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Different types of continuous functions and their applications

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Abstract: Examining a contemporary form of function reveals certain traits that facilitate the resolution of particular issues. As a result, in order to help with a variety of applications and solve a number of challenging problems, we generalize to β-continuity and almost weak-continuity and develop a novel class of functions entitled β-weak, β-strong, and almost β-weak continuity. This unique kind of function is provided certain characterizations and basic attributes. We analyze how comparable functions and β-weakly continuous functions vary and are similar. We also show how various forms of weak continuity relate to one another. Finally, the authors demonstrate that if a function has a regular range and a highly unconnected domain, then β-continuity is identical.

Keywords: β-continuous; almost weakly continuous; precontinuous, β-open sets, $\beta - \mathscr{T}_2$ -space, β-connected space, and β-weakly continuous.

1 Introduction

Because of the significance of extended topological spaces in applications and the related processes. In various approaches, many scientists have developed generalization of continuous function. Levine [12] defined weak continuity in 1961, which was regarded as among the most significant weak types of topological space nano-continuity. Abd El-Monsef et al. [2, 4] presented the class of β-continuity in 1983 and β-regular spaces in 1985 as a representation of each of Levine [13] semi-continuity and Rose [22, 23] weak and subweak-continuity, equivalently, Mashhour et al. [15, 16] precontinuity and α-continuity in 1982 and 1983. Jankovič [11] obtained another extension of almost weak-continuity and almost continuity in 1985. Continuity weak forms introduced in [8], and weak α-continuity established by Noiri [20]. In this direction, many researchers presented different ways to generalize the continuity of functions, as Azzam [6] presented weak forms of fanit continuity in 2018. Also, Azzam et al. [7] introduced almost α-completely regular and α-completely regular spaces. In 2019, Manoharan et al. [14] introduced a novel class of weak open sets via decomposition of continuity and R^*-interior. The writers present and examine a novel class of functions known as β-weak continuity, which is less robust than almost weak-continuity and β-continuity, as well as some of its characterizations in Section 2. Whenever, the third section provides a list of the fundamental topological characteristics of the new type. In addition, some of the separation axioms are used to define the domain (range) of this novel class of functions, This is covered in this note's fourth section.

Finally, Section 5 discusses the intersection of β-weak continuity and various types of continuity, and a new kind of topological structure names β-τ_2 space . Aside from anything else, some of the outcomes in [1, 5, 7, 9, 19, 21, 24-27] have been improved. Unless otherwise specified, for a space $\mathscr{N}$, no structure other than a topology $\mathscr{T}(\mathscr{N})$ is assumed. The interior, closure, and complement of any $W \sqsubseteq \mathscr{N}$ are indicated by $int(W)$, $cl(W)$, and W^C, respectively, relative to $\mathscr{T}(\mathscr{N})$.

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2 Preliminaries

We will demonstrate the preliminary steps that are required later in this particular section. The definitions that follow are where we start.

Definition 1.[2] *W is referred to as α-open [19] (resp. β-open [2], preopen [16] equivalent with almost-open [23]) if $W \sqsubseteq int(cl(int(W)))$ (resp. $W \sqsubseteq cl(int(cl(W))), W \sqsubseteq int(cl(W))$). The collection of all α-open (resp. β-open) sets of $\mathcal{N}$, written as $\alpha O(\mathcal{N})$ (resp. $\beta O(\mathcal{N})$). Also, β-$int(W)$ and β-$cl(W)$ means the β-interior and β-closure [5] of W which are defined likewise corresponding ones depending on β-openness and β-closeness.*

Definition 2.*A set W is referred to as α-open [?] (resp. β-open [?], preopen [?]) if $W \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(W)))$ (resp. $W \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(W))), W \subseteq \mathrm{int}(\mathrm{cl}(W))$). The collection of all α-open (resp. β-open) sets of a topological space $\mathcal{N}$ is written as $\alpha O(\mathcal{N})$ (resp. $\beta O(\mathcal{N})$). Also, β-$\mathrm{int}(W)$ and β-$\mathrm{cl}(W)$ denote the β-interior and β-closure [?] of W, which are defined correspondingly based on β-openness and β-closeness.*

Definition 3.*We refer to a function $f : (\mathcal{N},\mathcal{T}) \to (\mathcal{M},\sigma)$ as α-continuous [17] (resp. precontinuous [16] or almost-continuous [10], β-continuous [2]) if $f^{-1}(V)$ is α-open (resp. preopen or almost open, β-open) for each $V \in \sigma$).*

Definition 4.*We refer to a function $f : (\mathcal{N},\mathcal{T}) \to (\mathcal{M},\sigma)$ as weak [12] (resp. almost weakly [11], weakly α-[18]) continuous function if for each $V \in \sigma), f^{-1}(V) \sqsubseteq f^{-1}(Cl(V))$ (resp. $f^{-1}(V) \sqsubseteq int(cl(f^{-1}(cl(V))),$ $f^{-1}(V) \sqsubseteq int(cl(int(f^{-1}(cl(V))))$.*

Definition 5.[27] *For $\mathcal{N}$ carries topology $\mathcal{T}$ a point n is θ-closure point of $W \sqsubseteq \mathcal{N}$ (or $n \in \theta cl(W)$) if for each $U_n \in \mathcal{T}(\mathcal{N}), W \sqcap cl(U_n) \neq \phi$, where U_n means that U containing n.*

3 Characterizations

In this part, we establish numerous descriptions of β-weakly continuous functions.

Definition 6.*We refer to a function $f : (\mathcal{N},\tau) \to (\mathcal{M},\sigma)$ as β-weakly continuous at a point $n \in \mathcal{N}$ (briefly β-w.c.) at $n \in \mathcal{N}$ if for all $V \in \sigma$, there is $W_n \in \beta O(\mathcal{N})$ that $f(W_n) \sqsubseteq cl(V)$.*

Proposition 1.*$f : (\mathcal{N},\tau) \to (\mathcal{M},\sigma)$ is β-weakly continuous (β-w.c.) iff it is β-w.c. every time $n \in \mathcal{N}$.*

*Proof.*It is implied by Definition 2.1.

Theorem 1.*We refer to a function $f : (\mathcal{N},\tau) \to (\mathcal{M},\sigma)$, each of the characteristics listed below are identical.*
(i) f is β-w.c.
(ii) $f : (\mathcal{N},\beta O(\mathcal{N})) \to (\mathcal{M},\sigma)$ is weak continuous.
(iii) $f^{-1}(V) \sqsubseteq \beta$-$int(f^{-1}(cl(V)))\ \forall\ V \in \sigma$.
(iv) β-$cl(f^{-1}(V)) \sqsubseteq f^{-1}(cl(V))\ \forall\ V \in \sigma$.

Proof.$(i) \leftrightarrow (ii)$: Obvious.
$(i) \to (iii)$: Suppose $n \in f^{-1}(V)$, then by (i) there is $W_n \in \beta O(\mathcal{N})$ such that $f(n) \in f(W_n) \sqsubseteq cl(V)$. Thus $W_n \sqsubseteq f^{-1}(cl(V))$ and so $n \in W_n \sqsubseteq \beta$-$int(f^{-1}(cl(V)))$. Hence $f^{-1}(V) \sqsubseteq \beta$-$int(f^{-1}(cl(V)))$.
$(iii) \to (i)$ Let $n \in \mathcal{N}$ and $V_{f(n)} \in \sigma$, then $f^{-1}(V)_{f(n)} \sqsubseteq \beta$-$int(f^{-1}cl(V)_{f(n)} = W \in \beta O(\mathcal{M})$. Hence $f(n) \in f(W) \sqsubseteq f(\beta$-$int(f^{-1}cl(V)_{f(n)}))$, and therefore f is β-$w.c.$
$(i) \leftrightarrow (iv)$: It follows directly from the relationship between β-int and β-cl operations.
$(iii) \leftrightarrow (iv)$: Obvious.

Lemma 1.*$int(F \sqcup N) \sqsubseteq F \sqcup int(N)$ assuming a closed set F and any $N \sqsubseteq \mathcal{N}$.*

*Proof.*Since $F \sqsubseteq \mathcal{N}$ is closed, then $F^c \in \tau(\mathcal{N})$. One can show that $F^c \sqcap cl(N^c) \sqsubseteq cl(F^c \sqcap N^c)$ which leads to $F^c \sqcap (int(N))^c \sqsubseteq (F \sqcup int(N))^c$. So, $(F \sqcup int(N))^c \sqsubseteq (int(F \sqcup N))^c$. Hence $int(F \sqcup N) \sqsubseteq (F \sqcup int(N))$.

Proposition 2.*In a topological space $(\mathcal{N},\tau)$, for each subset N, we have*
(i) β-$cl(N) = N \sqcup int(cl(int(N))$;
(ii) β-$int(N) = N \sqcap cl(int(cl(N))$.

Proof.(i) Given that N is a β-closed set contained in β-$cl(N)$, therefore $int(cl(int(N))) \sqsubseteq int(cl(int(\beta\text{-}cl(N)))) \sqsubseteq \beta$-$cl(N)$ and so $N \sqcup int(cl(int(N))) \sqsubseteq \beta$-$cl(N)$. The other inclusion follows from the fact $int(cl(int[N \sqcup int(cl(int(N)))])) \sqsubseteq int(cl(int[N \sqcup cl(int(N))]))$ and Lemma 2.4 shows that $int(cl(int[N \sqcup int(cl(int(int(N))))])) \sqsubseteq int(cl[intN \sqcup cl[int(N)]))$, but $int(cl[int(N) \sqcup cl(int(N))]) = int[cl(int(N) \sqcup cl(int(N)))] = int(cl(int(N)))$. This means $int(cl(int[N \sqcup int(cl(int(N)))])) \sqsubseteq (N \sqcup int(cl(int(N)))$ that is $(N \sqcup int(cl(int(N))))$ is β-closed. Thus β-$cl(N \sqcup int(cl(int(N)))) = (N \sqcup int(cl(int(N)))$, which completes the proof.
(ii) Uses complementation and a duality property to directly follow.

Remember that if a subset N of $\mathcal{N}$ carries topology τ is both β-open and β-closed, it is referred to as β-regular.

$\beta R(\mathcal{N})$ is the whole family β-regular sets of $\mathcal{N}$. The intriguing outcome that follows will be crucial to the plot of the follow-up.

Lemma 2.*If and only if β-$cl(N) \in \beta R(\mathcal{N})$, then a subset N of a topological space $(\mathcal{N},\tau)$ is referred to as β-open.*

Proof. **Necessity**, Let $N \in \beta O(\mathscr{N})$. Next, we have $N \sqsubseteq cl(int(cl(N)))$ and hence $\beta\text{-}cl(N) \sqsubseteq \beta\text{-}cl(int(cl(N))) = cl(int(cl(N))) \sqsubseteq cl(int(cl(\beta\text{-}cl(N))))$. Therefore, $\beta\text{-}cl(N)$ is β-open and also β-closed. Hence $\beta\text{-}cl(N) \in \beta R(\mathscr{N})$.
sufficiency, Let $\beta\text{-}cl(N) \in \beta R(\mathscr{N})$. Next, we have $N \sqsubseteq \beta\text{-}cl(N) \sqsubseteq cl(int(cl(\beta\text{-}cl(N)))) \sqsubseteq cl(int(cl(cl(N)))) = cl(int(cl(A)))$. Therefore, we have $N \in \beta O(\mathscr{N})$.

Theorem 2. *Let $f : (\mathscr{N}, \tau) \to (\mathscr{M}, \sigma)$, then for each $V \in \sigma$, The statements below are interchangeable.*
(i) $f^{-1}(V) \sqsubseteq cl(int(cl(f^{-1}cl(V))))$;
(ii) $int(cl(int(f^{-1}cl(V)))) \sqsubseteq f^{-1}cl(V)$;
(iii) f is β-w.c.

Proof. Theorem 2.3 and Proposition 2.5 immediately follow.

Theorem 3. *Any function $f : (\mathscr{N}, \tau) \to (\mathscr{M}, \sigma)$ can be found the following assertions are interchangeable.*
(i) f is β-w.c.
(ii) $\beta\text{-}cl(f^{-1}(M)) \sqsubseteq f^{-1}(\theta\text{-}cl(M)$, for each $M \subseteq \mathscr{M}$.
(iii) $f(\beta\text{-}cl(N) \sqsubseteq (\theta\text{-}cl(f(N))$, for each $N \sqsubseteq \mathscr{N}$.
(iv) $int(cl(int(f^{-1}(M)))) \sqsubseteq f^{-1}(\theta\text{-}cl(M))$, for each $M \sqsubseteq \mathscr{M}$.
(v) $f(int(cl(int(N))) \sqsubseteq \theta - cl(f(N))$, for $N \subseteq \mathscr{N}$.

Proof. This comes directly from Theorems 2.3 and 2.7.

4 Essential Qualities

The purpose of Section 4 is to stimulate our paper into the fundamental features of weak continuity.

Lemma 3. [2] *If $H \in \tau^{\alpha}(\mathscr{N})$ and $V \in \beta O(\mathscr{N})$, then $H \sqcap V \in \beta O(\mathscr{N})$.*

Theorem 4. *Let $f : (\mathscr{N}, \tau) \to (\mathscr{M}, \sigma)$ be β-w.c., and $N \in \tau^{\alpha}(\mathscr{N})$, then $f \mid N$ is also β-w.c.*

Proof. Let $n \in N \in \tau^{\alpha}(\mathscr{N})$ by β-w.c. of f then, $\forall$ $V_{f(n)} \in \sigma$, there is $W_n \in \beta O(\mathscr{N})$ such that $f(W) \sqsubseteq cl(V)$. But $(W \sqcap N)_n \in \beta O(N)$, this show that $N \sqcap f(W) = (f \mid N)(W \sqcap N) \sqsubseteq cl(V)$. Consequently, $f \mid N$ is β-w.c.

Corollary 1. *β-w.c. is the restriction of a β-w.c. function to any open set.*

Theorem 5. *suppose $f : (\mathscr{N}, \tau) \to (\mathscr{M}, \sigma)$ is β-w.c., and $g : (\mathscr{M}, \sigma) \to (\mathscr{Z}, \eta)$ is continuous, then $g \circ f : (\mathscr{N}, \tau) \to (\mathscr{Z}, \eta)$ is β-w.c.*

Proof. Immediately by the property of each function f and g.

Remark. One can easily obtain the following facts:
(i) Theorem 3.4 need not be held by commutating f and g.
(ii) It is not necessary for β-w.c. functions to have β-w.c. composition.

Theorem 6. *Let $f : (\mathscr{N}, \tau) \to (\mathscr{M}, \sigma)$ be β-w.c., and $V \sqsubseteq Y$ is clopen, hence $f^{-1}(V)$ and $(f^{-1}(V))^c \in \beta O(\mathscr{N})$.*

Proof. This is evident from the two Statements (iii) and (iv) Theorem 2.3.

Theorem 7. *If $f : \mathscr{N} \to \mathscr{M}$ is β-w.c. iff its graph function $g : \mathscr{N} \to \mathscr{N} \times \mathscr{M}$ which defined as $g(n) = (n, f(n))$ for everyone $n \in \mathscr{N}$ is β-w.c.*

Proof. **Necessity,** Let's say f is β-w.c., for each $n \in \mathscr{N}$, and $g(n) \in W \in \tau(\mathscr{N} \times \mathscr{M})$. Next, there are $H \in \tau(\mathscr{N})$, and $V \in \tau(\mathscr{M})$ that $(n, f(n)) \in H \times V \subseteq W$. Considering f is β-w.c. Then, there is $G_n \in \beta O(\mathscr{N})$ that $f(G) \sqsubseteq cl(V)$. Put $Q = H \sqcap G \in \beta O(\mathscr{N})$ containing n and $g(Q) \sqsubseteq cl_{\mathscr{N} \times \mathscr{M}}(W)$. This indicates that g is β-w.c.
Sufficiency Consider g is β-w.c., $n \in \mathscr{N}$ and $V_{f(n)} \in \tau(\mathscr{M})$. Then, $g(n) \in \mathscr{N} \times V \in \tau(\mathscr{N} \times \mathscr{M})$. This means that there is $H_n \in \beta O(\mathscr{N})$ that $g(H) \sqsubseteq cl_{\mathscr{N} \times \mathscr{M}}(\mathscr{N} \times V) = \mathscr{N} \times cl_{\mathscr{M}}(V)$. Hence, $f(H) \sqsubseteq cl_{\mathscr{M}}(V)$, and so f is β-w.c.

Lemma 4. [16] *Consider $\{\mathscr{N}_i : i \in I\}$ as a family of spaces, $\mathscr{N} = \Pi \mathscr{N}_i$ is the product space and $N = \Pi_{i=1}^{n} N_i \times \Pi_{i \neq j} \mathscr{N}_j$ be a subset of $\mathscr{N}$ that is not empty, n is a positive integer and $N_i \sqsubseteq \mathscr{N}_i$. Then, $N_i \in \beta O(\mathscr{N}_i)(1 \leq i \leq m)$ if and only if $N \in \beta O(\mathscr{N})$.*

Theorem 8. *A function $f_\alpha : \mathscr{N}_\alpha \to \mathscr{M}_\alpha, \alpha \in \nabla$ is β-w.c. iff the product function $f : \Pi \mathscr{N}_\alpha \to \Pi \mathscr{M}_\alpha$ defined as $f\{n_\alpha\} = \{f_\alpha\{n_\alpha\}, \alpha \in \nabla\}$ is β-w.c.*

Proof. **Firstly,** Let us presume that f be β-w.c. for each $\alpha \in \nabla, n = \{n_\alpha\} \in \Pi \mathscr{N}_\alpha$ and $f(n) \in G \in \tau(\Pi \mathscr{M}_\alpha)$. Then, there is a basic open set ΠV_α that $f(n) \in \Pi V_\alpha \subseteq G$ where $\Pi V_\alpha = \Pi_{i=1}^{n} V_{\alpha i} \times \Pi_{\alpha \neq \alpha_i} \mathscr{M}_\alpha$ whenever $V_\alpha \in \tau(\mathscr{M}_\alpha)$ for each $\alpha = \alpha_1, \alpha_2, ..., \alpha_m$.
By assumption there is $W_\alpha \in \beta O(\mathscr{N}_\alpha)$ containing n_α such that $f_\alpha(W_\alpha) \sqsubseteq cl_{\mathscr{M}_\alpha}(V_\alpha)$.
Put $W = \Pi_{i=1}^{n} W_{\alpha i} \times \Pi_{\alpha \neq \alpha_i} \mathscr{N}_\alpha$ then, Lemma 3.8 shows that $n \in W \in \beta O(\Pi \mathscr{N}_\alpha)$ and so, $f(W) \sqsubseteq cl_{\Pi \mathscr{M}_\alpha}(G)$. Hence, f is β-w.c. The other direction follows immediately by homomorphism of projection function and Theorem 3.4.

Theorem 9. *We refer to a function $f : \mathscr{N} \to \Pi \mathscr{M}_\alpha, \alpha \in \nabla$ is β-w.c. if each $f_\alpha : \mathscr{N}_\alpha \to \Pi \mathscr{M}_\alpha, \alpha \in \nabla$ is β-w.c.*

Proof. This is derived straight from [3]'s Theorem 2.2.

5 Application Problems

Definition 7. *For $\mathscr{N}$ carries topology $\mathscr{T}$, it is called β-τ_2 [5], if each two distinct points of two distinct ones of $\mathscr{N}$ belongs to one of pair of disjoint β-open sets and $\mathscr{N}$ is β-connected if it can represented as a combination of two disjoin β-open sets.*

Theorem 10. *The domain of a β-w.c. function into a Urysohn space is β-τ_2.*

*Proof.*Suppose $f:(\mathscr{N},\tau)\to(\mathscr{M},\sigma)$ that β-*w.c.* injection, $\mathscr{M}$ be a Urysohn and $n_1\neq n_2\in\mathscr{N}$, then $f(n_1)\neq f(n_2)\in\mathscr{M}$. So, for each $V_i\in\sigma$ containing $f(n_i), i=1,2$, there is $W_{n_i}\sqsubseteq\beta O(\mathscr{N})$ such that $f(W_{n_i})\sqsubseteq cl_{\mathscr{M}}(V_i)$. Thus $f(\sqcap W_{n_i})\sqsubseteq\sqcap f(W_i)\sqsubseteq\sqcap cl_{\mathscr{M}}(V_i)=\phi$. Hence $\mathscr{N}$ is β-τ_2.

For $\mathscr{N}$ carries topology $\mathscr{T}$, it's claimed to be β-connected [5] if two nonempty disjoint β-open sets cannot be united to represent it.

Theorem 11.*If $f(\mathscr{N},\tau)\to(\mathscr{M},\sigma)$ is β-w.c. surjective and $\mathscr{N}$ is β-connected, then $\mathscr{M}$ is connected.*

*Proof.*Given the assumption that $\mathscr{M}$ is unconnected, there is $\phi\neq V_i\in\sigma(\mathscr{M}), i=1,2$, such that $\sqcap V_i=\phi$ and $\mathscr{M}=\sqcup V_i$. Thus, $V_i, i=1,2$ is clopen and therefore the third statement of Theorem 2.3 shows that $f^{-1}(V_i)\in\beta O(\mathscr{N}), i=1,2$ that $\sqcap f^{-1}(V_i)=\phi$ and $\mathscr{N}\sqcup f^{-1}(V_i)$, which contradicts that $\mathscr{N}$ is β-connected. Hence $\mathscr{M}$ is connected.

Theorem 12.*We refer to a function $f:\mathscr{N}\to\mathscr{M}$ is β-w.c. iff it is β-continuous, and $\mathscr{M}$ is regular.*

*Proof.*Given f is β-*w.c.*, $n\in\mathscr{N}$ and $V_{f(n)}\in\sigma(\mathscr{M})$. By regularity of $\mathscr{M}$, there is $G\in\sigma(\mathscr{M})$ that $f(n)\in G\sqsubseteq cl_{\mathscr{M}}(G)\sqsubseteq V$. Since f is β-*w.c.*, there exists $W_n\in\beta O(\mathscr{N})$ such that $f(W)\sqsubseteq cl_{\mathscr{M}}(G)\sqsubseteq V$. Then, f is β-continuous. The other direction is clear.

The subsequent lemma is applicable in this sequel and has clear proof.

Lemma 5.*In any topological space $(\mathscr{N},\tau), G\sqcap cl(N)\sqsubseteq cl(G\sqcap N)$, for any $N\sqsubseteq\mathscr{N}$ and $G\in\tau(\mathscr{N})$.*

Proposition 3.*The intersection of an β-open set and α-set is likewise β-open in that space.*

*Proof.*Let $N\in\tau^{\alpha}(\mathscr{N})$ and $M\in\beta O(\mathscr{N})$ then, $N\sqsubseteq int(cl(int(N)))$. So, $N\sqcap M\sqsubseteq int(cl(int(N)))\sqcap cl(int(cl(M)))$ [Lemma 4.5], but $int(cl(int(N)))\sqcap int(cl(M))=int[int(cl(N))\sqcap int(cl(M))]$. By using Lemma 4.5 again, we get $(int(cl(N))\sqcap int(cl(M)))$ $\sqsubseteq cl[int(A)\sqcap int(cl(M))]\sqsubseteq[int(N)\sqcap cl(M)]\sqsubseteq[int(N)\sqcap M]\sqsubseteq(N\sqcap int(cl(M))$. Hence, $N\sqcap M\in\beta O(\mathscr{N})$.

Theorem 13.*Let $\mathscr{M}$ be Hausdorff and $f_1:(\mathscr{N},\tau)\to(\mathscr{M},\sigma)$ be α-continuous if $f_2:(\mathscr{N},\tau)\to(\mathscr{M},\sigma)$ is β-w.c. and $f_1=f_2$ on a dense set then, $f_1=f_2$ on $\mathscr{N}$.*

*Proof.*Let $N=\{n\in\mathscr{N}: f_1(n)=f_2(n)\}$ and $n\notin N$ then, $f_1(n)\neq f_2(n)\}$ and there exist disjoint $V_i\in\sigma(\mathscr{M}), i=1,2$ that $f_i(n)\in V_i$. Hence, $V_1\sqcap Cl_{\mathscr{M}}(V_2)=\phi$. By hypothesis of f_1 and f_2, there exist $W_1\in\tau^{\alpha}(\mathscr{N})$ and $W_2\in\beta O(\mathscr{N})$ containing n such that $f_1(W_1)\sqsubseteq V_1$ and $f_2(W_2)\sqsubseteq Cl_{\mathscr{M}}(V_2)$. Therefore, $n\in\sqcap W_i\in\beta O(\mathscr{N}), i=1,2$ [Prposition 4.6] and $(\sqcap W_i)\sqcap N=\phi$, since, $\phi\neq\sqcap W_i\in\beta O(\mathscr{N})$ then, $int(cl(\sqcap W_i))\neq\phi$ and so $int(cl(\sqcap W_i))\sqcap N=\phi$. But $D\sqsubseteq N$ is dense, hence, $\mathscr{N}=cl(D)\sqsubseteq cl(N)$. This is a contradiction, leads to $N=\mathscr{N}$

*Remark.*Figure 1 show the connections results among continuity and different weak of continuity.

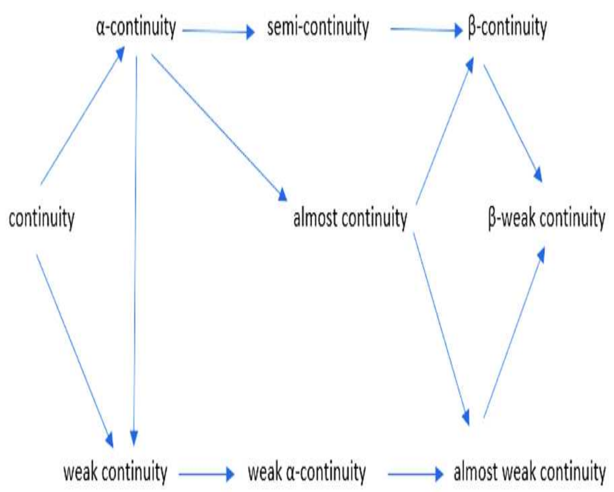

Fig. 1: The relationships between several weak forms of continuity and continuity result.

6 Examples and discussion

Figure 1 depicts the link between β-*w.c.* and other non-continuous functions of the same type. As shown in the following examples, the inverse implication does not have to be true.

*Example 1.*Let $\mathscr{N}=\{n_1,n_2,n_3\}$, $\tau_1=\{\mathscr{N},\varphi,\{n_1\},\{n_2\},\{n_1,n_2\}\}$, and $\tau_2=\{\mathscr{N},\varphi,\{n_1\},\{n_2,n_3\}\}$. The identity function from $(\mathscr{N},\tau)$ to $(\mathscr{N},\sigma)$ is β-*w.c.* but neither β-continuity nor almost weak continuity.

*Example 2.*Let $\mathscr{N}=\mathscr{M}=\{n_1,n_2,n_3\}$, $\tau=\{\mathscr{N},\varphi,\{n_3\}\}$, and $\sigma=\{\mathscr{M},\varphi,\{n_1\},\{n_2\},\{n_1,n_2\}\}$. Let $f:(\mathscr{N},\tau)\to(\mathscr{M},\sigma)$ be the identity function. Then, f is weakly α-continuous but it is not weakly continuous.

*Example 3.*Let $\mathcal{N}$={n_1,n_2,n_3,n_4}, τ={$\mathcal{N},\varphi,\{n_2\},\{n_3\},\{n_1,n_2\},\{n_2,n_3\},\{n_1,n_2,n_3\}$, $\{n_2,n_3,n_4\}$}. Let $f:(\mathcal{N},\tau)\to(\mathcal{N},\tau)$ be a function defined as follows:
$f(n_1)$=n_3, $f(n_2)$=n_4, $f(n_3)$=n_2, $f(n_4)$=n_1. Then, f is weakly continuous [17] and hence it is almost weakly continuous.

*Example 4.*Let $\mathcal{N}$=$\mathcal{M}$={n_1,n_2,n_3}, τ={$\mathcal{N},\varphi,\{n_1\},\{n_2\},\{n_1,n_2\}$}, and σ={$\mathcal{M},\varphi$, $\{n_1\},\{n_2,n_3\}$}. Let $f:(\mathcal{N},\tau)\to(\mathcal{M},\sigma)$ be the identity function. Then, f is semi-continuous but it is not weakly continuous [5].

*Remark.*In Examples 5.3 and 5.4, semi-continuity, almost-continuity, and weak-continuity are all on their own.

As demonstrated in [17], Theorem 3.1 and [20], Theorem 3.2, that a function is α-continuous iff it is semi-continuous and almost continuous which is equivalent to precontinuous [16]. Theorem 3.5 in [3] showed that $\tau(\mathcal{N})=\beta O(\mathcal{N})$ if $\mathcal{N}$ is submaximal and extremely disconnected, this fact shows that for a function $f:\mathcal{N}\to\mathcal{M}$, where $\mathcal{N}$ is submaximal and extremely disconnected iff it is β-continuous. Also, weakly continuous, α-continuous and continuity are equivalent for any function having a regular range [17]. Whenever [5], Theorem 5.9 shows under the same previous conditions that weakly α-continuity coincides with continuity.
The condition under which that β-continuity and β-*w.c.* are equivalent is established throughout Theorem 5.4, other equivalents show nextly.

Theorem 14.*Any function from an extremely disconnected space into any one is almost weakly continuous iff it is β-w.c.*

*Proof.*This is inferred from Theorem 2.5 and the meaning of extremely disconnectedness too.

Corollary 2.*If $f:\mathcal{N}\to\mathcal{M}$ where $\mathcal{N}$ is extremely disconnected and $\mathcal{M}$ is regular. Then, β-continuity, almost weakly continuity and β-w.c. for f are equivalent.*

*Proof.*This follows immediately from Theorems 4.4 and 5.2.

From the previous treatments, we can establish the following general result which has clear proof.

Theorem 15.*If $\mathcal{N}$ is submaximal, extremely disconnected and $\mathcal{M}$ is regular. Then, all types of continuity in Figure 1 are equivalent for any function $f:(\mathcal{N},\tau)\to(\mathcal{M},\sigma)$.*

7 Conclusion

Researchers are continually working to expand this topological structure and its related functions due to the relevance of the topological structure, which is used to measure several aspects that were challenging to assess with Euclidean geometry, such as intelligence, beauty, quality, and others. In order to go in this direction and obtain new forms of function continuity, we presented a new generalization of continuous function. We used the concepts of almost β-weakly continuous and β-weakly continuous as a result. This paper introduces the fundamental properties of β-weak continuity and their connections to other closely related types of non-continuous functions. We have provided some clear examples of key properties of weak kinds of continuity. We believe that the findings of this paper will promote future research on weak continuity by providing a comprehensive framework for its practical applications.

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