}_{c-1,0}(s)} = \sum_{k=0}^{c-2}\frac{u_{k,0}}{u_{c-1,0}}, \quad (47)$$

and

$$e^T[\lim_{s\to 0}(s\mathbf{I}\text{-}\mathbf{A})^{-1}]e_1 = \lim_{s\to 0}\sum_{k=0}^{c-2}\hat{a}_{k0}(s) = \sum_{k=0}^{c-2}\lim_{s\to 0}\left[\frac{\hat{u}_{c-1,1}(s)\hat{u}_{k,0}(s) - \hat{u}_{c-1,0}(s)\hat{u}_{k,1}(s)}{\mu\hat{u}_{c-1,0}(s)}\right] = \sum_{k=0}^{c-2}\left[\frac{u_{c-1,1}u_{k,0} - u_{c-1,0}u_{k,1}}{\mu u_{c-1,0}}\right]. \quad (48)$$

Using (47) and (48) in (46), we obtained

$$\pi_{c-1} = \frac{\vartheta\eta}{\vartheta+\eta}\left(1 - \vartheta\sum_{k=0}^{c-2}\frac{u_{c-1,1}u_{k,0} - u_{c-1,0}u_{k,1}}{\mu u_{c-1,0}}\right) \times \left\{\vartheta + \lambda\beta - \frac{\omega - \sqrt{\omega^2-\theta^2}}{2} + \mu\vartheta(c-1)\sum_{k=0}^{c-2}\frac{u_{k,0}}{u_{c-1,0}}\right\}^{-1}. \quad (49)$$

Similarly for $k = 0, 1, 2, ..., c-2$,

$$\pi_k = (c-1)\mu a_{k,c-2}\pi_{c-1} + \eta Q a_{k,0}, \quad (50)$$

and from (44) for $n = 1, 2, 3, \ldots$

$$\pi_{n+c-1} = \eta Q\gamma^n\left(\frac{\omega - \sqrt{\omega^2-\theta^2}}{\theta}\right)^n \times \left(1 - \vartheta\sum_{k=0}^{c-2}\frac{u_{c-1,1}u_{k,0} - u_{c-1,0}u_{k,1}}{\mu u_{c-1,0}}\right) \times \left\{\vartheta + \lambda\beta - \frac{\omega - \sqrt{\omega^2-\theta^2}}{2} + \mu\vartheta(c-1)\sum_{k=0}^{c-2}\frac{u_{k,0}}{u_{c-1,0}}\right\}^{-1}. \quad (51)$$

6 Conclusion

Indeed the given analysis of infinite buffer Markovian queueing system with c servers, balking and catastrophes and server failures is carried out for such a system have potential applications in many manufacturing systems and computer networks. The obtained formulas for the system queue length and failure distributions in both cases transient state and steady state can be easily used to estimate the reliability and the efficient of the considered model. This work provides a more general analysis to solve such models. Moreover, some other special cases can be easily obtained of our new model by a direct substitution.

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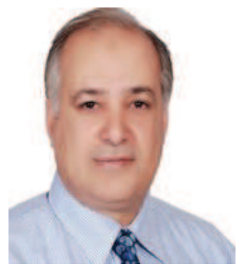

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