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On Solutions of Fractional Logistic Differential Equations

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Abstract: This paper presents an efficient computational technique based on the reproducing kernel theory for approximating the solutions of logistic differential equations of fractional order. The Caputo fractional derivative is utilized in the current approach. The numerical solution can be produced by taking the n-terms of the analytical solution. The convergence of the approximate solution to the analytical solution can be demonstrated with the help of numerical experiments. The numerical comparisons depict that the given method has high effectiveness, accuracy, and feasibility for fractional logistic differential equations.

Keywords: Fractional logistic differential equations, Caputo derivative, Hilbert space, reproducing kernel Hilbert space method, inner product.

1 Introduction

When we talk about fractional calculus (FC, for short), we are generalizing the notion of integer order differentiation and n-fold integration by including positive real order [1]. The most benefit behind studying fractional differential equations (FDEs, for short) is filling the void in scientific applications and engineering for material and memory transfer, describing, measuring and modeling genetic structures, the population growth on the ecosystems, the behavior of viscoelastic materials and others (see [2] and references therein). FDEs are not limited to these last processes, they were extensively used also in different areas like fractal geometry, electromagnetic, anomalous diffusion, signal processing, oscillation, optimal control, fluid mechanics and viscosity. With the science's development, FDEs are great tools to describe memory and hereditary properties of various processes and materials. When it comes to searching for problems whose analytical solutions are not available, the FC work is incomparable.

It is common for researchers to define the FDE, which is refer to its associated classical differential equation (CDE, for short) if it exists. The solution of FDE may not always correspond to its associated CDE. We mention here the fractional ordinary differential equation that we might not find its exact solution, whereas we do for its classical ordinary differential equation. Among the well-known FDEs, we are interested to mention the fractional logistic differential equation (FLDE, for short). In point of fact, Verhulst presented, for the first time, the so-called logistic equation (LE, for short) which has been used to describe the population growth.

Here's the standard LE form

$$Q'(\tau) = rQ(\tau)\left(1 - \frac{Q(\tau)}{K}\right).$$

Where K denotes the carrying capacity, maximum population growth rate is defined by $r > 0$, and Q mentions the population growth size that depends on time τ.

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Here, if we put $\vartheta(\tau) = \frac{Q(\tau)}{K}$, then we can express the logistic differential equation (LDE, for short) as

$$\vartheta'(\tau) = r\vartheta(\tau)(1-\vartheta). \tag{1}$$

The exact solution (ES, for short) of (1) appears in the following way [3]

$$\vartheta(\tau) = \frac{\lambda}{\lambda + (1-\lambda)e^{-r\tau}}, \tag{2}$$

where λ is the initial population data. Actually, (2) explains the rate, startling development, and rapidity of population growth but without including disease outbreaks, basic needs, and reducing food supplies. The FLDE has been applied in many important fields to model the adaptability of society to innovation, the social dynamics of replacement technologies [4], the epidemic disease spreading, and the growth of tumors [5]. Further, the solution of FLDE has attracted the interest of many authors and they made several attempts to obtain its exact, analytical or numerical solution. The technique of Carleman embedding is considered to obtain the ES of the FLDE by West [6]. Subsequently, Area et al. [7] had shown that the work proposed by West was invalid. However, D'Ovodio et al. [8] claimed that the ES given by West was successful only for the so-called modified fractional logistic equation. Vivek et al. [9] used the fractional Euler's method to obtain theapproximate solution solution of the FLDE. Pitolli and Pezza's fractional spline collocation approach for FLDE has an approximate solution presented in [10]. Alshammari et al. [11] used the fractional residual power series method (FRPSM) to investigate the numerical solution of FLDE. A numerical method for solving the FLDE is provided by Kaharuddin et al. [12].

In order to analyze the nonlinear FLDE whose ES doesn't exist, we develop an iterative reproducing kernel Hilbert space method (abbreviated RKHSM) in this paper. Solving a wide range of ODEs with classical or fractional derivatives, reproducing kernels, or RK for short, have received a lot of attention. The RK theory has its roots in Zaremba's studies [13]. This approach is a method that has been successfully used and is a representative of the RK concept. The most advantageous features of this approach are: 1) It guarantees the uniform convergence of the numerical solution and its derivatives. 2) It is mesh-free which means that it can be applied quickly and simply. Recent important applications of the RKHSM can be found in numerical analysis, computational mathematics, finance, probability and statistics, and machine learning [14, 15]. For example, fractional Bratu-type equations [16], fractional Bagley-Torvik equation [17], Riccati differential equations [18], nonlinear fractional Volterra integro-differential equations [19], nonlocal fractional boundary value problems [20], and forced Duffing equations [21] have all been solved accurately using RKHSM.

The reason why we made our study is to construct a RKHSM to obtaining numerical solutions of the FLDE with the help of RK theory.

To make the method's use simpler, certain fundamental ideas and basic concepts are presented in the second section of this work. The third section includes the FLDE presentation as well as the RKHSM description. A number of examples are provided in the fourth section to show the effectiveness of the approach and the precision of the solutions. The conclusion is then presented.

2 Reproducing kernel theory and fractional calculus requirements

Definition 1. *The formula for the Caputo fractional derivative of function $\phi(\tau)$ is [17]*

$$(D^{\wp}\phi)(\tau) := (J^{n-\wp}\phi^{(n)})(\tau) = \frac{1}{\Gamma(n-\wp)}\int_0^{\tau}(\tau-\eta)^{-1-\wp+n}\frac{\mathrm{d}^n\phi(\eta)}{\mathrm{d}\eta^n}d\eta, \tag{3}$$

for $\phi \in C^n_{-1}$, $\tau > 0$, $\wp \in (n-1, n]$, and $n \in \mathbb{N}$.

Definition 2. *Allow $\mathfrak{H}$ to represent a Hilbert space on $\mathscr{X}$ that is not empty set. If a function $\mathfrak{Q} : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ meets the following two conditions, we refer to it as a reproducing kernel of $\mathfrak{H}$: [15]*

1. $\mathfrak{Q}(\cdot,\tau) \in \mathfrak{H}$, $\forall \tau \in \mathscr{X}$.
2. $\langle \phi, \mathfrak{Q}(\cdot,\tau)\rangle = \phi(\tau)$, $\forall \phi \in \mathfrak{H}$, $\forall \tau \in \mathfrak{H}$.

Remark.

- Reproducing Property (also known as RP) is a common term to refer to condition (2).
- If a Hilbert space $\mathfrak{H}$ has a reproducing kernel function (RKF, for short) $\mathfrak{Q}$, it is referred to as a "reproducing kernel Hilbert space" (RKHS, for short).

Definition 3. *We write the following to defining the space* $W_2^2[0,T]$ *[18]*

$$W_2^2[0,T]=\{\kappa(\tau)|\kappa^{(j)} \text{ are absolutely continuous functions on } [0,T], \kappa''\in L^2[0,T], j=0,1, \text{and } \kappa(0)=0\}.$$

The following is the norm of the space $W_2^2[0,T]$

$$\|\kappa\|_{W_2^2}=\sqrt{\langle\kappa,\kappa\rangle_{W_2^2}},$$

in which $\kappa,\upsilon\in W_2^2[0,T]$ *and the inner product* $\langle\bullet,\star\rangle_{W_2^2}$ *is given by*

$$\langle\kappa,\upsilon\rangle_{W_2^2}=\sum_{j=0}^{1}\kappa^{(j)}(0)\upsilon^{(j)}(0)+\int_0^T\kappa''(\tau)\upsilon''(\tau)d\tau. \tag{4}$$

Theorem 1. $S_\eta(\tau)$ *is the RKF connected to* $W_2^2[0,T]$, *as determined by*

$$S_\eta(\tau)=\begin{cases} s(\eta,\tau)\ ,\tau\le\eta,\\ s(\tau,\eta)\ ,\tau>\eta,\end{cases} \tag{5}$$

where $s(\eta,\tau)=\tau\eta+\frac{1}{2}\tau^2\eta-\frac{1}{6}\tau^3$.

Proof. By Definition 3, we have

$$\langle\kappa,S_\eta\rangle_{W_2^2}=\kappa(0)S_\eta(0)+\kappa'(0)S_\eta'(0)+\int_0^T\kappa''(\tau)S_\eta''(\tau)d\tau. \tag{6}$$

When we apply integration by parts twice to (6), we get

$$\begin{aligned}\langle\kappa,S_\eta\rangle_{W_2^2}=&\kappa(0)S_\eta(0)+\kappa'(0)\left(S_\eta'(0)-S_\eta''(0)\right)+\kappa'(T)S_\eta''(T)-\kappa(T)S_\eta^{(3)}(T)\\ &+\kappa(0)S_\eta^{(3)}(0)+\int_0^T\kappa(\tau)S_\eta^{(4)}(\tau)d\tau.\end{aligned} \tag{7}$$

Taking

$$\begin{aligned}&S_\eta'(0)-S_\eta''(0)=0,\\ &S_\eta(0)=0,\\ &S_\eta''(T)=0,\\ &S_\eta^{(3)}(T)=0,\end{aligned} \tag{8}$$

then we find that (with the help of RP) :

$$S_\eta^{(4)}(\tau)=\delta(\tau-\eta), \tag{9}$$

the Dirac-Delta function is denoted by δ. Here, for $\tau\ne\eta$, (9) becomes

$$S_\eta^{(4)}(\tau)=0. \tag{10}$$

Consequently,

$$S_\eta(\tau)=\begin{cases}\gamma_1(\eta)+\gamma_2(\eta)\tau+\gamma_3(\eta)\tau^2+\gamma_4(\eta)\tau^3\ ,\tau\le\eta,\\ \sigma_1(\eta)+\sigma_2(\eta)\tau+\sigma_3(\eta)\tau^2+\sigma_4(\eta)\tau^3\ ,\tau>\eta,\end{cases} \tag{11}$$

where The coefficients $\gamma_i(\eta)$ and $\sigma_i(\eta), i=1,\ldots,4$ will be found.
The Dirac-Delta function property allows us to discover that

$$S_{\eta^+}(\eta)=S_{\eta^-}(\eta), \tag{12}$$

$$S_{\eta^+}'(\eta)=S_{\eta^-}'(\eta), \tag{13}$$

$$S_{\eta^+}''(\eta)=S_{\eta^-}''(\eta), \tag{14}$$

$$S_{\eta^+}^{(3)}(\eta)-S_{\eta^-}^{(3)}(\eta)=1. \tag{15}$$

The unknown coefficients can be found by using (8), (12)-(15), and (11) with its derivatives

$$\gamma_1(\eta)=0, \qquad \sigma_1(\eta)=-\frac{\eta^3}{6},$$
$$\gamma_2(\eta)=\eta, \qquad \sigma_2(\eta)=\frac{1}{2}\eta(2+\eta),$$
$$\gamma_3(\eta)=\frac{\eta}{2}, \qquad \sigma_3(\eta)=0,$$
$$\gamma_4(\eta)=-\frac{1}{6}, \qquad \sigma_4(\eta)=0.$$

Theorem 2. *The RKF, $S_\eta(\tau)$, of $W_2^2[0,T]$ is obtained as follows:*

$$S_\eta(\tau)=\begin{cases}\tau\eta+\frac{1}{2}\tau^2\eta-\frac{1}{6}\tau^3 \ ,\tau\le\eta,\\ \eta\tau+\frac{1}{2}\eta^2\tau-\frac{1}{6}\eta^3 \ ,\tau>\eta.\end{cases} \tag{16}$$

Proof. We need to show that

$$\langle\kappa,S_\eta\rangle_{W_2^2}=\kappa(\eta).$$

We have

$$\langle\kappa,S_\eta\rangle_{W_2^2}=\kappa(0)S_\eta(0)+\kappa'(0)S'_\eta(0)+\int_0^T\kappa''(\tau)S''_\eta(\tau)d\tau.$$

We use integration by parts for reaching

$$\langle\kappa,S_\eta\rangle_{W_2^2}=\kappa(0)S_\eta(0)+\kappa'(0)\left(S'_\eta(0)-S''_\eta(0)\right)+\kappa'(T)S''_\eta(T)-\int_0^T\kappa'(\tau)S^{(3)}_\eta(\tau)d\tau. \tag{17}$$

Since $S_\eta(\tau)\in W_2^2[0,T]$, we have

$$S_\eta(0)=0. \tag{18}$$

Then we get:

$$\langle\kappa,S_\eta\rangle_{W_2^2}=\kappa'(T)S''_\eta(T)+\kappa'(0)(S'_\eta(0)-S''_\eta(0))-\int_0^T\kappa'(\tau)S^{(3)}_\eta(\tau)d\tau. \tag{19}$$

$S'_\eta(0),S''_\eta(0)$, and $S''_\eta(T)$ must be calculated. We have

$$S'_\eta(\tau)=\begin{cases}\eta+\tau\eta-\frac{1}{2}\tau^2 \ ,\tau\le\eta,\\ \eta+\frac{1}{2}\eta^2 \ ,\tau>\eta.\end{cases}$$

$$S''_\eta(\tau)=\begin{cases}\eta-\tau \ ,\tau\le\eta,\\ 0 \ ,\tau>\eta.\end{cases}$$

Therefore, we obtain

$$S'_\eta(0)=\eta, \qquad S''_\eta(0)=\eta, \qquad S''_\eta(T)=0.$$

As a result,

$$\langle\kappa,S_\eta\rangle_{W_2^2}=-\int_0^T\kappa'(\tau)S^{(3)}_\eta(\tau)d\tau. \tag{20}$$

We have

$$S^{(3)}_\eta(\tau)=\begin{cases}-1 \ ,\tau\le\eta,\\ 0 \ ,\tau>\eta.\end{cases}$$

Then, we get

$$\begin{aligned}\langle\kappa,S_\eta\rangle_{W_2^2}&=-\int_0^\eta\kappa'(\tau)S^{(3)}_\eta(\tau)d\tau-\int_\eta^T\kappa'(\tau)S^{(3)}_\eta(\tau)d\tau\\ &=\int_0^\eta\kappa'(\tau)d\tau\\ &=\kappa(\eta)-\kappa(0).\end{aligned}$$

Since $\kappa(\tau) \in W_2^2[0,T]$, $\kappa(0)=0$. As a result, we achieve the desired outcome as:

$$\left\langle \kappa, S_\eta \right\rangle_{W_2^2} = \kappa(\eta).$$

The proof is finished with this.

Definition 4. *We write the following to defining the space $W_2^1[0,T]$ [22]*

$$W_2^1[0,T] = \{\kappa(\tau) | \kappa \text{ is absolutely continuous function on } [0,T], \kappa' \in L^2[0,T]\}.$$

The following is the norm of the space $W_2^1[0,T]$

$$\|\kappa\|_{W_2^1} = \sqrt{\langle \kappa, \kappa \rangle_{W_2^1}}, \tag{21}$$

in which $\kappa, \upsilon \in W_2^1[0,T]$ and the inner product $\langle \bullet, \star \rangle_{W_2^1}$ is given by

$$\langle \kappa, \upsilon \rangle_{W_2^1} = \kappa(0)\upsilon(0) + \int_0^T \kappa'(\tau)\upsilon'(\tau) d\tau, \tag{22}$$

Theorem 3. *$R_\eta(\tau)$ is the RKF connected to $W_2^1[0,T]$, as determined by*

$$R_\eta(\tau) = \begin{cases} 1+\tau \ , \tau \le \eta, \\ 1+\eta \ , \tau > \eta. \end{cases} \tag{23}$$

Proof. By Definition 4 provides us with

$$\left\langle \kappa, R_\eta \right\rangle_{W_2^1} = \kappa(0) R_\eta(0) + \int_0^T \kappa'(\tau) R'_\eta(\tau) d\tau. \tag{24}$$

When we apply integration by parts to (24), we get

$$\left\langle \kappa, R_\eta \right\rangle_{W_2^1} = \kappa(0) R_\eta(0) + \kappa(T) R'_\eta(T) - \kappa(0) R'_\eta(0) - \int_0^T \kappa(\tau) R''_\eta(\tau) d\tau. \tag{25}$$

Taking

$$\begin{aligned} R_\eta(0) - R'_\eta(0) = 0, \\ R'_\eta(T) = 0, \end{aligned} \tag{26}$$

then we discover that (with the aid of RP):

$$-R''_\eta(\tau) = \delta(\tau - \eta), \tag{27}$$

the Dirac-Delta function is denoted by δ. For $\tau \neq \eta$, (27) becomes

$$R''_\eta(\tau) = 0. \tag{28}$$

As a result,

$$R_\eta(\tau) = \begin{cases} \gamma_1(\eta) + \gamma_2(\eta)\tau \ , \tau \le \eta, \\ \sigma_1(\eta) + \sigma_2(\eta)\tau \ , \tau > \eta, \end{cases} \tag{29}$$

where The unknown coefficients $\gamma_i(\eta)$ and $\sigma_i(\eta), i = 1,2$ will be found.
The Dirac-Delta function property allows us to discover that

$$R_{\eta^+}(\eta) = R_{\eta^-}(\eta), \tag{30}$$

$$R'_{\eta^+}(\eta) - R'_{\eta^-}(\eta) = -1. \tag{31}$$

The coefficients can be found by using (26), (30), (31), and (29) with its derivative

$$\begin{aligned} \gamma_1(\eta) &= 1, & \sigma_1(\eta) &= 1+\eta, \\ \gamma_2(\eta) &= 1, & \sigma_2(\eta) &= 0. \end{aligned}$$

3 Problem formulation and RKHSM's solution

Considering the following FLDE:

$$\begin{cases} D^{\wp}\vartheta(\tau) = r\vartheta(\tau)(1-\vartheta(\tau)), \; \tau \in (0,T], \quad 0 < \wp \leq 1, \\ \vartheta(0) = \lambda. \end{cases} \tag{32}$$

Here $D^{\wp}$ is the fractional derivative of order $\wp$ in Caputo sense and the initial population size is given by $\lambda \in (0,1)$.

It is important to note that the RKHSM algorithm will be straightforward to use once we homogenize the initial condition $\vartheta(0) = \lambda$. In order to do this, the following change of variable is carried out:

$$\hbar(\tau) = \vartheta(\tau) - \lambda.$$

Thus, (32) becomes

$$\begin{cases} D^{\wp}\hbar(\tau) + r(2\lambda-1)\,\hbar(\tau) = \Phi(\tau), \; \tau \in (0,T], \quad 0 < \wp \leq 1, \\ \hbar(0) = 0, \end{cases} \tag{33}$$

where $\Phi(\tau) = -r(\lambda(\lambda-1) + \hbar^2(\tau))$.

The first step that provide the representation of solution of (33) is to introduce a linear operator $\Im : W_2^2[0,T] \to W_2^1[0,T]$ as

$$\Im\hbar(\tau) = D^{\wp}\hbar(\tau) + r(2\lambda-1)\,\hbar(\tau). \tag{34}$$

Theorem 4. $\Im : W_2^2[0,T] \to W_2^1[0,T]$ *is a bounded linear operator.*

Proof. The linearity of the operator $\Im$ is easily discernible. Thus, we may directly demonstrate $\Im$'s boundedness, which means that we will demonstrate

$$\|\Im\hbar\|_{W_2^1} \leq \Theta\,\|\hbar\|_{W_2^2}, \text{ with } \Theta > 0.$$

From the expressions (21) and (22), we reach

$$\|\Im\hbar(\tau)\|_{W_2^1}^2 = \langle \Im\hbar(\tau), \Im\hbar(\tau)\rangle_{W_2^1} = [\Im\hbar(0)]^2 + \int_0^T \left[\Im\hbar'(\tau)\right]^2 d\tau.$$

The RP has allowed us to write

$$\hbar(\tau) = \langle \hbar(\diamond), S_\tau(\diamond)\rangle_{W_2^2}.$$

In addition,

$$\Im\hbar(\tau) = \langle \hbar(\diamond), \Im S_\tau(\diamond)\rangle_{W_2^2},$$
$$\Im\hbar'(\tau) = \langle \hbar(\diamond), \partial_\tau(\Im S_\tau(\diamond))\rangle_{W_2^2}.$$

The continuity of $S_\tau(\diamond)$ and the Schwarz inequality enable us to deduce

$$|\Im\hbar(\tau)| = \left|\langle \hbar(\diamond), \Im S_\tau(\diamond)\rangle_{W_2^2}\right| \leq \|\hbar\|_{W_2^2}\,\|\Im S_\tau(\diamond)\|_{W_2^2} \leq \Theta_1\,\|\hbar\|_{W_2^2}. \tag{35}$$

In the same way,

$$\left|\Im\hbar'(\tau)\right| \leq \Theta_2\,\|\hbar\|_{W_2^2}.$$

Hence

$$\begin{aligned} \|\Im\hbar(\tau)\|_{W_2^2}^2 &\leq \Theta_1^2\,\|\hbar\|_{W_2^2}^2 + \int_0^T \Theta_2^2\,\|\hbar\|_{W_2^2}^2\, d\tau \\ &= \left(\Theta_1^2 + T\Theta_2^2\right)\|\hbar\|_{W_2^2}^2. \end{aligned} \tag{36}$$

From (36), we conclude that $\|\Im\hbar(\tau)\|_{W_2^1} \leq \Theta\,\|\hbar\|_{W_2^2}$, where $\Theta = \Theta_1^2 + T\Theta_2^2$.

Applying (34) allows us to rewrite (33) as the following

$$\begin{cases} \Im\hbar(\tau) = \Phi(\tau),\ \tau \in (0,T], \\ \hbar(0) = 0, \end{cases} \tag{37}$$

here, $\Phi(\tau) = -\frac{r}{k}(\lambda(\lambda-k)+\hbar^2(\tau))$.

In order to carry out the processes, we build the orthogonal function system of $W_2^2[0,T]$ by letting $\rho_i(\tau) = R_{\tau_i}(\tau)$ and $\psi_i(\tau) = \Im^*\rho_i(\tau)$, where

- The formal adjoint of $\Im$ is $\Im^*$.
- $\{\tau_i\}_{i=1}^{\infty}$ is a countable set that is dense in $[0,T]$.
- $R_{\tau_i}(\tau)$ stands for the RKF connected to $W_2^1[0,T]$.

The process of Gram-Schmidt allows us to write the following orthonormal system $\{\overline{\psi}_i\}_{i=1}^{\infty}$ in $W_2^2[0,T]$:

$$\overline{\psi}_i(\tau) = \sum_{k=1}^{i} \nu_{ik}\psi_k(\tau), \quad \nu_{ii} > 0,\ i \in \{1,2,3,\dots\}. \tag{38}$$

Where $\{\psi_i\}_{i=1}^{\infty}$ is a representation of the orthonormal system in $W_2^2[0,T]$. And, for the orthogonalization coefficients ν_{ik} we have :

$$\nu_{ij} = \begin{cases} \frac{1}{\|\psi_1\|}, & \text{for} \quad i = j = 1, \\ \frac{1}{e_i}, & \text{for} \quad i = j \neq 1, \\ -\frac{1}{e_i}\sum_{k=j}^{i-1} C_{ik}\nu_{kj}, & \text{for} \quad i > j, \end{cases} \tag{39}$$

where $e_i = \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} C_{ik}^2}, C_{ik} = \langle \psi_i, \overline{\psi}_k \rangle_{W_2^2}$.

Remark. The following provides a quicker method to obtain the expression of $\psi_i(\tau)$

$$\psi_i(\tau) = \langle \Im^*\rho_i(\eta), S_\tau(\eta)\rangle_{W_2^2} = \langle \rho_i(\eta), \Im S_\tau(\eta)\rangle_{W_2^1} = \langle R_{\tau_i}(\eta), \Im S_\tau(\eta)\rangle_{W_2^1} = \Im_\eta S_\tau(\eta)|_{\eta=\tau_i}.$$

$\Im_\eta$ is used here to indicate that $\Im$ is applied to η.

Theorem 5. *$\{\psi_i\}_{i=1}^{\infty}$ is the complete system of $W_2^2[0,T]$ if $\{\tau_i\}_{i=1}^{\infty}$ is a dense set on $[0,T]$.*

Proof. It is evident that $\psi_i \in W_2^2$. Consequently, for each fixed $\hbar \in W_2^2$, we put

$$\langle \hbar(\tau), \psi_i(\tau)\rangle_{W_2^2} = 0,\ i = 1,2,\dots.$$

Since

$$\langle \hbar(\tau), \psi_i(\tau)\rangle_{W_2^2} = \langle \hbar(\tau), \Im^*\rho_i(\vartheta)\rangle_{W_2^2} = \langle \Im\hbar(\tau), \rho_i(\tau)\rangle_{W_2^1} = \Im\hbar(\tau_i) = 0,$$

and $\{\tau_i\}_{i=1}^{\infty}$ is a dense set on $[0,T]$, $\Im\hbar(\tau) = 0$. We can write $\hbar(\tau) = 0$ by using the fact that $\Im^{-1}$ exists.

Lemma 1. *Assume $\hbar(\tau) \in W_2^2[0,T]$, then*

$$\left\|\hbar^{(i)}(\tau)\right\|_C \le \Upsilon \|\hbar(\tau)\|_{W_2^2}, \quad i \in \{0,1\},$$

where Υ is non-negative and $\|\hbar(\tau)\|_C = \max_{0\le\tau\le1} |\hbar(\tau)|$.

Proof. For all $\tau \in [0,T]$ we can write

$$\hbar^{(i)}(\tau) = \left\langle \hbar(\diamond), \partial_\tau^{(i)} S_\tau(\diamond)\right\rangle_{W_2^2}, \quad i = 0,1.$$

As a result of using the expression for $\partial_\tau^{(i)} S_\tau(\diamond)$,

$$\left\|\partial_\tau^{(i)} S_\tau\right\|_{W_2^2} \le \Upsilon_i, \quad i \in \{0,1\}.$$

Thus,

$$\left|\hbar^{(i)}(\tau)\right| = \left|\left\langle \hbar(\diamond), \partial_\tau^{(i)} S_\tau(\diamond)\right\rangle_{W_2^2}\right| \le \left\|\partial_\tau^{(i)} S_\tau\right\|_{W_2^2} \|\hbar\|_{W_2^2} \le \Upsilon_i \|\hbar\|_{W_2^2},\ i \in \{0,1\}. \tag{40}$$

Where $\Upsilon = \max_{i=0,1}\{\Upsilon_i\}$. The lemma 1 is then a consequence of (40).

Theorem 6. *Assuming that* $\{\tau_i\}_{i=1}^{\infty}$ *is dense on* $[0,T]$ *and* (37) *has a unique solution on* W_2^2*, then the solution of* (37) *is*

$$\hbar(\tau)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\Phi(\tau_k)\overline{\psi}_i(\tau). \tag{41}$$

And the solution of (32) *is shown below*

$$\vartheta(\tau)=\left(\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\Phi(\tau_k)\overline{\psi}_i(\tau)\right)+\lambda. \tag{42}$$

Proof. First, we observe that the orthonormal basis $\{\bar{\psi}_i(\tau)\}_{i=1}^{\infty}$ in $W_2^2[0,T]$ is complete, we obtain

$$\begin{aligned}\hbar(\tau)&=\sum_{i=1}^{\infty}\langle\hbar(\tau),\bar{\psi}_i(\tau)\rangle_{W_2^2}\,\bar{\psi}_i(\tau)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\,\langle\hbar(\tau),\psi_k(\tau)\rangle_{W_2^2}\,\bar{\psi}_i(\tau)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\,\langle\hbar(\tau),\Im^*\rho_k(\tau)\rangle_{W_2^2}\,\bar{\psi}_i(\tau)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\,\langle\Im\hbar(\tau),\rho_k(\tau)\rangle_{W_2^1}\,\bar{\psi}_i(\tau)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\,\langle\Im\hbar(\tau),R_\tau(\tau_k)\rangle_{W_2^1}\,\bar{\psi}_i(\tau)\\&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\Phi(\tau_k)\bar{\psi}_i(\tau),\end{aligned}$$

with $\Phi(\tau_k)=\Im\hbar(\tau_k)$.
Additionally, we mention that it is simple to obtain (42) from $\vartheta(\tau)=\hbar(\tau)+\lambda$ and (41).

In this case, the approximate solution $\hbar_n(\tau)$ is obtained if we take a finite many terms in (41).

$$\hbar_n(\tau)=\sum_{i=1}^{n}\sum_{k=1}^{i}\nu_{ik}\Phi(\tau_k)\overline{\psi}_i(\tau). \tag{43}$$

$W_2^2[0,T]$ is a Hilbert space, hence it is obvious that

$$\sum_{i=1}^{\infty}\sum_{k=1}^{i}\nu_{ik}\Phi(\tau_k)\overline{\psi}_i(\tau)<\infty.$$

We come to the conclusion that $\hbar_n(\tau)$ is convergent in the norm.

Theorem 7. *The numerical solution and its derivative converge uniformly.*

Proof. We present the following estimate for all $\tau\in[0,T]$ to show the uniform convergence of $\hbar_n(\tau)$

$$\begin{aligned}|\hbar_n-\hbar|&=\left|\langle\hbar_n-\hbar,S_\tau\rangle_{W_2^2}\right|\\&\le\|S_\tau\|_{W_2^2}\,\|\hbar_n-\hbar\|_{W_2^2}\\&\le\mathscr{C}_0\,\|\hbar_n-\hbar\|_{W_2^2},\end{aligned}$$

where $\mathscr{C}_0$ is a constant.
Along the same lines,

$$\left|\hbar_n'-\hbar'\right|\le\|\partial_\tau S_\tau\|_{W_2^2}\,\left\|\hbar_n'-\hbar'\right\|_{W_2^2}.$$

We have the following from the uniform boundedness of $\partial_\tau S_\tau(\diamond)$

$$\|\partial_\tau S_\tau\|_{W_2^2} \leq \mathscr{C}_1,$$

where $\mathscr{C}_1$ is a positive constant. Consequently,

$$\left|\hbar_n' - \hbar'\right| \leq \mathscr{C}_1 \left\|\hbar_n' - \hbar'\right\|_{W_2^2}.$$

As a result, we have successfully proven the theorem.

4 Numerical applications

For fractional logistic differential equations, the iterative RKHS technique is used. All calculations are performed using Maple.

Example 1. Considering the FLDE as

$$\begin{cases} D^{\wp}\vartheta(\tau) = \frac{1}{2}\vartheta(\tau)(1-\vartheta(\tau)),\ \tau > 0, \quad 0 < \wp \leq 1, \\ \vartheta(0) = \frac{1}{2}. \end{cases} \tag{44}$$

The exact solution in this example is

$$\vartheta(\tau) = \frac{1}{1+e^{-\frac{1}{2}\tau}}.$$

To use the RKHS approach, the initial conditions must be homogenized. As a result, we apply the following transformation:

$$\vartheta(\tau) = \hbar(\tau) + \frac{1}{2}.$$

Thus,

$$\begin{cases} D^{\wp}\hbar(\tau) = \frac{1}{2}\left(\frac{1}{4} - \hbar^2(\tau)\right),\ 0 < \tau \leq 1, \\ \hbar(0) = 0. \end{cases} \tag{45}$$

Here, the RKHSM is used to obtain several interesting results, taking $\tau_i = i/n,\ i \in \{1,2,\ldots,n\}$. In Table 1, we fix the fractional derivative at $\wp = 1$ then we compared our results with those obtained by ES, fractional residual power series method (FRPSM, for short) [11] and optimal homotopy asymptotic method (OHAM, for short) [23]. Table 2 shows the approximate solution (AS), exact solution (ES), relative error (RE) , and absolute error (AE) of Example 1 at various $\tau \in [0,1]$. Whereas in Table 3, we show the approximate solutions of Example 1, obtained at different $\tau \in [0,1]$ when $\wp \in \{0.5, 0.75, 0.9, 1\}$. To display of efficiency for method, we take low numbers. Additionally, increasing the number of points in the domain will reduce the absolute error. We observe that the results for different values of $\wp$ are found to be in good agreement with each other from the acquired findings presented in Tables 1, 2, and 3.

Table 1. Comparison of approximate solutions of RKHSM at $\wp = 1$ by other methods for example 1.

τ	Exact solution	RKHSM	FRPSM	OHAM	AE in FRPSM	AE in OHAM	AE in RKHSM
0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0	0	0
0.3	0.5374298454	0.5374297936	0.5374298457	0.5374288935	3.0×10^{-10}	9.5190×10^{-7}	5.18×10^{-8}
0.5	0.5621765008	0.5621764494	0.5621765137	0.5621790838	1.3×10^{-8}	2.5830×10^{-6}	5.14×10^{-8}
0.8	0.5986876601	0.5986876097	0.5986880000	0.5986911508	3.4×10^{-7}	3.4907×10^{-6}	5.04×10^{-8}
1	0.6224593312	0.6224592820	0.6224609375	0.6224603770	1.6×10^{-6}	1.0458×10^{-6}	4.92×10^{-8}

Table 2. RKHSM's results for example 1 when $\wp = 1$.

τ	Exact solution	Approximate solution	Absolute error	Relative error
0	0.5000000000	0.5000000000	0	0
0.1	0.5124973965	0.5124973448	5.17×10^{-8}	1.008785612×10^{-7}
0.2	0.5249791875	0.5249791356	5.19×10^{-8}	9.886106199×10^{-8}
0.3	0.5374298454	0.5374297936	5.18×10^{-8}	9.638467317×10^{-8}
0.4	0.5498339973	0.5498339458	5.15×10^{-8}	9.366463379×10^{-8}
0.5	0.5621765008	0.5621764494	5.14×10^{-8}	9.143036026×10^{-8}
0.6	0.5744425168	0.5744424652	5.16×10^{-8}	8.982622019×10^{-8}
0.7	0.5866175790	0.5866175282	5.08×10^{-8}	8.659815495×10^{-8}
0.8	0.5986876601	0.5986876097	5.04×10^{-8}	8.418413032×10^{-8}
0.9	0.6106392340	0.6106391840	5.00×10^{-8}	8.188140757×10^{-8}
1	0.6224593312	0.6224592820	4.92×10^{-8}	7.904130846×10^{-8}

Table 3. Numerical solutions of example 1 obtained with our proposed method for several values τ and $\wp$.

τ	$\wp = 1$	$\wp = 0.9$	$\wp = 0.75$	$\wp = 0.5$
0	0.5000000000	0.5000000000	0.5000000000	0.5000000002
0.2	0.5249791356	0.5303427356	0.5402164227	0.5618865679
0.4	0.5498339458	0.5565573124	0.5675289097	0.5870099732
0.6	0.5744424652	0.5810975297	0.5909853651	0.6056555389
0.8	0.5986876097	0.6043821308	0.6120094608	0.6208971276
1	0.6224592820	0.6265674601	0.6312088845	0.6339330445

Table 4. Comparison of the RKHSM's approximate solutions for example 2 versus those from other techniques when $\wp = 1$.

τ	Exact solution	RKHSM	FRPSM	VIM	AE in FRPSM	AE in VIM	AE in RKHSM
0	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0	0	0
0.3	0.3502029635	0.3502029364	0.3502029634	0.3502013889	1.00×10^{-10}	1.5746×10^{-6}	2.71×10^{-8}
0.5	0.3616644631	0.3616644354	0.3616644609	0.3616576645	2.15×10^{-8}	6.7986×10^{-6}	2.77×10^{-8}
0.8	0.3791524530	0.3791524268	0.3791524170	0.3791275720	3.60×10^{-7}	2.4881×10^{-5}	2.62×10^{-8}
1	0.3909913151	0.3909912851	0.3909911774	0.3909465021	1.38×10^{-7}	4.4813×10^{-5}	3.00×10^{-8}

Table 5. RKHSM's results for example 2 when $\wp = 1$.

τ	Exact solution	Approximate solution	Absolute error	Relative error
0	0.3333333333	0.3333333333	0	0
0.1	0.3389118421	0.3389118172	2.49×10^{-8}	7.3470433×10^{-8}
0.2	0.3445354618	0.3445354357	2.61×10^{-8}	7.5754175×10^{-8}
0.3	0.3502029635	0.3502029364	2.71×10^{-8}	7.7383697×10^{-8}
0.4	0.3559130712	0.3559130443	2.69×10^{-8}	7.5580253×10^{-8}
0.5	0.3616644631	0.3616644354	2.77×10^{-8}	7.6590328×10^{-8}
0.6	0.3674557720	0.3674557446	2.74×10^{-8}	7.4566797×10^{-8}
0.7	0.3732855868	0.3732855590	2.78×10^{-8}	7.4473810×10^{-8}
0.8	0.3791524530	0.3791524268	2.62×10^{-8}	6.9101491×10^{-8}
0.9	0.3850548747	0.3850548463	2.84×10^{-8}	7.3755721×10^{-8}
1	0.3909913151	0.3909912851	3.00×10^{-8}	7.6728047×10^{-8}

Table 6. Numerical solutions of example 2 obtained with our proposed method for several values τ and $\wp$.

τ	$\wp = 1$	$\wp = 0.9$	$\wp = 0.75$	$\wp = 0.5$
0	0.3333333333	0.3333333333	0.3333333333	0.3333333332
0.2	0.3445354357	0.3469629737	0.3515091793	0.3618341548
0.4	0.3559130443	0.3590802992	0.3643979840	0.3744013481
0.6	0.3674557446	0.3707605468	0.3758855842	0.3841703940
0.8	0.3791524268	0.3822073883	0.3865817337	0.3924932058
1	0.3909912851	0.3935053054	0.3967393860	0.3998879500

Example 2. Considering the FLDE as

$$\begin{cases} D^{\wp}\vartheta(\tau) = \frac{1}{4}\vartheta(\tau)\left(1-\vartheta(\tau)\right), \ \tau > 0, \quad 0 < \wp \leq 1, \\ \vartheta(0) = \frac{1}{3}. \end{cases} \tag{46}$$

The exact solution in this example is

$$\vartheta(\tau) = \frac{1}{1+2e^{-\frac{1}{4}\tau}}.$$

To use the RKHS approach, the initial conditions must be homogenized. As a result, we apply the following transformation:

$$\vartheta(\tau) = \hbar(\tau) + \frac{1}{3}.$$

Thus,

$$\begin{cases} D^{\wp}\hbar(\tau) - \frac{1}{2}\hbar(\tau) = \frac{1}{4}\left(\frac{1}{3} - \hbar^2(\tau)\right), \ 0 < \tau \leq 1, \\ \hbar(0) = 0. \end{cases} \tag{47}$$

Here, the RKHSM is used to obtain several interesting results, taking $\tau_i = i/n, \ i \in \{1,2,\ldots,n\}$. In Table 4, we fix the fractional derivative at $\wp = 1$ then we compared our results with those obtained by ES, FRPSM [11] and variational iteration method (VIM, for short). Table 5 shows the AS, ES, RE, and AE of Example 1 at various $\tau \in [0,1]$. Whereas in Table 6, we show the approximate solutions of Example 1, obtained at different $\tau \in [0,1]$ when $\wp \in \{0.5, 0.75, 0.9, 1\}$. To display of efficiency for method, we take low numbers. Additionally, increasing the number of points in the domain will reduce the absolute error. We observe that the results for different values of $\wp$ are found to be in good agreement with each other from the acquired findings presented in Tables 4, 5, and 6.

5 Conclusion

We have presented an efficient RKHSM for the FLDE. The numerical solution ϑ_n and its derivative converge uniformly. The linear operator's boundedness has been established, and it is demonstrated that the RKHSM has good convergence . The FL problem's outcomes also confirm the RKHSM method's effectiveness and practicality.

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