Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/030212

The Expected Number of Maxima of a Random Algebraic Polynomial with Independently Normally Distributed Random Variables

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Received: 17 Jan. 2015, Revised: 10 Feb. 2015, Accepted: 11 Feb. 2015.
Published online: 1 Jul. 2015.

Abstract: The expected number of maxima of the curve representing the algebraic polynomial of the form given below $a_0\binom{n-1}{0}^{\frac{1}{2}} + a_1\binom{n-1}{1}^{\frac{1}{2}} x + a_2\binom{n-1}{2}^{\frac{1}{2}} x^2 + \dots\dots\dots\dots\dots a_{n-1}\binom{n-1}{n-1}^{\frac{1}{2}} x^{n-1}$,
Here $a_j\{j=(1,2,\dots\dots\dots\dots\dots,n-1)\}$ are independent standard random variables. This paper is an attempt to provide an asymptotic estimate for the expected number of maxima for non-zero mean $\mu\binom{n-1}{0}^{\frac{1}{2}}$ and variance σ^2. The result shows a significant difference in mathematical behavior between our polynomial and the one which was previously studied.

Key Words: Random algebraic polynomial, Exceedance measure, Expected number of maxima.

1 Introduction

Let us consider the random algebraic polynomial as:

$$P(x) = \sum_{j=0}^{n-1} a_j x^j \qquad (1)$$

Where $a_j\,(j=0,1,2,\dots\dots\dots\dots n)$ is a sequence of independent normally distributed random variables with mean 0 and variance 1. Let $EN(\alpha,\beta)$ be the average number of maxima of $P(x)$ in the interval (α,β). For the above polynomial first of all Das [1] obtained that the average number of maxima is asymptotic to $\frac{(\sqrt{3}+1)}{2\pi}\log n$ for n sufficiently large. Subsequently Farahmand [4] obtained for non-zero mean, the expected number of maxima is asymptotic to $\frac{(\sqrt{3}+1)}{4\pi}\log n$, when n is large. However little is known about random polynomials with non-identical coefficients. Motivated by their close relation with physics, reported by Edelman and Kostlan [2], we assume that the coefficients a_j have non-identical mean $\mu\binom{n-1}{0}^{\frac{1}{2}}$ and variance σ^2. It is same as considering the polynomial of the form

$$Q(x) = \sum_{j=0}^{n-1} a_j \binom{n-1}{j}^{\frac{1}{2}} x^j \qquad (2)$$

For the above coefficients as standard normal random variables, Farahmand [3] has shown that the average number of maxima is asymptotic to $\sqrt{n-1}$. Again Farahmand [5] has shown that for sufficiently large n, the expected number of maxima which occurs below the x-axis is asymptotic to $O(1)$. Therefore our paper emphasizes on the average number of maxima for the above polynomial $Q(x)$, when the coefficients are independently normally distributed with mean $\mu\binom{n-1}{0}^{\frac{1}{2}}$ and variance σ^2, which is a mere generalization of the results of the above. Hence we have the following theorem.

Theorem 1: If the coefficients of $Q(x)$ in (2) are independently normally distributed random variables with

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mean $\mu\binom{n-1}{j}^{\frac{1}{2}}$ and variance σ^2, then for sufficiently large n, the mathematical expectation of the number of maxima of $Q(x)$ satisfies,

$$EN\left(-\infty,-1\right)=\frac{(2\pi+1)}{2\sqrt{2\pi}}\mathrm{O}(n)\approx EN\left(1,\infty\right)$$

$$EN\left(-1,0\right)=\frac{1}{2\pi}\mathrm{O}(n)\left[2\tan^{-1}(1-\varepsilon)+\tan^{-1}(1)\right]$$

$$EN\left(0,1\right)=\frac{\left(\pi\sqrt{2}+1\right)}{2\pi}\mathrm{O}(n)$$

2A Formula for the Expected Number of Maxima

Let

$$\Phi(t)=(2\pi)^{\frac{-1}{2}}\int_{-\infty}^{t}\exp^{-\left(\frac{y^2}{2}\right)}dy$$

Then by using the formula for the expected number of maxima given by Cramer &Leadbetter [P.242], the polynomial $Q(t)$ can be written

Let:

$$EN\left(\alpha,\beta\right)=\int_{\alpha}^{\beta}\left(\frac{B}{A}\right)\left(1-\rho^2\right)^{\frac{1}{2}}\Phi\left(\frac{m_1}{A}\right)\left[\varphi(\eta)+\eta\Phi(\eta)\right]dx \tag{3}$$

Where

$$m_1=E\left(Q'(x)\right)$$
$$= E\left[\sum_{j=0}^{n-1}a_j\binom{n-1}{j}^{\frac{1}{2}}jx^{j-1}\right]$$
$$=\mu(n-1)(1+x)^{n-2} \tag{4}$$

$$m_2=E\left(Q''(x)\right)$$
$$= E\left[\sum_{j=0}^{n-1}a_j\binom{n-1}{j}^{\frac{1}{2}}j(j-1)x^{j-2}\right]$$
$$=\mu(n-1)(n-2)(1+x)^{n-3} \tag{5}$$

$$A^2=V\left(Q'(x)\right)^{n-3}$$
$$=V\left[\sum_{j=0}^{n-1}a_j\binom{n-1}{j}^{\frac{1}{2}}jx^{j-1}\right]$$
$$=\sigma^2(n-1)\left(1+nx^2-x^2\right)\left(1+x^2\right) \tag{6}$$

$$B^2=V\left(Q''(x)\right)$$
$$=V\left[\sum_{j=0}^{n-1}a_j\binom{n-1}{j}^{\frac{1}{2}}j(j-1)x^{j-2}\right]$$
$$=\left[\sigma^2(n-1)(n-2)x^4+4(n-2)x^2+2\right]\left(1+x^2\right)^{n-5} \tag{7}$$

$$C=Cov\left(Q'(x),Q''(x)\right)$$
$$=\sum_{j=0}^{n-1}a_j\binom{n-1}{j}j^2(j-1)x^{2j-3} \tag{8}$$

$$\rho=\frac{C}{AB} \tag{9}$$

$$\eta=B^{-1}\left(1-\rho^2\right)^{-\frac{1}{2}}=\frac{\left(Cm_1-A^2m_2\right)}{A\Delta} \tag{10}$$

$$\Delta^2=A^2B^2-C^2 \tag{11}$$

From (3) and $\Phi(t)=\frac{1}{2}+(\pi)^{-\frac{1}{2}}erf\left(\frac{t}{\sqrt{2}}\right)$,

We have the following formula

$$EN(\alpha,\beta)=\int_{\alpha}^{\beta}\left(\frac{\Delta}{2\pi A^2}\right)\times \exp\left[\frac{\left(-B^2m_1^2+2Cm_1m_2-A^2m_2^2\right)}{2\Delta^2}\right]dx$$
$$+\int_{\alpha}^{\beta}\left(Cm_1-A^2m_2\right)\left(\sqrt{2}\pi A^3\right)^{-1}\exp\left(\frac{-m_1^2}{2A^2}\right)\times erf\left[\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A\Delta\right)^{-1}\right]dx$$
$$\le\int_{\alpha}^{\beta}\left(\frac{\Delta}{2\pi A^2}\right)\exp\left[\frac{\left(-B^2m_1^2+2Cm_1m_2-A^2m_2^2\right)}{2\Delta^2}\right]dx$$
$$+\int_{\alpha}^{\beta}\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A^3\right)^{-1}\exp\left(\frac{-m_1^2}{2A^2}\right)dx$$

$$\text{As, } erf\left[\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A\Delta\right)^{-1}\le\sqrt{\pi}\right]$$

$$EN(\alpha,\beta)=\int_{\alpha}^{\beta}I_1(x)dx+\int_{\alpha}^{\beta}I_2(x)dx \tag{12}$$

Where

$$I_1(x)=\left(\frac{\Delta}{2\pi A^2}\right)\times \exp\left[\frac{\left(-B^2m_1^{\ 2}+2Cm_1m_2-A^2m_2^{\ 2}\right)}{2\Delta^2}\right] \quad (13)$$

$$I_2(x)=\left(Cm_1-A^2m_2\right)\left(\sqrt{2}\pi A^3\right)^{-1}\exp\left(\frac{-m_1^{\ 2}}{2A^2}\right) \quad (14)$$

3Proof of the Theorem

To find out the expected number of maxima, we divided the real line into two parts i.e.

$$(0,1-\varepsilon),(1-\varepsilon,1),(1,\infty)\ \&$$
$$(-1+\varepsilon,0),(-1,-1+\varepsilon),(-\infty,-1), \text{ Where } \varepsilon > 0$$

For $0 < x < 1-\varepsilon$

$$m_1=\frac{\mu}{(1+x)^2}(n-1)(1+x)^n$$
$$=\frac{\mu}{(1+x)^2}\mathrm{O}(n) \quad (15)$$

$$m_2=\frac{\mu}{(1+x)^3}(n-1)(n-2)(1+x)^n$$
$$=\frac{\mu}{(1+x)^3}\mathrm{O}(n^2) \quad (16)$$

$$A^2=\frac{\sigma^2}{(1+x^2)^3}(n-1)(1+nx^2-x^2)(1+x^2)^n$$
$$=\frac{\sigma^2}{(1+x^2)^3}\mathrm{O}(n^2) \quad (17)$$

$$B^2=\frac{\sigma^2}{(1+x^2)^5}(n-1)(n-2)\times$$
$$\left[(n-1)(n-2)x^4+4(n-2)x^2+2\right](1+x^2)^n$$
$$=\frac{\sigma^2}{(1+x^2)^5}\mathrm{O}(n^4) \quad (18)$$

$$C=\frac{\sigma^2x}{(1+x^2)^5}(n-1)(n-2)$$
$$\times\left[(n-1)x^2+2\right](1+x^2)^n$$

$$=\frac{\sigma^2x}{(1+x^2)^4}\mathrm{O}(n^3) \quad (19)$$

$$\Delta^2=A^2B^2-C^2$$
$$=\frac{\sigma^4(1-x^2)}{(1+x^2)^8}\mathrm{O}(n^6) \quad (20)$$

$$\frac{\Delta}{A^2}=\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\mathrm{O}(n\ ) \quad (21)$$

And

Now $I_1(0,(1-\varepsilon))=\int_0^{1-\varepsilon}\left(\frac{\Delta}{2\pi A^2}\right)\times$

$$\exp\left[\frac{\left(-B^2m_1^{\ 2}+2Cm_1m_2-A^2m_2^{\ 2}\right)}{2\Delta^2}\right]dx$$

$$=\frac{1}{2\pi}\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\mathrm{O}(n)\times$$

$$\exp\left[\frac{\frac{-\sigma^2}{(1+x^2)_5}\mathrm{O}(n^4)\frac{\mu^2}{(1+x)_4}\mathrm{O}(n^2) +2\frac{\sigma^2x}{(1+x^2)_4}\mathrm{O}(n^3)\frac{\mu}{(1+x)_2}\mathrm{O}(n)\frac{\mu}{(1+x)_3}\mathrm{O}(n^2) -\frac{\sigma^2}{(1+x^2)_3}\mathrm{O}(n^2)\frac{\mu^2}{(1+x)_6}\mathrm{O}(n^4)}{\frac{2\sigma^4(1-x^2)}{(1+x^2)^8}\mathrm{O}(n^6)}\right]dx$$

$$=\frac{1}{2\pi}\mathrm{O}(n)\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\times$$

$$\exp\left[\frac{\frac{-\mu^2\sigma^2}{(1+x^2)^5(1+x)^4}O(n^6)+\frac{2\mu^2\sigma^2 x}{(1+x^2)^4(1+x)^5}O(n^6)-\frac{\mu^2\sigma^2}{(1+x^2)^3(1+x)^6}O(n^6)}{\frac{2\sigma^4(1-x^2)}{(1+x^2)^8}O(n^6)}\right]dx$$

$$=\frac{1}{2\pi}O(n)\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\times$$

$$\exp\left[\frac{\frac{-\mu^2\sigma^2}{(1+x^2)^3(1+x)^4}O(n^6)\times\left(\frac{1}{(1+x^2)^2}-\frac{2x}{(1+x^2)(1+x)}+\frac{1}{(1+x)2}\right)}{\frac{2\sigma^4(1-x^2)}{(1+x^2)^8}O(n^6)}\right]dx$$

$$=\frac{1}{2\pi}O(n)\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\times$$

$$\exp\left[\frac{-\mu^2(1+x^2)^3(-3x^2-2x^3-3x^4)}{2\sigma^2(1-x^2)(1+x)^6}\right]dx$$

$$\langle\ \frac{1}{2\pi}O(n)\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\times$$

$$\exp\left[\frac{8\mu^2(1+x^2)^3}{2\sigma^2(1-x^2)(1+x)^6}\right]dx$$

$$as,\ (3x^2+2x^3+3x^4)\langle\ 8$$

$$=\frac{1}{2\pi}O(n)\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\times$$

$$\exp\left[\frac{4\mu^2(1+x^2)^3}{\sigma^2(1-x^2)(1+x)^6}\right]dx$$

$$\leq\frac{1}{2\pi}O(n)\int_0^{1-\varepsilon}\frac{(1-x^2)^{\frac{1}{2}}}{(1+x^2)}\frac{1}{(1-x^2)^{\frac{1}{2}}}dx$$

Where $\frac{4\mu^2}{\sigma^2}\leq\frac{(1-x^2)(1+x)^6\log(1-x^2)^{\frac{1}{2}}}{(1+x^2)^3}$

$$=\frac{1}{2\pi}O(n)\int_0^{1-\varepsilon}\frac{1}{(1+x^2)}dx$$

$$=\frac{1}{2\pi}O(n)\left[\tan^{-1}x\right]_0^{1-\varepsilon}$$

$$=\frac{1}{2\pi}O(n)\tan^{-1}(1-\varepsilon) \tag{22}$$

And

$$I_2(0,(1-\varepsilon))$$

$$=\int_0^{1-\varepsilon}(Cm_1-A^2m_2)(\sqrt{2}A^3)^{-1}\exp\left(\frac{-m_1^2}{2A^2}\right)dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_0^{1-\varepsilon}\left[\frac{\left(\frac{\sigma^2 x}{(1+x^2)_4}O(n^3)\frac{\mu}{(1+x)2}O(n)-\frac{\sigma^2}{(1+x^2)_3}O(n^2)\frac{\mu}{(1+x)3}O(n^2)\right)}{\frac{\sigma^3}{(1+x^2)^{\frac{9}{2}}}O(n^3)}\right]\times$$

$$\exp\left[\frac{\frac{-\mu^2}{(1+x)^4}O(n^2)}{\frac{2\sigma^2}{(1+x^2)^3}O(n^2)}\right]dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_0^{1-\varepsilon}\left[\frac{\begin{pmatrix}\frac{\mu\sigma^2 x}{(1+x^2)^4(1+x)^2}O(n^4)\\ -\frac{\mu\sigma^2}{(1+x^2)^3(1+x)^3}O(n^4)\end{pmatrix}}{\frac{\sigma^3}{(1+x^2)^{\frac{9}{2}}}O(n^3)}\right]\times$$

$$\exp\left[\frac{-\mu^3(1+x^2)^3}{2\sigma^2(1+x)^4}\right]dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_0^{1-\varepsilon}\left[\frac{\frac{\mu\sigma^2}{(1+x^2)^3(1+x)^2}O(n^4)\left(\frac{x}{(1+x^2)}-\frac{1}{1+x}\right)}{\frac{\sigma^3}{(1+x^2)^{\frac{9}{2}}}O(n^3)}\right]\times$$

$$\exp\left[\frac{-\mu^3(1+x^2)^3}{2\sigma^2(1+x)^4}\right]dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^{1-\varepsilon}\left[\left(\frac{\mu}{\sigma}\right)\frac{(x-1)(1+x^2)^{\frac{1}{2}}}{(1+x)^3}\right]\times$$

$$\exp\left[\frac{-\mu^3(1+x^2)^3}{2\sigma^2(1+x)^4}\right]dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^{1-\varepsilon}\left[-\left(\frac{\mu}{\sigma}\right)\frac{(1-x)(1+x^2)^{\frac{1}{2}}}{(1+x)^3}\right]\times$$

$$\exp\left[\frac{-\mu^3(1+x^2)^3}{2\sigma^2(1+x)^4}\right]dx$$

Where $\frac{\mu^2}{2\sigma^2}\geq\frac{(1+x)^4}{(1+x^2)^3}\log(1+x^2)^{\frac{1}{2}}$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^{1-\varepsilon}\frac{-(1+x^2)\left[\log(1+x^2)\right]^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}\times$$

$$\left[\frac{(1-x)(1+x^2)^{\frac{1}{2}}}{(1+x)^3}\frac{1}{(1+x^2)^{\frac{1}{2}}}dx\right]$$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^{1-\varepsilon}\frac{-(1-x)\left[\log(1+x^2)\right]^{\frac{1}{2}}}{(1+x)(1+x^2)^{\frac{3}{2}}}dx$$

$$\langle\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^{1-\varepsilon}-\left[\frac{\log(1+x^2)}{(1+x^2)^3}\right]^{\frac{1}{2}}dx$$

As $\frac{1-x}{1+x}\langle 1$

$$\langle\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^{1-\varepsilon}-\left[\frac{1}{(1+x^2)}\right]dx$$

As

$$\log\frac{x}{y}\langle\frac{x}{y}$$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)(-1)\left[\tan^{-1}x\right]_0^{1-\varepsilon}$$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)\left(-\tan^{-1}(1-\varepsilon)\right)$$

$$\approx\left(\frac{1}{\sqrt{2}}\right)O(n)\tan^{-1}(1-\varepsilon) \quad (23)$$

So, $EN\ (0,1-\varepsilon)=I_1(0,1-\varepsilon)+I_2(0,1-\varepsilon)$

$$=\frac{1}{2\pi}O(n)\tan^{-1}(1-\varepsilon)+\left(\frac{1}{\sqrt{2}}\right)O(n)\tan^{-1}(1-\varepsilon) \quad (24)$$

Next for, 1-$\varepsilon < x <$ 1, proceeding as in above for the case 0 < $x <$ 1-ε, we obtain:

$$I_1(1-\varepsilon,1)=\int_{1-\varepsilon}^{1}\left(\frac{\Delta}{2\pi A^2}\right)\times \exp\left[\frac{\left(-B^2m_1^{\ 2}+2Cm_1m_2-A^2m_2^{\ 2}\right)}{2\Delta^2}\right]dx$$

$$=\frac{1}{2\pi}\mathrm{O}(n)\int_{1-\varepsilon}^{1}\frac{1}{\left(1+x^2\right)}dx$$

$$=\frac{1}{2\pi}\mathrm{O}(n)\left[\tan^{-1}x\right]_{1-\varepsilon}^{1}$$

$$=\frac{1}{2\pi}\mathrm{O}(n)\left[\tan^{-1}(1)-\tan^{-1}(1-\varepsilon)\right] \qquad (25)$$

And

$$I_2(1-\varepsilon,1)=\int_{1-\varepsilon}^{1}\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A^3\right)^{-1}\times \exp\left(\frac{-m_1^{\ 2}}{2A^2}\right)dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)\mathrm{O}(n)\int_{1-\varepsilon}^{1}\left[-\left(\frac{\mu}{\sigma}\right)\frac{(1-x)\left(1+x^2\right)^{\frac{1}{2}}}{(1+x)^3}\right] \exp\left[\frac{-\mu^3\left(1+x^2\right)^3}{2\sigma^2(1+x)^4}\right]dx$$

$$\langle\left(\frac{1}{\sqrt{2}}\right)\mathrm{O}(n)\int_{0}^{1-\varepsilon}-\left[\frac{1}{\left(1+x^2\right)}\right]dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)\mathrm{O}(n)(-1)\left[\tan^{-1}x\right]_{0}^{1-\varepsilon}$$

$$=\left(\frac{1}{\sqrt{2}}\right)\mathrm{O}(n)\left[\tan^{-1}(1-\varepsilon)-\tan^{-1}(1)\right] \qquad (26)$$

So, $EN(1-\varepsilon,1)=I_1(1-\varepsilon,1)+I_2(1-\varepsilon,1)$

$$=\frac{1}{2\pi}\mathrm{O}(n)\left[\tan^{-1}(1)-\tan^{-1}(1-\varepsilon)\right]+ \left(\frac{1}{\sqrt{2}}\right)\mathrm{O}(n)\left[\tan^{-1}(1-\varepsilon)-\tan^{-1}(1)\right] \qquad (27)$$

Next for, $1<x<\infty$, let $x=\dfrac{1}{y}$ so, $0<y<1$

Therefore

$$m_1=\frac{\mu}{(1+x)^2}(n-1)(1+x)^n$$

$$=\frac{\mu y^2}{(1+y)^2}(n-1)\frac{(1+y)^n}{y^n}$$

$$=\frac{\mu y^2}{(1+y)^2}\mathrm{O}(n) \qquad (28)$$

$$m_2=\frac{\mu}{(1+x)^3}(n-1)(n-2)(1+x)^n$$

$$=\frac{\mu y^3}{(1+y)^3}(n-1)(n-2)\frac{(1+y)^n}{y^n}$$

$$=\frac{\mu y^3}{(1+y)^3}\mathrm{O}\left(n^2\right) \qquad (29)$$

$$A^2=\frac{\sigma^2}{\left(1+x^2\right)^3}(n-1)\left(1+nx^2-x^2\right)\left(1+x^2\right)^n$$

$$=\frac{\sigma^2}{\left(1+x^2\right)^3}(n-1)\left(1-x^2\right)\left(1+x^2\right)^n+ \frac{\sigma^2}{\left(1+x^2\right)^3}(n-1)\left(nx^2\right)\left(1+x^2\right)^n$$

$$=\frac{\sigma^2y^4\left(y^2-1\right)}{\left(1+y^2\right)^3}\frac{(n-1)\left(1+y^2\right)^n}{y^{2n}}+ \frac{\sigma^2y^4}{\left(1+y^2\right)^3}\frac{(n-1)(n)\left(1+y^2\right)^n}{y^{2n}}$$

$$=\frac{\sigma^2y^4\left(y^2-1\right)}{\left(1+y^2\right)^3}\mathrm{O}(n)+\frac{\sigma^2y^4}{\left(1+y^2\right)^3}\mathrm{O}\left(n^2\right)$$

$$=\frac{\sigma^2y^4}{\left(1+y^2\right)^3}\left[\frac{y^2-1}{n}+1\right]\mathrm{O}\left(n^2\right)$$

$$\approx\frac{\sigma^2y^4}{\left(1+y^2\right)^3}\mathrm{O}\left(n^2\right) \qquad (30)$$

$$B^2=\frac{\sigma^2}{\left(1+x^2\right)^5}(n-1)(n-2)\times$$

$$\left[(n-1)(n-2)x^4+4(n-2)x^2+2\right](1+x^2)^n$$

$$=\frac{\sigma^2 y^6}{(1+y^2)^5}(n-1)(n-2)\times$$

$$\left[(n-1)(n-2)+4(n-2)y^2+2y^4\right](1+y^2)^n$$

$$=\frac{\sigma^2 y^6}{(1+y^2)^5}\mathrm{O}(n^4) \tag{31}$$

$$C=\frac{\sigma^2 x}{(1+x^2)^4}(n-1)(n-2)\left[(n-1)x^2+2\right](1+x^2)^n$$

$$=\frac{\sigma^2 y^7}{(1+y^2)^4}(n-1)(n-2)\times$$

$$\left[\frac{(n-1)+2y^2}{y^2}\right]\frac{(1+y^2)^n}{y^{2n}}$$

$$=\frac{\sigma^2 y^5}{(1+y^2)^4}\mathrm{O}(n^3) \tag{32}$$

$$\Delta^2=A^2B^2-C^2=\frac{2\sigma^4 y^{10}}{(1+y^2)^8}\mathrm{O}(n^6) \tag{33}$$

and

$$\frac{\Delta}{A^2}=\frac{\sqrt{2}y}{(1+y^2)}\mathrm{O}(n) \tag{34}$$

Now $I_1(1,\infty)=I_1(0,1)=\int_0^1\left(\frac{\Delta}{2\pi A^2}\right)\times$

$$\exp\left[\frac{\left(-B^2m_1^2+2Cm_1m_2-A^2m_2^2\right)}{2\Delta^2}\right]dx$$

$$=\frac{1}{2\pi}\int_0^1\frac{\sqrt{2}y}{(1+y^2)}\mathrm{O}(n)\quad\times$$

$$\exp\left[\frac{\left(\begin{array}{l}\frac{-\sigma^2y^6}{(1+y^2)^5}\mathrm{O}(n^4)\frac{\mu^2y^4}{(1+y)^4}\mathrm{O}(n^2)\\ +2\frac{\sigma^2y^5}{(1+y^2)^4}\mathrm{O}(n^3)\frac{\mu y^2}{(1+y)^2}\\ \mathrm{O}(n)\frac{\mu y^3}{(1+x)^3}\mathrm{O}(n^2)\\ -\frac{\sigma^2y^4}{(1+y^2)^3}\mathrm{O}(n^2)\frac{\mu^2y^6}{(1+y)^6}\mathrm{O}(n^4)\end{array}\right)}{\frac{4\sigma^4y^{10}}{(1+y^2)^8}\mathrm{O}(n^6)}\right]\left(\frac{-1}{y^2}\right)dy$$

$$=\frac{1}{\sqrt{2}\pi}\int_1^\infty\frac{-1}{y(1+y^2)}\mathrm{O}(n)\quad\times$$

$$\exp\left[\frac{\frac{-\mu^2\sigma^2y^{10}\mathrm{O}(n^6)}{(1+y^2)^3(1+y)^4}\left(\frac{1}{(1+y^2)^2}-\frac{1}{(1+y^2)(1+y)}+\frac{1}{(1+y)^2}\right)}{\frac{4\sigma^4y^{10}\mathrm{O}(n^6)}{(1+y^2)^8}}\right]dy$$

$$=\frac{-1}{\sqrt{2}\pi}\int_1^\infty\frac{1}{y(1+y^2)}\mathrm{O}(n)\quad\times$$

$$\exp\left[\frac{-\mu^2\sigma^2y^{10}\mathrm{O}(n^6)(y^4-2y^3+y^2)(1+y^2)^8}{4\sigma^4y^{10}\mathrm{O}(n^6)(1+y^2)^5(1+y)^6}\right]dy$$

$$=\frac{-1}{\sqrt{2}\pi}\int_1^\infty\frac{1}{y(1+y^2)}\mathrm{O}(n)\quad\times$$

$$\exp\left[\frac{-\mu^2\left(y^4-2y^3+y^2\right)\left(1+y^2\right)^3}{4\sigma^2\,(1+y)^6}\right]dy$$

$$=\frac{-1}{\sqrt{2}\pi}\int_1^\infty\frac{1}{y\left(1+y^2\right)}O(n)\quad\times$$

$$\exp\left[\frac{\mu^2\left(1+y^2\right)^3\left(2y^3-y^4-y^2\right)}{4\sigma^2\,(1+y)^6}\right]dy$$

$$\langle\ \frac{-1}{2\sqrt{2}\pi}\int_1^\infty\frac{1}{y}O(n)\exp\left[\frac{\mu^2\left(1+y^2\right)^3}{2\sigma^2\,(1+y)^6}\right]dy$$

Where

$$\left(2y^3-y^4-y^2\right)\langle\ 2$$

And

$$\frac{-1}{\left(1+y^2\right)}\langle\ \frac{-1}{2}$$

$$=\frac{-1}{2\sqrt{2}\pi}O(n)\int_1^\infty\frac{1}{y}\times y\,dy$$

Where

$$\frac{\mu}{2\sigma^2}\le\frac{(1+y)^6}{\left(1+y^2\right)^3}\log y$$

$$=\frac{-1}{2\sqrt{2}\pi}O(n)$$

$$\approx\frac{1}{2\sqrt{2}\pi}O(n)\qquad(35)$$

And

$$I_2(1,\infty)=\int_1^\infty\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A^3\right)^{-1}$$

$$\exp\left(\frac{-m_1^2}{2A^2}\right)dx$$

$$\Rightarrow I_2(0,1)$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_0^1\left[\frac{\left(\begin{array}{c}\frac{\sigma^2y^5}{(1+y^2)_4}O\left(n^3\right)\frac{\mu y^2}{(1+y)_2}O(n)\\-\frac{\sigma^2y^4}{(1+y^2)_3}O\left(n^2\right)\frac{\mu y^3}{(1+y)_3}O\left(n^2\right)\end{array}\right)}{\frac{\sigma^3y^6}{\left(1+y^2\right)^{\frac{9}{2}}}O\left(n^3\right)}\right]\times$$

$$\exp\left[\frac{\frac{-\mu^2y^4}{(1+y)^4}O\left(n^2\right)}{\frac{2\sigma^2y^4}{\left(1+y^2\right)^3}O\left(n^2\right)}\right]\left(\frac{-1}{y^2}\right)dy$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_0^1\frac{\frac{-\mu\sigma^2y^7O\left(n^4\right)}{\left(1+y^2\right)^3(1+y)^2y^2}\left[\begin{array}{c}\frac{1}{\left(1+y^2\right)}\\-\frac{1}{(1+y)}\end{array}\right]}{\frac{\sigma^3y^6}{\left(1+y^2\right)^{\frac{9}{2}}}O\left(n^3\right)}\times$$

$$\exp\left[\frac{-\mu^2\left(1+y^2\right)^3}{2\sigma^2(1+y)^4}\right]dy$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_0^1\frac{-\mu y\,(1-y)\left(1+y^2\right)^{\frac{1}{2}}O(n)}{\sigma(1+y)^3}\times$$

$$\exp\left[\frac{-\mu^2\left(1+y^2\right)^3}{2\sigma^2(1+y)^4}\right]dy$$

$$\langle\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^1\left(\frac{-\mu}{\sigma}\right)\frac{\left(1+y^2\right)^{\frac{1}{2}}}{(1+y)^3}\times$$

$$\exp\left[\frac{-\mu^2\left(1+y^2\right)^3}{2\sigma^2(1+y)^4}\right]dy\quad\text{Where }(1-y)<1$$

$$\leq\left(\frac{1}{\sqrt{2}}\right)O(n)\times$$

$$\int_0^1\frac{-\sqrt{2}(1+y)^2\left[\log(1+y)\right]^{\frac{1}{2}}}{\left(1+y^2\right)^{\frac{3}{2}}}\frac{\left(1+y^2\right)^{\frac{1}{2}}}{(1+y)^3}(1+y)dy$$

Where

$$\left(\frac{\mu^2}{2\sigma}\right)\geq\frac{(1+y)^4}{\left(1+y^2\right)^3}\log(1+y)$$

$$=\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^1\frac{-\left[\log(1+y)\right]^{\frac{1}{2}}}{\left(1+y^2\right)}dy$$

$$<\left(\frac{1}{\sqrt{2}}\right)O(n)\int_0^1\frac{-(1+y)^{\frac{1}{2}}}{\left(1+y^2\right)}dy$$

As

$$\frac{-(1+y)^{\frac{1}{2}}}{\left(1+y^2\right)}\langle\frac{-1}{\sqrt{2}}\langle-1$$

$$<\left(\frac{1}{\sqrt{2}}\right)(-1)O(n)\int_0^1 dy$$

$$=\left(\frac{-1}{\sqrt{2}}\right)O(n)$$

$$\approx\left(\frac{1}{\sqrt{2}}\right)O(n) \tag{36}$$

So, $EN\,(1,\infty)=I_1(1,\infty)+I_2(1,\infty)$

$$=\frac{1}{2\sqrt{2}\pi}O(n)+\frac{1}{\sqrt{2}}O(n) \tag{37}$$

Now for, $-1+\varepsilon<x<0$

$$=m_1=\frac{\mu}{(1+x)^2}(n-1)(1+x)^n=0 \tag{38}$$

$$m_2=\frac{\mu}{(1+x)^3}(n-1)(n-2)(1+x)^n=0 \tag{39}$$

$$A^2=\frac{\sigma^2}{\left(1+x^2\right)^3}(n-1)\left(1+nx^2-x^2\right)\left(1+x^2\right)^n$$

$$=\frac{\sigma^2}{\left(1+x^2\right)^3}O\left(n^2\right) \tag{40}$$

$$B^2=\frac{\sigma^2}{\left(1+x^2\right)^5}(n-1)(n-2)\times$$

$$\left[(n-1)(n-2)x^4+4(n-2)x^2+2\right]\left(1+x^2\right)^n$$

$$=\frac{\sigma^2}{\left(1+x^2\right)^5}O\left(n^4\right) \tag{41}$$

$$C=\frac{\sigma^2 x}{\left(1+x^2\right)^5}(n-1)(n-2)\left[(n-1)x^2+2\right]\left(1+x^2\right)^n$$

$$=\frac{\sigma^2 x}{\left(1+x^2\right)^4}O\left(n^3\right) \tag{42}$$

$$\Delta^2=A^2B^2-C^2$$

$$=\frac{\sigma^4\left(1-x^2\right)}{\left(1+x^2\right)^8}O\left(n^6\right) \tag{43}$$

And

$$\frac{\Delta}{A^2}=\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\left(1+x^2\right)}O(n) \tag{44}$$

Now

$$I_1(-1+\varepsilon,0)=\int_{-1+\varepsilon}^{0}\left(\frac{\Delta}{2\pi A^2}\right)$$

$$=\frac{1}{2\pi}\int_{-1+\varepsilon}^{0}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\left(1+x^2\right)}O(n)\times$$

$$\exp\left[\frac{\left(-B^2m_1^2+2Cm_1m_2-A^2m_2^2\right)}{2\Delta^2}\right]dx$$

Where exponential part tends to be 1 as both m_1 and m_2 equal to zero.

$$<\frac{1}{2\pi}O(n)\int_{-1+\varepsilon}^{0}\frac{1}{\left(1+x^2\right)}dx$$

$$=\frac{1}{2\pi}O(n)\left[\tan^{-1}x\right]_{-1+\varepsilon}^{0}$$

$$=\frac{1}{2\pi}O(n)\left[-\tan^{-1}(\varepsilon-1)\right]$$

$$=\approx\frac{1}{2\pi}O(n)\left[\tan^{-1}(\varepsilon-1)\right] \quad (45)$$

And

$$I_2(-1+\varepsilon,0)=\int_{-1+\varepsilon}^{0}\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A^3\right)^{-1}\exp\left(\frac{-m_1^2}{2A^2}\right)=0 \quad (46)$$

As m_1 and m_2 both equals to zero

So,

$$EN(-1+\varepsilon,0)=I_1(-1+\varepsilon,0)+I_2(-1+\varepsilon,0)$$

$$==\frac{1}{2\pi}O(n)\left[\tan^{-1}(\varepsilon-1)\right] \quad (47)$$

Next for, -1 < x < -1+ ε , proceeding as in above for the case -1+ ε < x < 0, we obtain

$$m_1=\frac{\mu}{(1+x)^2}(n-1)(1+x)^n=0 \quad (48)$$

$$m_2=\frac{\mu}{(1+x)^3}(n-1)(n-2)(1+x)^n=0 \quad (49)$$

$$A^2=\frac{\sigma^2}{(1+x^2)^3}(n-1)\left(1+nx^2-x^2\right)\left(1+x^2\right)^n$$

$$=\frac{\sigma^2}{(1+x^2)^3}O\left(n^2\right) \quad (50)$$

$$B^2=\frac{\sigma^2}{(1+x^2)^5}(n-1)(n-2)$$

$$\left[(n-1)(n-2)x^4+4(n-2)x^2+2\right]\left(1+x^2\right)^n \quad (50)$$

$$=\frac{\sigma^2}{(1+x^2)^5}O\left(n^4\right)$$

$$C=\frac{\sigma^2 x}{(1+x^2)^5}(n-1)(n-2)\left[(n-1)x^2+2\right]\left(1+x^2\right)^n$$

$$=\frac{\sigma^2 x}{(1+x^2)^4}O\left(n^3\right) \quad (52)$$

$$\Delta^2=A^2B^2-C^2=\frac{\sigma^4\left(1-x^2\right)}{(1+x^2)^8}O\left(n^6\right) \quad (53)$$

And

$$=\frac{\Delta}{A^2}=\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\left(1+x^2\right)}O(n) \quad (54)$$

Now

$$I_1(-1,-1+\varepsilon)=\int_{-1}^{-1+\varepsilon}\left(\frac{\Delta}{2\pi A^2}\right)\times$$

$$\exp\left[\frac{\left(-B^2m_1^2+2Cm_1m_2-A^2m_2^2\right)}{2\Delta^2}\right]dx$$

$$=\frac{1}{2\pi}\int_{-1}^{-1+\varepsilon}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\left(1+x^2\right)}O(n)$$

$$<\frac{1}{2\pi}O(n)\int_{-1}^{-1+\varepsilon}\frac{1}{\left(1+x^2\right)}dx$$

$$\text{as}\left(1-x^2\right)^{\frac{1}{2}}\langle 1$$

$$=\frac{1}{2\pi}O(n)\left[\tan^{-1}x\right]_{-1}^{-1+\varepsilon}$$

$$=\frac{1}{2\pi}O(n)\left[\tan^{-1}(\varepsilon-1)-\tan^{-1}(1)\right] \quad (55)$$

And

$$I_2(-1,-1+\varepsilon)=\int_{-1}^{-1+\varepsilon}\left(Cm_1-A^2m_2\right)\left(\sqrt{2}A^3\right)^{-1}\times$$

$$\exp\left(\frac{-m_1^2}{2A^2}\right)dx=0 \quad (55)$$

So, $EN(-1,-1+\varepsilon)=I_1(-1,-1+\varepsilon)+I_2(-1,-1+\varepsilon)$

$$=\frac{1}{2\pi}O(n)\left[\tan^{-1}(\varepsilon-1)-\tan^{-1}(1)\right] \quad (57)$$

At the end for the interval $-\infty<x<-1$,

Let $x=\frac{1}{y}$ so, -1 < y < 0.

Now -1 < y < 0 can be split into

-1 < y < $\frac{-1}{2}$ and $\frac{-1}{2}$ < y < 0.

For $-1\langle y\langle\frac{-1}{2}$

$$m_1=\frac{\mu y^2}{(1+y)^2}(n-1)\frac{(1+y)^n}{y^n}=0 \quad (58)$$

$$m_2 = \frac{\mu}{(1+x)^3}(n-1)(n-2)(1+x)^n$$

$$= \frac{\mu y^3}{(1+y)^3}(n-1)(n-2)\frac{(1+y)^n}{y^n} = 0 \quad (59)$$

$$A^2 = \frac{\sigma^2}{(1+x^2)^3}(n-1)(1+nx^2-x^2)(1+x^2)^n =$$

$$\frac{\sigma^2}{(1+x^2)^3}(n-1)(1-x^2)(1+x^2)^n +$$

$$\frac{\sigma^2}{(1+x^2)^3}(n-1)(nx^2)(1+x^2)^n$$

$$= \frac{\sigma^2 y^4(y^2-1)}{(1+y^2)^3}\frac{(n-1)(1+y^2)^n}{y^{2n}} +$$

$$\frac{\sigma^2 y^4}{(1+y^2)^3}\frac{(n-1)(n)(1+y^2)^n}{y^{2n}}$$

$$= \frac{\sigma^2 y^4(y^2-1)}{(1+y^2)^3}\mathrm{O}(n) + \frac{\sigma^2 y^4}{(1+y^2)^3}\mathrm{O}(n^2)$$

$$= \frac{\sigma^2 y^4}{(1+y^2)^3}\left[\frac{y^2-1}{n}+1\right]\mathrm{O}(n^2)$$

$$\approx \frac{\sigma^2 y^4}{(1+y^2)^3}\mathrm{O}(n^2) \quad (60)$$

$$B^2 = \frac{\sigma^2}{(1+x^2)^5}(n-1)(n-2)\times$$

$$\left[(n-1)(n-2)x^4+4(n-2)x^2+2\right](1+x^2)^n$$

$$= \frac{\sigma^2 y^6}{(1+y^2)^5}(n-1)(n-2)\times$$

$$\left[(n-1)(n-2)+4(n-2)y^2+2y^4\right](1+y^2)^n$$

$$= \frac{\sigma^2 y^6}{(1+y^2)^5}\mathrm{O}(n^4) \quad (61)$$

$$C = \frac{\sigma^2 x}{(1+x^2)^4}(n-1)(n-2)\left[(n-1)x^2+2\right](1+x^2)^n$$

$$\left[\frac{(n-1)+2y^2}{y^2}\right]\frac{(1+y^2)^n}{y^{2n}} \quad (62)$$

$$= \frac{\sigma^2 y^7}{(1+y^2)^4}(n-1)(n-2)\times$$

$$= \frac{\sigma^2 y^5}{(1+y^2)^4}\mathrm{O}(n^3)$$

$$\Delta^2 = A^2B^2 - C^2 = \frac{2\sigma^4 y^{10}}{(1+y^2)^8}\mathrm{O}(n^6) \quad (63)$$

And,

$$\frac{\Delta}{A^2} = \frac{\sqrt{2}y}{(1+y^2)}\mathrm{O}(n) \quad (64)$$

Now,

$$I_1\left(-1,-\frac{1}{2}\right)$$

$$= \int_{-1}^{-\frac{1}{2}}\left(\frac{\Delta}{2\pi A^2}\right)\exp\left[\frac{\left(-B^2m_1^2+2Cm_1m_2-A^2m_2^2\right)}{2\Delta^2}\right]dx$$

$$= \frac{1}{2\pi}\int_{-1}^{-\frac{1}{2}}\frac{\sqrt{2}y}{(1+y^2)}\mathrm{O}(n)\left(\frac{-1}{y^2}\right)dy$$

Where exponential part tends to be 1 as both m_1 and m_2 equal to zero.

$$= \frac{\mathrm{O}(n)}{2\pi}\int_{-1}^{-\frac{1}{2}}\frac{-1}{y(1+y^2)}dy$$

$$= \frac{\mathrm{O}(n)}{2\sqrt{2}\pi}\int_{-1}^{-\frac{1}{2}}dy$$

Where, $\dfrac{-1}{y(1+y^2)} \langle \dfrac{1}{2}$

$$= \frac{1}{4\sqrt{2}\pi}\mathrm{O}(n) \quad (65)$$

And

$$I_2\left(-1,-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} \left(Cm_1 - A^2 m_2\right)\left(\sqrt{2}A^3\right)^{-1} \exp\left(\frac{-m_1^2}{2A^2}\right) dx = 0 \quad (66)$$

As both m_1 and m_2 is equal to zero.

So, $EN\left(-1,-\frac{1}{2}\right) = I_1\left(-1,-\frac{1}{2}\right) + I_2\left(-1,-\frac{1}{2}\right)$

$$= \frac{1}{4\sqrt{2\pi}} O(n) \quad (67)$$

Finally for $\frac{-1}{2} \langle y \langle 0$

$$m_1 = \frac{\mu y^2}{(1+y)^2} O(n) \quad (68)$$

$$m_2 = \frac{\mu y^3}{(1+y)^3} O(n^2) \quad (69)$$

$$A^2 = \frac{\sigma^2 y^4}{(1+y^2)^3} O(n^2) \quad (70)$$

$$B^2 = \frac{\sigma^2 y^6}{(1+y^2)^5} O(n^4) \quad (71)$$

$$C = \frac{\sigma^2 y^5}{(1+y^2)^4} O(n^3) \quad (72)$$

$$\Delta^2 = A^2 B^2 - C^2 = \frac{2\sigma^4 y^{10}}{(1+y^2)^8} O(n^6) \quad (73)$$

$$\text{and } \frac{\Delta}{A^2} = \frac{\sqrt{2}y}{(1+y^2)} O(n) \quad (74)$$

Now $I_1\left(-\frac{1}{2},0\right)$

$$= \int_{-\frac{1}{2}}^{0} \left(\frac{\Delta}{2\pi A^2}\right) \exp\left[\frac{\left(-B^2 m_1^2 + 2Cm_1 m_2 - A^2 m_2^2\right)}{2\Delta^2}\right] dx$$

$$= \frac{1}{2\pi} \int_{-\frac{1}{2}}^{0} \frac{\sqrt{2}y}{(1+y^2)} O(n) \times \exp\left[\frac{\mu^2 (1+y^2)^3 (2y^3 - y^4 - y^2)}{4\sigma^2 (1+y)^6}\right]\left(\frac{-1}{y^2}\right) dy$$

$$= \frac{1}{\sqrt{2\pi}} O(n) \int_{-\frac{1}{2}}^{0} \frac{-1}{y(1+y^2)} \times \exp\left[\frac{\mu^2 (1+y^2)^3 (2y^3 - y^4 - y^2)}{4\sigma^2 (1+y)^6}\right] dy$$

where $\frac{\mu^2}{4\sigma^2} \le \frac{(1+y)^6 \log y}{(1+y^2)^3 (2y^3 - y^4 - y^2)}$

$$= \frac{1}{\sqrt{2\pi}} O(n) \int_{-\frac{1}{2}}^{0} \frac{-1}{y} y\, dy$$

$$\text{As}\left(\frac{-1}{1+y^2}\right) < -1 = \frac{-1}{\sqrt{2\pi}} O(n) \int_{-\frac{1}{2}}^{0} dy$$

$$= \frac{1}{\sqrt{2\pi}} O(n) \int_{-\frac{1}{2}}^{0} (-1) dy$$

$$= \frac{1}{\sqrt{2\pi}} O(n) [-y]_{-\frac{1}{2}}^{0}$$

$$= \frac{1}{\sqrt{2\pi}} O(n) [-y]_{-\frac{1}{2}}^{0}$$

$$= \frac{1}{2\sqrt{2\pi}} O(n) \quad (75)$$

And

$$I_2\left(\frac{-1}{2},0\right) = \int_{\frac{-1}{2}}^{0} \left(Cm_1 - A^2 m_2\right)\left(\sqrt{2}A^3\right)^{-1} \exp\left(\frac{-m_1^2}{2A^2}\right) dx$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_{-\frac{1}{2}}^{0}\left[\frac{\left(\begin{array}{l}\frac{\sigma^2 y^5}{(1+y^2)_4}O(n^3)\frac{\mu y^2}{(1+y)_2}O(n)\\ -\frac{\sigma^2 y^4}{(1+y^2)_3}O(n^2)\frac{\mu y^3}{(1+y)_3}O(n^2)\end{array}\right)}{\frac{\sigma^3 y^6}{(1+y^2)^{\frac{9}{2}}}O(n^3)}\right]\times$$

$$\exp\left[\frac{\frac{-\mu^2 y^4}{(1+y)^4}O(n^2)}{\frac{2\sigma^2 y^4}{(1+y^2)^3}O(n^2)}\right]\left(\frac{-1}{y^2}\right)dy$$

$$=\left(\frac{1}{\sqrt{2}}\right)\times$$

$$\int_{-\frac{1}{2}}^{0}\left[\frac{\frac{-\mu\sigma^2 y^7 O(n^4)}{(1+y^2)^3(1+y)^2 y^2}}{\frac{\sigma^3 y^6}{(1+y^2)^{\frac{9}{2}}}O(n^3)}\right]\left[\frac{1}{(1+y^2)}-\frac{1}{(1+y)}\right]\times$$

$$\exp\left[\frac{-\mu^2(1+y^2)^3}{2\sigma^2(1+y)^4}\right]dy$$

$$=\left(\frac{1}{\sqrt{2}}\right)\int_{-\frac{1}{2}}^{0}\frac{-\mu y(1-y)(1+y^2)^{\frac{1}{2}}O(n)}{\sigma(1+y)^3}\times$$

$$\exp\left[\frac{-\mu^2(1+y^2)^3}{2\sigma^2(1+y)^4}\right]dy$$

$$=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{3}{2}\right)O(n)\int_{-\frac{1}{2}}^{0}\left(\frac{-\mu}{\sigma}\right)\frac{(1+y^2)^{\frac{1}{2}}}{(1+y)^3}\times$$

$$\exp\left[\frac{-\mu^2(1+y^2)^3}{2\sigma^2(1+y)^4}\right]dy$$

Where *(1-y) < 1*

$$\le\left(\frac{3}{2\sqrt{2}}\right)O(n)\int_{-\frac{1}{2}}^{0}\frac{-\sqrt{2}(1+y)^2\left[\log(1+y)\right]^{\frac{1}{2}}}{(1+y^2)^{\frac{3}{2}}}\times$$

$$\frac{(1+y^2)^{\frac{1}{2}}}{(1+y)^3}(1+y)dy$$

Where $\left(\frac{\mu^2}{2\sigma^2}\right)\ge\frac{(1+y)^4}{(1+y^2)^3}\log(1+y)$

$$=\left(\frac{3}{2}\right)O(n)\int_{-\frac{1}{2}}^{0}\frac{-\left[\log(1+y)\right]^{\frac{1}{2}}}{(1+y^2)}dy$$

$$<\left(\frac{3}{2}\right)O(n)\int_{-\frac{1}{2}}^{0}\frac{-(1+y)^{\frac{1}{2}}}{(1+y^2)}dy$$

As $\frac{-(1+y)^{\frac{1}{2}}}{(1+y^2)}\langle\frac{-4}{5}$

$$<\left(\frac{3}{2}\right)\left(\frac{-4}{5}\right)O(n)\int_{-\frac{1}{2}}^{0}dy$$

$$=\left(\frac{-6}{5}\right)O(n)\left[y\right]_{-\frac{1}{2}}^{0}$$

$$=\left(\frac{-6}{5}\right)O(n)\times\frac{1}{2}$$

$$\approx\left(\frac{3}{5}\right)O(n) \qquad (76)$$

So, $EN\left(-\frac{1}{2},0\right)=I_1\left(-\frac{1}{2},0\right)+I_2\left(-\frac{1}{2},0\right)$

$$=\frac{1}{2\sqrt{2\pi}}O(n)+\frac{3}{5}O(n) \qquad (77)$$

From (67) and (77) we obtain

$EN(-\infty,-1)$

$$=EN(-1,0)+EN\left(-1,-\frac{1}{2}\right)+EN\left(-\frac{1}{2},0\right)$$

$$=\frac{1}{4\sqrt{2\pi}}O(n)+\frac{1}{2\sqrt{2\pi}}O(n)+\frac{3}{5}O(n) \qquad (78)$$

Now using (24) and (27) we get,

$$EN\,(0,1) = EN\,\left(0,1-\varepsilon\right) + EN\,\left(1-\varepsilon,1\right)$$
$$= \frac{1}{2\pi}\mathrm{O}\left(n\right)\tan^{-1}\left(1-\varepsilon\right) + \left(\frac{1}{\sqrt{2}}\right)\mathrm{O}\left(n\right)\tan^{-1}\left(1-\varepsilon\right)$$
$$+\frac{1}{2\pi}\mathrm{O}\left(n\right)\left[\tan^{-1}\left(1\right) - \tan^{-1}\left(1-\varepsilon\right)\right]$$
$$+\left(\frac{1}{\sqrt{2}}\right)\mathrm{O}\left(n\right)\left[\tan^{-1}\left(1-\varepsilon\right) - \tan^{-1}\left(1\right)\right]$$
$$\approx \frac{1}{2\pi}\mathrm{O}\left(n\right)\left[2\tan^{-1}\left(1-\varepsilon\right) + \tan^{-1}\left(1\right)\right]$$
$$+\left(\frac{1}{\sqrt{2}}\right)\mathrm{O}\left(n\right)\left[2\tan^{-1}\left(1-\varepsilon\right) + \tan^{-1}\left(1\right)\right]$$
$$= \left[2\tan^{-1}\left(1-\varepsilon\right) + \tan^{-1}\left(1\right)\right]\left[\frac{1}{2\pi} + \frac{1}{\sqrt{2}}\right]\mathrm{O}\left(n\right)$$
$$= \left[\frac{\pi\sqrt{2}+1}{2\pi}\right]\left[2\tan^{-1}\left(1-\varepsilon\right) + \tan^{-1}\left(1\right)\right]\mathrm{O}\left(n\right) \tag{79}$$

Using (47) and (57) we obtain,

$$EN\,\left(-1,0\right) = EN\,\left(-1+\varepsilon,0\right) + EN\,\left(-1,-1+\varepsilon\right)$$
$$= \frac{1}{2\pi}\mathrm{O}\left(n\right)\left[-\tan^{-1}\left(\varepsilon-1\right)\right]$$
$$+\frac{1}{2\pi}\mathrm{O}\left(n\right)\left[\tan^{-1}\left(\varepsilon-1\right) - \tan^{-1}\left(1\right)\right]$$
$$= \frac{1}{2\pi}\mathrm{O}\left(n\right)\left[2\tan^{-1}\left(\varepsilon-1\right) - \tan^{-1}\left(1\right)\right] \tag{80}$$

Lastly using (37) we obtain,

$$EN\,\left(1,\infty\right) = \frac{1}{2\sqrt{2}\pi}\mathrm{O}\left(n\right) + \frac{1}{\sqrt{2}}\mathrm{O}\left(n\right)$$
$$= \frac{1}{2\pi}\left[\frac{2\pi+1}{\sqrt{2}}\right]\mathrm{O}\left(n\right)$$
$$\approx EN\,\left(-\infty,-1\right) \tag{81}$$

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