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# Solitary Wave Solutions of the Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-Law Nonlinearity

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**Abstract:** In this paper, we employ the extended tanh function method to find the exact traveling wave solutions and solitary wave solutions involving parameters of the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. These studies reveal that the Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-law nonlinearity has a rich variety of solutions

**Keywords:** The extended tanh function method; The Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity; Traveling wave solutions; Solitary wave solutions.

## 1 Introduction

It is well known that many models in mathematics and physics are described by nonlinear partial differential equations (NLPDEs). The theory of solitons has contributed to understanding many experiments and complex phenomena in mathematical physics. Thus, it is of interest to evaluate new solitary wave solutions of these equations. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. for example, the modified simple equation method [1,2,3,4], $\left(\frac{G'}{G}\right)$-expansion method [5,6,7], modified $\left(\frac{G'}{G}\right)$-expansion method [8,9], extended Jacobi elliptic function method [10,11], exp $-\varphi(\xi)$-expansion method [12,13,14], extended exp $-\varphi(\xi)$-expansion method [15], Riccati-Bernoulli Sub-ODE method [16,17], the extended tanh- fuction method [18,19,20,21,22] and so on.
In this paper, we shall use the extended tanh function method to find the exact and solitary wave solutions of Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity [23]. This equation is an alternative to the Korteweg-de Vries (KdV) equation, and describes the unidi-rectional propagation of small-amplitude long waves on the surface of the water in a channel. The BBM equation is not only convenient for shallow water waves but also for hydromagnetic waves, acoustic waves, and therefore it has more advantages compared with the KdV equation. The main idea of this method is finding the exact solutions of any models which can be expressed by a polynomial of $\phi(\xi)$ which satisfies the nonlinear ordinary differential equations $\phi' = b + \phi^2(\xi)$ where $\xi = x - ct$ while $a, c$ are arbitrary constants to be determined later. The degree of the polynomial can be calculated by the homogenous balance between the highest order derivatives and the nonlinear terms. Equating the coefficients of the same power of $\phi(\xi)$ we get the system of algebraic equation. The constants of the polynomial can be determined by solving the system of algebraic equation by any computer program.
The rest of this paper is arranged as follows: Description of the method ( the extended tanh function method ) in Section 2. We use this method to find the exact solutions of the nonlinear evolution equations pointed out above

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and also we make the comparison between our results and another result that obtained by three different method in Section 3. Conclusions are given in Section 4.

## 2 Basic steps of the modified extended tanh-function method

Consider the following nonlinear evolution equation

$$F(u,u_t,u_x,u_{tt},u_{xx},....)=0, \quad (1)$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following,we give the main steps of this method :

**Step 1.** We use the wave transformation

$$u(x,t)=u(\xi), \qquad \xi=x-ct, \quad (2)$$

where c is a constant, to reduce Eq.(1) to the following ODE:

$$P(u,u',u'',u''',.....)=0, \quad (3)$$

where P is a polynomial in $u(\xi)$ and its total derivatives,while $\left\{{}'=\frac{d}{d\xi}{}'\right\}$.

**Step 2.** Suppose the solution of Eq.(3) has the form:

$$u(\xi)=a_0+\sum_{i=1}^{N}\left(a_i\,\phi^i+b_i\,\phi^{-i}\right), \quad (4)$$

where $a_i$, $b_i$ are constants to be determined, such that $a_N\neq 0$ or $b_N\neq 0$ and $\phi$ satisfies the Riccati equation

$$\phi'=b+\phi^2, \quad (5)$$

**Step 3.** Determine the positive integer $N$ in Eq.(4) by balancing the highest order derivatives and the nonlinear terms.

**Step 4.** Substitute Eq.(4) along Eq.(5) into Eq.(3) and collecting all the terms of the same power $\phi^i$, $i=N,N-1,...,1-N,-N$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of $a_i$ and $b_i$.

**Step 5.** Substituting these values and the solutions of Eq.(5) into Eq.(4) we obtain the exact solutions of Eq.(1).

## 3 The Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-Law Nonlinearity

Consider the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity [23]

$$u_t+au_x+\left(b_2u^{2n}+b_3u\right)u_x+cu_{xx}+ku_{xxt}=0, \quad (6)$$

Where $a$ represents the strength of a defection or drifting, $b_2,b_3$ measure the strength of the two nonlinear terms with the exponent $n$ being the power law nonlinearity parameter while the parameters $c,k$ are the dissipative diffraction coefficient. When $(n=1)$, Eq. (6) can be written as the following:

$$u_t+au_x+\left(b_2u^2+b_3u\right)u_x+cu_{xx}+ku_{xxt}=0, \quad (7)$$

using the transformation $u(\xi)=u(x,t)$ where $(\xi=x-ct)$, we get

$$(a-c_1)u'+\left(b_2u^2+b_3u\right)u'+cu''-kc_1u'''=0. \quad (8)$$

Balancing $u^2u'$ and $u'''$ we obtain $(2M+M+1=M+3)\Rightarrow(M=1)$, so that the solution of Eq. (8) can be written in the form:

$$u(\xi)=a_0+a_1\phi+\frac{b_1}{\phi}. \quad (9)$$

Substituting Eq. (9) and its derivatives into Eq. (8) and collecting all term with the same power of $\left(\phi^i\right)$ where $(i=4,3,2,1,0,-1,-2,-3,-4)$ and equating them to zero, we obtain the system of algebraic equations. Solving this system of equation, we obtain:

**Case 1.**

$$a=\frac{16bkb_2^2b_3^4-12b^4c^2k+3b^2kb_3^4+2b_2^2b_3^4}{12kb^2b_2b_3^2},$$

$$a_0=\frac{-b_3^2+2bc}{2b_3b_2},a_1=\frac{-b_3}{b},b_1=-ba_1,c_1=\frac{b_2b_3^2}{6kb^2}.$$

So that, the exact traveling wave solutions of Eq. (8) will be in the form

$$u(\xi)=\frac{-b_3^2+2bc}{2b_3b_2}-\frac{b_3}{b}\phi-\frac{ba_1}{\phi}. \quad (10)$$

The solitary wave solutions will be in the form:

When $(b<0)$,we get:

$$u(\xi)=\tfrac{-b_3^2+2bc}{2b_3b_2}+\tfrac{b_3\sqrt{-b}}{b}tanh(\sqrt{-b}\xi) \quad (11)$$

$$-a_1\sqrt{-b}coth(\sqrt{-b}\xi), \quad (12)$$

or

$$u(\xi)=\tfrac{-b_3^2+2bc}{2b_3b_2}+\tfrac{b_3\sqrt{-b}}{b}coth(\sqrt{-b}\xi) \quad (13)$$

$$-a_1\sqrt{-b}tanh(\sqrt{-b}\xi). \quad (14)$$

When $(b>0)$,we get:

$$u(\xi)=\frac{-b_3^2+2bc}{2b_3b_2}-\frac{b_3\sqrt{b}}{b}tan(\sqrt{b}\xi)-a_1\sqrt{b}cot(\sqrt{b}\xi), \quad (15)$$

or

$$u(\xi)=\frac{-b_3^2+2bc}{2b_3b_2}-\frac{b_3\sqrt{b}}{b}cot(\sqrt{b}\xi)-a_1\sqrt{-b}tan(\sqrt{b}\xi), \quad (16)$$

When $(b=0)$,we get:

The trivial solution.

**Case 2.**

$$a = \frac{4bkb_2^2b_3^4 - 12b^4c^2k + 3b^2kb_3^4 + 2b_2^2b_3^4}{12kb^2b_2b_3^2},$$

$$a_0 = \frac{-b_3^2 + 2bc}{2b_3b_2}, a_1 = 0, b_1 = \pm\sqrt{\frac{6ckb^2}{b_2}}, c_1 = \frac{b_2b_3^2}{6kb^2}.$$

So that, the exact traveling wave solutions of Eq. (8) will be in the form

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \pm \sqrt{\frac{6ckb^2}{b_2}}\frac{1}{\phi}. \quad (17)$$

The solitary wave solutions will be in the form:
When $(b < 0)$,we get:

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \mp \sqrt{\frac{-6ckb}{b_2}} coth(\sqrt{-b}\xi), \quad (18)$$

or

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \mp \sqrt{\frac{-6ckb}{b_2}} tanh(\sqrt{-b}\xi). \quad (19)$$

When $(b > 0)$,we get:

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \pm \sqrt{\frac{6ckb}{b_2}} cot(\sqrt{b}\xi), \quad (20)$$

or

$$u(\xi) = \frac{-b_3^2 + 2bc}{2b_3b_2} \pm \sqrt{\frac{6ckb}{b_2}} tan(\sqrt{b}\xi). \quad (21)$$

When $(b = 0)$,we get:

$$u(\xi) = \frac{-b_3^2}{2b_3b_2}. \quad (22)$$

**Case 3.**

$$a = \frac{2c^2b_2(2bk+1)}{3kb_3^2}, a_0 = a_1 = 0,$$

$$b_1 = \pm b\sqrt{\frac{6c_1k}{b_2}}, c_1 = \frac{2c^2b_2}{3kb_3^2}.$$

So that, the exact traveling wave solutions of Eq. (8) will be in the form

$$u(\xi) = \pm b\sqrt{\frac{6c_1k}{b_2}}\frac{1}{\phi}. \quad (23)$$

The solitary wave solutions will be in the form:
When $(b < 0)$,we get:

$$u(\xi) = \mp\sqrt{\frac{-6c_1kb}{b_2}} coth(\sqrt{-b}\xi), \quad (24)$$

or

$$u(\xi) = \mp\sqrt{\frac{-6c_1kb}{b_2}} tanh(\sqrt{-b}\xi). \quad (25)$$

When $(b > 0)$,we get:

$$u(\xi) = \pm\sqrt{\frac{-6c_1kb}{b_2}} cot(\sqrt{b}\xi), \quad (26)$$

or

$$u(\xi) = u(\xi) = \pm\sqrt{\frac{-6c_1kb}{b_2}} tan(\sqrt{b}\xi). \quad (27)$$

When $(b = 0)$,we get:

The trivial solution.

**Remark:**
All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

## Comparison:

In this research, a good numerous comparison between our results and that obtained by another researchers in [23]. they used a several methods to obtain exact and solitary wave solutions for the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity.
We obtain from our comparison:
**Solutions in case 1.**
Eq.(11) is similar to Eq. (31) in [23] when $\left(\Delta = \frac{c(-b_3^2+2bc)}{a_1b_2b_3}, b = \frac{-\Delta^2}{16}, a_1 = \frac{-b_3}{b}, \lambda = c\right)$. While Eqs.(13), (15), (16) are new form of solution for Eq.(8).
**Solutions in case 2.**
Eqs.(18), (19) is similar to Eqs. (27), (28) in [23] when $\left(\Delta = 2\sqrt{-b}, a_1^2 = \frac{6c^3k}{b_2}, c = \frac{2b_3b_2\sqrt{-b}}{-b_3^2+2bc}\right)$. While Eqs.(20), (21), (22) are new form of solution for Eq.(8).
**Solutions in case 3.**
Eqs.(24), (25) is similar to Eqs. (48), (49) in [23] when $\left(n = 1, b = \frac{-v^2}{2}, c_1 = \frac{2(a-vc)}{v^2(1+kv^2)}, \xi_0 = 0\right)$. While Eqs.(26), (27) are new form of solution for Eq.(8).

## 4 Conclusion

The extended tanh function method has been applied in this paper to find the exact traveling wave solutions and then the solitary wave solutions the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. Let us compare our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity are new and different from those obtained in [23]. The obtained exact solutions can be used as benchmarks against the numerical simulations in theoretical physics and fluid mechanics.

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## Author's contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied.

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## References

[1] Khan, Kamruzzaman, M. Ali Akbar and Norhashidah Hj Mohd Ali. The modified simple equation method for exact and solitary wave solutions of nonlinear evolution equation: the GZK-BBM equation and right-handed non commutative Burgers equations. ISRN Mathematical Physics 2013 (2013).

[2] Younis, Muhammad. A New approach for the exact solutions of nonlinear equations of fractional order via modified simple equation method. Applied Mathematics 5.13 (2014): 1927.

[3] Zahran, Emad H.M. and Mostafa M. A. Khater. The modified simple equation method and its applications for solving some nonlinear evolutions equations in mathematical physics. Jokull journal 64.5 (2014): 297-312.

[4] Mostafa M. A. Khater. The Modified Simple Equation Method and its Applications in Mathematical Physics and Biology. Global Journal of Science Frontier Research 15.4 (2015).

[5] Ebadi, Ghodrat, and Anjan Biswas. Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear diffusion equations with nonlinear source. Journal of the Franklin Institute 347.7 (2010): 1391-1398.

[6] Wang, Mingliang, Xiangzheng Li, and Jinliang Zhang. The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letters A 372.4 (2008): 417-423.

[7] Mostafa M. A. Khater. On the New Solitary Wave Solution of the Generalized Hirota-Satsuma Couple KdV System. Global Journal of Science Frontier Research 15.4 (2015).

[8] Zhang, Sheng, Jing-Lin Tong, and Wei Wang. A generalized-expansion method for the mKdV equation with variable coefficients. Physics Letters A 372.13 (2008): 2254-2257.

[9] Mostafa M. A. Khater, The Two-Variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-Expansion Method for Solving Nonlinear Dynamics of Microtubles-A New Model. Global Journal of Science Frontier Research 15.2 (2015).

[10] Fan, Engui, and Jian Zhang. Applications of the Jacobi elliptic function method to special-type nonlinear equations. Physics Letters A 305.6 (2002): 383-392.

[11] Zhang, Huiqun. Extended Jacobi elliptic function expansion method and its applications. Communications in Nonlinear Science and Numerical Simulation 12.5 (2007): 627-635.

[12] Mostafa M. A. Khater. Exact traveling wave solutions for the generalized Hirota-Satsuma couple KdV system using the exp $(-\phi(\xi))$-expansion method. Cogent Mathematics 3.1 (2016): 1172397.

[13] Bekir, Ahmet, and Ahmet Boz. Exact solutions for a class of nonlinear partial differential equations using exp-function method. International Journal of Nonlinear Sciences and Numerical Simulation 8.4 (2007): 505-512.

[14] Abdelrahman, A. E., Emad HM Zahran, and Mostafa MA Khater. Exact traveling wave solutions for power law and Kerr law non linearity using the exp $(-\phi(\xi))$-expansion method. Global Journal of Science Frontier Research 14.4 (2014).

[15] Mostafa M. A. Khater. Extended exp $(-\phi(\xi))$-Expansion method for Solving the Generalized Hirota-Satsuma Coupled KdV System. Global Journal of Science Frontier Research 15.7 (2015).

[16] Yang, Xiao-Feng, Zi-Chen Deng, and Yi Wei. A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application. Advances in Difference Equations 2015.1 (2015): 1-17.

[17] Emad H. M. Zahran and Mostafa M. A. Khater, An Infinite Sequence of Exact Solutions of The Reaction-convection-diffusion Equation According To A Riccati Bernoulli Sub-ODE Method , Journal of Research in Applied Sciences. Vol., 2(4): 101-107 (2015).

[18] Fan, Engui. Extended tanh-function method and its applications to nonlinear equations. Physics Letters A 277.4 (2000): 212-218.

[19] Elwakil, S. A., et al. Modified extended tanh-function method for solving nonlinear partial differential equations. Physics Letters A 299.2 (2002): 179-188.

[20] Zahran, Emad HM, and Mostafa M. A. Khater. Modified extended tanh-function method and its applications to the Bogoyavlenskii equation. Applied Mathematical Modelling 40.3 (2016): 1769-1775.

[21] El-Wakil, S. A., and M. A. Abdou. New exact travelling wave solutions using modified extended tanh-function method. Chaos, Solitons & Fractals 31.4 (2007): 840-852.

[22] Malfliet, Willy, and Willy Hereman. The tanh method: I. Exact solutions of nonlinear evolution and wave equations. Physica Scripta 54.6 (1996): 563.

[23] Wang, G. W., et al. Shock waves and other solutions to the Benjamin-Bona-Mahoney-Burgers equation with dual-power law nonlinearity. Acta Phys. Pol. A 126.6 (2014): 1221-1225.

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