Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/050202

On Cyclic (α,β)-Admissible Generalized Contraction Mappings in Generalized Metric Spaces

Eskandar Ameer[1,2,*]

[1] Department of Mathematics, International Islamic University, Islamabad - 44000, Pakistan
[2] Department of Mathematics, Taiz University, Taiz, Yemen.

Received: 7 Jan. 2017, Revised: 24 may. 2017, Accepted: 8 Jun. 2017
Published online: 1 Jul. 2017

Abstract: The aim of this paper is to present new fixed point results in the framework of Branciari metric spaces. Some examples are presented to support the results proved herein. These results unify, generalize and complement the results of Jleli and Samet [J. Inequal. Appl. 2014, Article ID38(2014)]. We also provide an example of our result, where the comparable result in the existing literature is not applicable. As an application of our results, we obtain a fixed point results involving a cyclic mapping, not necessarily continuous.

Keywords: Generalized metric space; fixed point; cyclic (α,β)-admissible mapping.

1 Introduction

One of the most important result in metric fixed point theory is Banach contraction principle ([3]) which states that, if a metric space X is complete and $T:X\to X$ satisfies

$$d(Tx,Ty)\le kd(x,y)$$

for all x,y in X and for some $k\in(0,1)$. Then T has a unique fixed point.

Extensions of Banach contraction principle have been obtained either by generalizing the domain of the mapping or by extending the contractive condition on the mappings (see [1]-[30]).

In metric fixed point theory, contractive conditions on mappings play vital role in finding the solution of fixed point problems.

Rakotch [29] extended this principle replacing the contraction factor k in above inequality with some function on $X\times X$ taking values in $(0,1)$.

Geraghty [16] introduced the class of mappings:

$$S=\left\{\psi:[0,\infty)\to[0,1):\lim_{n\to\infty}t_n=0 \text{ whenever } \lim_{n\to\infty}\psi(t_n)=1\right\}$$

and obtained an interesting extension of Banach contraction principle as follows.

Theorem 1. *Let X be a complete metric space and $T:X\to X$. If there exists $\psi\in S$ such that*

$$d(Tx,Ty)\le\psi(d(x,y))d(x,y)$$

holds for all $x,y\in X$. Then T has a unique fixed point $x^*\in X$ and for each $x\in X$, the sequence $\{T^n(x)\}$ (called Picard sequence) converges to x^*.

Nadler [23] replaced the range of a mapping with $CB(X)$ and proved the multivalued version of Banach contraction principle using the Hausdorff metric.

In 2000, Branciari [2] introduced the concept of generalized metric spaces. Since then, several fixed point results have been obtained in the setup of such spaces (see [4],[5],[7]-[14]) and the references therein.

In this paper, we obtain several fixed point results of cyclic (α,β)- admissible mappings satisfying generalized contractive conditions and hence unify the comparable results in the existing literature.

In the sequel, the letters $\mathbb{N}$, and $\mathbb{R}$ will denote the set of natural numbers, and the set of all real numbers, respectively.

Consistent with [2] and [13], the following definitions and results will be needed in the sequel.

Definition 1.[2] *Let X be a nonempty set and $d:X\times X\longrightarrow[0,\infty)$. If for any $x,y\in X$ and $u,v\in X$, each of them different from x and y, the following conditions hold:*

* Corresponding author e-mail: eskandarameer@yahoo.com

(i) $d(x,y)=0$ if and only if $x=y$;
(ii) $d(x,y)=d(y,x)$;
(iii) $d(x,y)\leq d(x,u)+d(u,v)+d(v,y)$.
The pair (X,d) is called a generalized metric space.

Definition 2. *Let (X,d) be a generalized metric space. (a) A sequence $\{x_n\}$ in X is said to be convergent to $x\in X$ if $d(x_n,x)\longrightarrow 0$ as $n\longrightarrow\infty$. In this case, we say that $x_n\longrightarrow x$ as $n\longrightarrow\infty$. (b) A sequence $\{x_n\}$ in X is said to be Cauchy if $d(x_n,x_m)\longrightarrow 0$ as $n,m\longrightarrow\infty$.(c) (X,d) is said to be complete if and only if every Cauchy sequence in X converges to some point in X.*

Lemma 1. *[5] Let (X,d) be a generalized metric space, and $\{x_n\}$ a Cauchy sequence in X. If $d(x_n,x)\longrightarrow 0$ as $n\longrightarrow\infty$ for some $x\in X$. Then $d(x_n,y)\longrightarrow d(x,y)$ as $n\longrightarrow\infty$ for all $y\in X$. In particular, $\{x_n\}$ does not converge to y if $y\neq x$.*

We set

$$\Theta=\{\theta:(0,\infty)\longrightarrow(1,\infty):\ \theta \text{ satisfies } \Theta 1,\Theta 2 \text{ and } \Theta 3\},$$

where
$(\Theta 1)$ θ is nondecreasing,
$(\Theta 2)$ for each sequence $\{t_n\}$ in $(0,\infty)$, $\lim\limits_{n\longrightarrow\infty}\theta(t_n)=1$ if and only if $\lim\limits_{n\longrightarrow\infty}t_n=0^+$, and
$(\Theta 3)$ there exists $r\in(0,1)$ and $\ell\in(0,\infty]$ such that $\lim\limits_{t\longrightarrow 0^+}\dfrac{\theta(t)-1}{t^r}=\ell$.

Example 1. [13] Let $i\in\{1,2,3\}$. Define $\theta_i:(0,\infty)\longrightarrow(1,\infty)$ by

(1) $\theta_1(t)=e^{\sqrt{t}}$,
(2) $\theta_2(t)=e^{\sqrt{te^t}}$,
(3) $\theta_3(t)=2-\frac{2}{\pi}\arctan\left(\frac{1}{t^\gamma}\right)$, $0<\gamma<1, t>0$.
Then $\theta_i\in\Theta$.

Recently, Jleli et al. [13] obtained the following generalizations of the Banach contraction principle.

Theorem 2. *[13] Let (X,d) be a complete generalized metric space and $T:X\longrightarrow X$. If there exist $\theta\in\Theta$ and $k\in(0,1)$ such that for any $x,y\in X$*

$$d(Tx,Ty)\neq 0 \text{ implies that } \theta(d(Tx,Ty))\leq(\theta(d(x,y))^k.$$

Then T has a unique fixed point.

Theorem 3. *[14] Let (X,d) be a complete generalized metric space and $T:X\longrightarrow X$. If there exist $\theta\in\Theta$ and $k\in(0,1)$ such that for any $x,y\in X$*

$$d(Tx,Ty)\neq 0 \text{ implies that } \theta(d(Tx,Ty))\leq(\theta(M(x,y))^k,$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

and θ is continuous. Then T has a unique fixed point.

Definition 3. *[1] Let X be a nonempty set and $\alpha,\beta:X\longrightarrow[0,\infty)$. A selfmapping T on X is called a cyclic (α,β)-admissible mapping if for any $x\in X$,*

$$\alpha(x)\geq 1 \text{ implies that } \beta(Tx)\geq 1,$$
$$\text{and } \beta(x)\geq 1 \text{ implies that } \alpha(Tx)\geq 1.$$

The aim of this paper is to extend Theorems 2 and 3 using the concept of cyclic (α,β)-admissible mappings.

2 Main Results

We start with the following result.

Theorem 4. *Let (X,d) be a complete generalized metric space, $\alpha,\beta:X\longrightarrow[0,\infty)$ and $T:X\longrightarrow X$ cyclic (α,β)-admissible mapping. Suppose that there exists $\theta\in\Theta$ and $k\in(0,1)$ such that for any for $x,y\in X$ with $d(Tx,Ty)\neq 0$, we have*

$$\alpha(x)\beta(y)\theta(d(Tx,Ty))\leq[\theta(R(x,y))]^k,$$

where

$$R(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\}$$

and θ is continuous. If there exists $x_0\in X$ such that $\alpha(x_0)$, $\beta(x_0)$, $\beta(Tx_0)\geq 1$, and one of the following conditions holds:

(i) T is continuous,
(ii) if $\{x_n\}$ is a sequence in X such that $x_n\longrightarrow x$ as $n\longrightarrow\infty$ and $\beta(x_n)\geq 1$ for all $n\in\mathbb{N}$, then $\beta(x)\geq 1$.
Then T has a fixed point. Furthermore, if $\alpha(x)$, $\beta(x)\geq 1$ for every fixed point $x\in X$, then T has a unique fixed point.

Proof. Let x_0 be a given point in X such that $\alpha(x_0)$, $\beta(x_0)$, $\beta(Tx_0)\geq 1$. Define a sequence $\{x_n\}$ in X by $x_n=Tx_{n-1}=T^nx_0$ for all $n\in\mathbb{N}$. If there exists some $n_0\in\mathbb{N}$ for which $T^{n_0}x_0=T^{n_0+1}x_0$, then $T^{n_0}x_0$ is a fixed point of T and hence the result follows. Assume that $d\left(T^nx_0,T^{n+1}x_0\right)>0$ for every $n\in\mathbb{N}$. Since T is cyclic (α,β)-admissible mapping,

$\alpha(x_0)\geq 1$ implies that $\beta(Tx_0)=\beta(x_1)\geq 1$ and
$\beta(x_0)\geq 1$ implies that $\alpha(Tx_0)=\alpha(x_1)\geq 1$.

Continuing this way, we have $\alpha(T^nx_0)\geq 1$ and $\beta(T^nx_0)\geq 1$ for all $n\in\mathbb{N}$ and hence

$$\alpha\left(T^{n-1}x_0\right)\beta(T^nx_0)\geq 1 \text{ and} \tag{2.1}$$

$$\alpha\left(T^{n-1}x_0\right)\beta\left(T^{n+1}x_0\right)\geq 1 \tag{2.2}$$

hold for all $n\in\mathbb{N}$. By given assumption, we have

$$\begin{aligned}&\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)\\ &\leq\alpha\left(T^{n-1}x_0\right)\beta(T^nx_0)\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)\end{aligned}$$

$$\leq \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0,T^nx_0\right), d\left(T^{n-1}x_0,TT^{n-1}x_0\right), \\ d\left(T^nx_0,TT^nx_0\right), \\ \frac{d\left(T^{n-1}x_0,TT^{n-1}x_0\right)d\left(T^nx_0,TT^nx_0\right)}{1+d\left(T^{n-1}x_0,T^nx_0\right)} \end{array}\right\}\right)\right]^k$$

$$= \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0,T^nx_0\right), d\left(T^nx_0,T^{n+1}x_0\right), \\ \frac{d\left(T^{n-1}x_0,T^nx_0\right)d\left(T^nx_0,T^{n+1}x_0\right)}{1+d\left(T^{n-1}x_0,T^nx_0\right)} \end{array}\right\}\right)\right]^k$$

$$= \left[\theta\left(\max\left\{d\left(T^{n-1}x_0,T^nx_0\right), d\left(T^nx_0,T^{n+1}x_0\right)\right\}\right)\right]^k. \tag{2.3}$$

If for some $n\in\mathbb{N}$,

$$\max\left\{\begin{array}{c} d\left(T^{n-1}x_0,T^nx_0\right), \\ d\left(T^nx_0,T^{n+1}x_0\right) \end{array}\right\} = d\left(T^nx_0,T^{n+1}x_0\right),$$

then we have

$$\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right) \leq \left[\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)\right]^k,$$

and hence

$$\ln\left[\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)\right] \leq k\ln\left[\theta\left(d(T^nx_0,T^{n+1}x_0)\right)\right],$$

a contradiction. Therefore

$$\max\left\{d\left(T^{n-1}x_0,T^nx_0\right), d\left(T^nx_0,T^{n+1}x_0\right)\right\} = d\left(T^{n-1}x_0,T^nx_0\right)$$

for all $n\in\mathbb{N}$. From (2.3), we have

$$\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right) \leq \left[\theta\left(d\left(T^{n-1}x_0,T^nx_0\right)\right)\right]^k \text{ for all } n\in\mathbb{N}.$$

Now

$$\begin{aligned} 1 &\leq \theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right) \leq \left[\theta\left(d\left(T^{n-1}x_0,T^nx_0\right)\right)\right]^k \\ &\leq \left[\theta\left(d\left(T^{n-2}x_0,T^{n-1}x_0\right)\right)\right]^{k^2} \\ &\leq \ldots \leq \left[\theta\left(d(x_0,Tx_0)\right)\right]^{k^n}. \end{aligned} \tag{2.4}$$

On taking limit as $n\longrightarrow\infty$, we get

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right) = 1,$$

and hence

$$\lim_{n\longrightarrow\infty} d\left(T^nx_0,T^{n+1}x_0\right) = 0.$$

As $\theta\in\Theta$, there exist $r\in(0,1)$ and $\ell\in(0,\infty]$ such that

$$\lim_{n\longrightarrow\infty}\frac{\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1}{\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r} = \ell. \tag{2.5}$$

If $\ell<\infty$ then set $B=\frac{\ell}{2}>0$. There exists $n_0\in\mathbb{N}$ such that

$$\left|\frac{\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1}{\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r}-\ell\right| \leq B \text{ for all } n\geq n_0,$$

which implies that

$$\frac{\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1}{\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r} \geq \ell-B = B$$

for all $n\geq n_0$. Thus

$$n\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1\right],$$

for all $n\geq n_0$, where $A=\frac{1}{B}$. Suppose that $\ell=\infty$. Let $B>0$ be an arbitrary positive number. By (2.5), there exists $n_0\in\mathbb{N}$ such that

$$\frac{\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1}{\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r} \geq B \text{ for all } n\geq n_0.$$

That is,

$$n\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1\right],$$

for all $n\geq n_0$, where $A=\frac{1}{B}$. Thus in all cases, there exist $A>0$ and $n_0\in\mathbb{N}$ such that

$$n\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r \leq An\left[\theta\left(d\left(T^nx_0,T^{n+1}x_0\right)\right)-1\right],$$

for all $n\geq n_0$. By (2.4),

$$n\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r \leq An\left(\left[\theta\left(d(x_0,Tx_0)\right)\right]^{k^n}-1\right), \tag{2.6}$$

for all $n\geq n_0$. On taking limit as $n\longrightarrow\infty$, we obtain that

$$\lim_{n\longrightarrow\infty} n\left[d\left(T^nx_0,T^{n+1}x_0\right)\right]^r = 0.$$

Thus, there exists $n_1\in\mathbb{N}$ such that

$$d\left(T^nx_0,T^{n+1}x_0\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n\geq n_1. \tag{2.7}$$

Now, we claim that T has a periodic point. Assume on contrary that for all $n,m\in\mathbb{N}$ with $n\neq m$, $T^nx_0\neq T^mx_0$. Now

$$\begin{aligned} &\theta\left(d\left(T^nx_0,T^{n+2}x_0\right)\right) \\ &\leq \alpha\left(T^{n-1}x_0\right)\beta\left(T^{n+1}x_0\right)\theta\left(d\left(T^nx_0,T^{n+2}x_0\right)\right) \\ &\leq \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0,T^{n+1}x_0\right), d\left(T^{n-1}x_0,TT^{n-1}x_0\right), \\ d\left(T^{n+1}x_0,TT^{n+1}x_0\right), \\ \frac{d\left(T^{n-1}x_0,TT^{n-1}x_0\right)d\left(T^{n+1}x_0,TT^{n+1}x_0\right)}{1+d\left(T^{n-1}x_0,T^{n+1}x_0\right)} \end{array}\right\}\right)\right]^k \\ &= \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0,T^{n+1}x_0\right), d\left(T^{n-1}x_0,T^nx_0\right), \\ d\left(T^{n+1}x_0,T^{n+2}x_0\right), \\ \frac{d\left(T^{n-1}x_0,T^nx_0\right)d\left(T^{n+1}x_0,T^{n+2}x_0\right)}{1+d\left(T^{n-1}x_0,T^{n+1}x_0\right)} \end{array}\right\}\right)\right]^k \\ &= \left[\theta\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x_0,T^{n+1}x_0\right), d\left(T^{n-1}x_0,T^nx_0\right), \\ d\left(T^{n+1}x_0,T^{n+2}x_0\right) \end{array}\right\}\right)\right]^k \end{aligned} \tag{2.8}$$

Since θ is nondecreasing, we obtain from (2.8) that

$$\theta\left(d\left(T^nx_0,T^{n+2}x_0\right)\right) \leq \left[\max\left\{\begin{array}{c} \theta\left(d\left(T^{n-1}x_0,T^{n+1}x_0\right)\right), \\ \theta\left(d\left(T^{n-1}x_0,T^nx_0\right)\right), \\ \theta\left(d\left(T^{n+1}x_0,T^{n+2}x_0\right)\right) \end{array}\right\}\right]^k. \tag{2.9}$$

Let I be the set of all those $n\in\mathbb{N}$ such that the following holds:

$$u_n = \max\left\{\begin{array}{c}\theta\left(d\left(T^{n-1}x_0,T^{n+1}x_0\right)\right),\theta\left(d\left(T^{n-1}x_0,T^{n}x_0\right)\right),\\ \theta\left(d\left(T^{n+1}x_0,T^{n+2}x_0\right)\right)\end{array}\right\}$$
$$= \theta\left(d\left(T^{n-1}x_0,T^{n+1}x_0\right)\right).$$

If $|I|<\infty$, then there $N\in\mathbb{N}$ such that for all $n\geq N$,

$$\max\left\{\begin{array}{c}\theta\left(d\left(T^{n-1}x_0,T^{n+1}x_0\right)\right),\theta\left(d\left(T^{n-1}x_0,T^{n}x_0\right)\right),\\ \theta\left(d\left(T^{n+1}x_0,T^{n+2}x_0\right)\right)\end{array}\right\}$$
$$= \max\left\{\theta\left(d\left(T^{n-1}x_0,T^{n}x_0\right)\right),\ \theta\left(d\left(T^{n+1}x_0,T^{n+2}x_0\right)\right)\right\}.$$

It follows from (2.9) that

$$1\leq\theta\left(d\left(T^{n}x_0,T^{n+2}x_0\right)\right)$$
$$\leq\left[\max\left\{\begin{array}{c}\theta\left(d\left(T^{n-1}x_0,T^{n}x_0\right)\right),\\ \theta\left(d\left(T^{n+1}x_0,T^{n+2}x_0\right)\right)\end{array}\right\}\right]^k$$

for all $n\geq N$. On taking limit as $n\longrightarrow\infty$ in the above inequality and by

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(T^{n}x_0,T^{n+1}x_0\right)\right)=1,$$

we obtain that

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(T^{n}x_0,T^{n+2}x_0\right)\right)=1.$$

If $|I|=\infty$, we can find a subsequence $\{a_n\}$ of $\{u_n\}$ such that for a large enough n, we have

$$a_n=\theta\left(d\left(T^{n-1}x_0,T^{n+1}x_0\right)\right).$$

By (2.9), we get that

$$1\leq\theta\left(d\left(T^{n}x_0,T^{n+2}x_0\right)\right)\leq\left[\theta\left(d\left(T^{n-1}x_0,T^{n+1}x_0\right)\right)\right]^k$$
$$\leq\left[\theta\left(d\left(T^{n-2}x_0,T^{n}x_0\right)\right)\right]^{k^2}\leq\ldots\leq\left[\theta\left(d\left(x_0,T^{2}x_0\right)\right)\right]^{k^n}$$

for large n. Taking limit as $n\longrightarrow\infty$ in the above inequality, we obtain that

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(T^{n}x_0,T^{n+2}x_0\right)\right)=1, \tag{2.10}$$

and hence $\lim_{n\longrightarrow\infty}d\left(T^{n}x_0,T^{n+2}x_0\right)=0$ in both cases. Following arguments similar to those given above, there exists $n_2\in\mathbb{N}$ such that

$$d\left(T^{n}x_0,T^{n+2}x_0\right)\leq\frac{1}{n^{\frac{1}{r}}}\ \text{ for all } n\geq n_2. \tag{2.11}$$

Put $h=\max\{n_0,n_1\}$. We consider the following cases:
Case 1: If $m>2$ is odd, then $m=2L+1$ for some $L\geq 1$. From (2.7), we have

$$\begin{aligned}d\left(T^{n}x_0,T^{n+m}x_0\right)&\leq d\left(T^{n}x_0,T^{n+1}x_0\right)+d\left(T^{n+1}x_0,T^{n+2}x_0\right)\\&\quad+\ldots+d\left(T^{n+2L}x_0,T^{n+2L+1}x_0\right)\\&\leq\frac{1}{n^{\frac{1}{r}}}+\frac{1}{(n+1)^{\frac{1}{r}}}+\ldots+\frac{1}{(n+2L)^{\frac{1}{r}}}\\&\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}},\end{aligned}$$

for all $n\geq h$.
Case 2: If $m>2$ is even, then $m=2L$ for some $L\geq 2$. From (2.7) and (2.11), we have

$$\begin{aligned}d\left(T^{n}x_0,T^{n+m}x_0\right)&\leq d\left(T^{n}x_0,T^{n+2}x_0\right)+d\left(T^{n+2}x_0,T^{n+3}x_0\right)\\&\quad+\ldots+d\left(T^{n+2L-1}x_0,T^{n+2L}x_0\right)\\&\leq\frac{1}{n^{\frac{1}{r}}}+\frac{1}{(n+2)^{\frac{1}{r}}}+\ldots+\frac{1}{(n+2L-1)^{\frac{1}{r}}}\\&\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}}\end{aligned}$$

for all $n\geq h$. Thus , for all $n\geq h$, and $m\in\mathbb{N}$, $d\left(T^{n}x_0,T^{n+m}x_0\right)\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}}$. As $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}}$ is convergent (indeed $\frac{1}{r}>1$), $\{T^{n}x_0\}$ is a Cauchy sequence. By completeness of X, there exists $z\in X$ such that $T^{n}x_0\longrightarrow z$ as $n\longrightarrow\infty$. If T is continuous, then

$$z=\lim_{n\longrightarrow\infty}T^{n+1}x_0=\lim_{n\longrightarrow\infty}T\left(T^{n}x_0\right)=T\left(\lim_{n\longrightarrow\infty}T^{n}x_0\right)=Tz.$$

Now if condition (ii) holds, then $\beta(z)\geq 1$. Without any loss of generality, we can assume that $T^{n}x_0\neq z$ for some large enough n. If $d(z,Tz)>0$, then by given assumption, we have

$$\begin{aligned}&\theta\left(d\left(T^{n+1}x_0,Tz\right)\right)\\&\leq\alpha\left(T^{n}x_0\right)\beta(z).\theta\left(d\left(T^{n+1}x_0,Tz\right)\right)\\&\leq\left[\theta\left(\max\left\{\begin{array}{c}d\left(T^{n}x_0,z\right),\ d\left(T^{n}x_0,T^{n+1}x_0\right),\\ d(z,Tz),\frac{d\left(T^{n}x_0,T^{n+1}x_0\right)d(z,Tz)}{1+d\left(T^{n}x_0,z\right)}\end{array}\right\}\right)\right]^k\\&=\left[\theta\left(\max\left\{\begin{array}{c}d\left(T^{n}x_0,z\right),\ d\left(T^{n}x_0,T^{n+1}x_0\right),\\ d(z,Tz)\end{array}\right\}\right)\right]^k.\end{aligned}$$

On taking limit as $n\longrightarrow\infty$ in the above inequality, we obtain

$$\theta\left(d(z,Tz)\right)\leq\left[\theta\left(d(z,Tz)\right)\right]^k<\theta\left(d(z,Tz)\right),$$

a contradiction. Hence z is a periodic point of T of period q (say). If $q>1$ and $d(z,Tz)>0$, then we have

$$\begin{aligned}\theta\left(d(z,Tz)\right)&=\theta\left(d\left(T^{q}z,T^{q+1}z\right)\right)\\&\leq\alpha\left(T^{q-1}z\right)\beta\left(T^{q}z\right).\theta\left(d\left(T^{q}z,T^{q+1}z\right)\right)\\&\leq\left[\theta\left(d(z,Tz)\right)\right]^{k^q}<\theta\left(d(z,Tz)\right),\end{aligned}$$

a contradiction. Thus the set of fixed points of T is non-empty. To prove uniqueness: Suppose that $z,u\in X$ are two fixed points of T such that $d(z,u)=d(Tz,Tu)>0$. By given hypothesis, $\alpha(z)\geq 1$ and $\beta(u)\geq 1$. Now

$$\begin{aligned}\theta\left(d(z,u)\right)&=\theta\left(d(Tz,Tu)\right)\leq\alpha(z)\beta(u).\theta\left(d(Tz,Tu)\right)\\&\leq\left[\theta\left(d(z,u)\right)\right]^k<\theta\left(d(z,u)\right),\end{aligned}$$

gives a contradiction. Hence the result follows.

*Example 2.*Let $X=\{0,1,2,3,4\}$. Define $d: X\times X\longrightarrow\mathbb{R}$ by

$$\begin{aligned}
&d(x,x)=0, \text{ for all } x\in X,\\
&d(1,2)=d(2,1)=3,\\
&d(2,3)=d(3,2)=d(1,3)=d(3,1)=1, \text{ and}\\
&d(x,y)=|x-y|, \qquad \text{otherwise.}
\end{aligned}$$

Note that (X,d) is a complete generalized metric space, but not a metric space. Indeed,

$$3=d(1,2)>d(1,3)+d(3,2)=1+1=2.$$

Let $T: X\longrightarrow X$ be defined by

$$T(x)=\begin{cases}2 & \text{if } x\in\{0,1,2,3\}\\ 0 & \text{if } x=4.\end{cases}$$

Define

$$\alpha(x)=\beta(x)=\begin{cases}1 & \text{if } x\in\{0,1,2,3\},\\ 0 & \text{otherwise.}\end{cases}$$

Also define $\theta:(0,\infty)\longrightarrow(1,\infty)$ by $\theta(t)=e^{\sqrt{t}}$. Clearly, $\theta\in\Theta$ and T is a cyclic (α,β)-admissible mapping. Now if $\{x_n\}$ is a sequence in X such that $x_n\longrightarrow x$ as $n\longrightarrow\infty$ and $\beta(x_n)\geq 1$. Then $\beta(x)\geq 1$. For $x\in\{0,1,2,3\}$ and $y=4$, we have

$$\begin{aligned}\alpha(x)\beta(1).\theta(d(T(x),T(1))) &= \alpha(x)\beta(1).\theta\left(d\left(\frac{1}{2},0\right)\right)\\ &\leq[\theta(R(x,1))]^k,\end{aligned}$$

for all $k\in(0,1)$. Thus, all the conditions of Theorem 4 are satisfied. Moreover, $x=2$ is a fixed point of T.

Following are some corollaries of Theorem 4.

Theorem 5.*Let (X,d) be a complete metric space, $\alpha,\beta: X\longrightarrow[0,\infty)$ and $T: X\longrightarrow X$ cyclic (α,β)-admissible mapping. Suppose that there exists $\theta\in\Theta$ and $k\in(0,1)$ such that for any $x,y\in X$ with $d(Tx,Ty)\neq 0$, we have*

$$\alpha(x)\beta(y)\theta(d(Tx,Ty))\leq[\theta(R(x,y))]^k,$$

where

$$R(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\}$$

and θ is continuous. If there exists $x_0\in X$ such that $\alpha(x_0)$, $\beta(x_0)$, $\beta(Tx_0)\geq 1$, and one of the following conditions holds:

(4.1) T is continuous,

(4.2) if $\{x_n\}$ is a sequence in X such that $x_n\longrightarrow x$ as $n\longrightarrow\infty$ and $\beta(x_n)\geq 1$ for all $n\in\mathbb{N}$, then $\beta(x)\geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x),\beta(x)\geq 1$ for every fixed point $x\in X$, then T has a unique fixed point.

Theorem 6.*Let (X,d) be a complete generalized metric space, and $T: X\longrightarrow X$. If there exists $\theta\in\Theta$ and $k\in(0,1)$ such that for any $x,y\in X$ with $d(Tx,Ty)\neq 0$, we have*

$$\theta(d(Tx,Ty))\leq[\theta(R(x,y))]^k,$$

where

$$R(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\}$$

and θ is continuous. Then T has a unique fixed point.

*Proof.*The result follows by setting $\alpha(x)=\beta(x)=1$ for all $x\in X$ in Theorem 4.

The following corollary is a fixed point result in [14].

Corollary 1.*[14] Let (X,d) be a complete generalized metric space, and $T: X\longrightarrow X$. If there exists $\theta\in\Theta$ and $k\in(0,1)$ such that for any for any $x,y\in X$ with $d(Tx,Ty)\neq 0$, we have*

$$\theta(d(Tx,Ty))\leq[\theta(M(x,y))]^k,$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

and θ is continuous. The n T has a unique fixed point.

Theorem 7.*Let (X,d) be a complete generalized metric space, $\alpha,\beta: X\longrightarrow[0,\infty)$ and $T: X\longrightarrow X$ cyclic (α,β)-admissible mapping . Suppose that there exists $\theta\in\Theta$ and $k\in(0,1)$ such that for any $x,y\in X$ with $d(Tx,Ty)\neq 0$, we have*

$$\alpha(x)\beta(y)\theta(d(Tx,Ty))\leq[\theta(d(x,y))]^k,$$

where θ is continuous. If there exists $x_0\in X$ such that $\alpha(x_0)$, $\beta(x_0)$, $\beta(Tx_0)\geq 1$, and one of the following conditions holds:

(i) T is continuous,

(ii) if $\{x_n\}$ is a sequence in X such that $x_n\longrightarrow x$ as $n\longrightarrow\infty$ and $\beta(x_n)\geq 1$ for all $n\in\mathbb{N}$, then $\beta(x)\geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x),\beta(x)\geq 1$ for every fixed point $x\in X$, then T has a unique fixed point.

Theorem 8.*Let (X,d) be a complete metric space, $\alpha,\beta: X\longrightarrow[0,\infty)$ and $T: X\longrightarrow X$ cyclic (α,β)-admissible mapping . Suppose that there exists $\theta\in\Theta$ and $k\in(0,1)$ such that for any $x,y\in X$ with $d(Tx,Ty)\neq 0$, we have*

$$\alpha(x)\beta(y)\theta(d(Tx,Ty))\leq[\theta(d(x,y))]^k,$$

where θ is continuous. If there exists $x_0\in X$ such that $\alpha(x_0)$, $\beta(x_0)$, $\beta(Tx_0)\geq 1$, and one of the following conditions holds:

(i) T is continuous,

(ii) if $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$.

Then T has a fixed point. Furthermore, if $\alpha(x), \beta(x) \geq 1$ for every fixed point $x \in X$, then T has a unique fixed point.

*Example 3.*Let $X = [0,1]$ and $d : X \times X \longrightarrow \mathbb{R}$ given by $d(x,y) = |x-y|$ for all $x,y \in X$. Let $T : X \longrightarrow X$ be defined by

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,1) \\ 0 & \text{if } x = 1. \end{cases},$$

Suppose that

$$\alpha(x) = \beta(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & \text{otherwise} \end{cases}.$$

Define $\theta : (0,\infty) \longrightarrow (1,\infty)$ by $\theta(t) = e^{\sqrt{te^t}}$. Clearly, $\theta \in \Theta$ and T is a cyclic (α,β)-admissible mapping. Now if $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\beta(x_n) \geq 1$. Then $\beta(x) \geq 1$. For $x \in [0,1)$, $y = 1$, we have

$$\begin{aligned} \alpha(x)\beta(1).\theta(d(T(x),T(1))) &= \alpha(x)\beta(1).\theta\left(d\left(\frac{1}{2},0\right)\right) \\ &\leq [\theta(d(x,1))]^k \text{ where } k \in (0,1). \end{aligned}$$

Thus, all the conditions of Theorem 7 (or Theorem 8) are satisfied. Moreover, $x = \dfrac{1}{2}$ is a fixed point of T. Note that the result in [13] can not applied to T. Indeed, for $x = \frac{1}{2}$, $y = 1$, we have

$$\begin{aligned} \theta\left(d\left(T\left(\frac{1}{2}\right),T(1)\right)\right) &= \theta\left(d\left(\frac{1}{2},0\right)\right) = \theta\left(\frac{1}{2}\right) \\ &= e^{\sqrt{\frac{1}{2}e^{\frac{1}{2}}}} \nleq \left[e^{\sqrt{\frac{1}{2}e^{\frac{1}{2}}}}\right]^k \\ &= \left[\theta\left(d\left(\frac{1}{2},1\right)\right)\right]^k, \end{aligned}$$

for all $k \in (0,1)$.

Set $\alpha(x) = \beta(x) = 1$ for all $x \in X$ in Theorem 7 to obtain the following result.

Corollary 2.*[13] Let (X,d) be a complete generalized metric space and $T : X \longrightarrow X$. Suppose that there exists $\theta \in \Theta$ and $k \in (0,1)$ such that for any $x,y \in X$ with $d(x,y) \neq 0$, we have $\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k$. Then T has a unique fixed point.*

3 Some cyclic contractions via cyclic (α,β)-admissible mapping

Kirk et al. [6] introduced the concept of cyclic mappings and cyclic contractions as follows:

Let A and B be nonempty subsets of a metric space (X,d).

Definition 4.*[6] A mapping $T : A \cup B \longrightarrow A \cup B$ is called (a) cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$ (b) cyclic contraction if there exists $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x \in A$ and $y \in B$.*

Note that a Banach-contraction mapping is continuous while a cyclic contraction need not be continuous. This signifies the role of cyclic mappings in metric fixed point theory ([17]-[28]).

In this section, we apply Theorem 4 to prove fixed point results involving a cyclic mapping in the setup of generalized metric spaces.

Theorem 9.*Let A and B be two closed subsets of a complete generalized metric space (X,d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \longrightarrow A \cup B$ a cyclic mapping. If for all $x \in A$ and $y \in B$, we have*

$$\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

$\theta \in \Theta$ is continuous and $k \in (0,1)$. Then T has a unique fixed point in $A \cap B$.

*Proof.*Define $\alpha, \beta : X \longrightarrow [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases} \text{ and } \beta(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{otherwise} \end{cases}.$$

For $x \in A$ and $y \in B$, we have

$$\alpha(x)\beta(y).\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}.$$

Also note that T is a cyclic (α,β)-admissible mapping. As $A \cap B \neq \emptyset$, there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$, $\beta(x_0) \geq 1$ and $\beta(Tx_0) \geq 1$. Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. Thus $x_n \in B$ for all $n \in \mathbb{N}$ and hence $x \in B$ which implies $\beta(x) \geq 1$. Thus, all the conditions of Theorem 4 are satisfied. So T has a unique fixed point z (say) in $A \cup B$. If $z \in A$, then $z = Tz \in B$. Similarly, if $z \in B$, then $z = Tz \in A$. Therefore $z \in A \cap B$.

Corollary 3.*Let A and B be two closed subsets of a complete metric space (X,d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \longrightarrow A \cup B$ a cyclic mapping. If for all $x \in A$ and $y \in B$, we have*

$$\theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

$\theta \in \Theta$ is continuous and $k \in (0,1)$. Then T has a unique fixed point in $A \cap B$.

*Example 4.*Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x-y|$ and $T : A \cup B \longrightarrow A \cup B$ defined by $Tx = -\frac{x}{6}$ where $A = [-1,0]$ and $B = [0,1]$. Define $\theta : (0,\infty) \longrightarrow (1,\infty)$ by $\theta(t) = e^t$. Note that for all $x \in A$ and $y \in B$, we have

$$\theta\left(d\left(Tx,Ty\right)\right) = e^{|Tx-Ty|} = e^{\frac{|x-y|}{6}} = \left[e^{|x-y|}\right]^{\frac{1}{6}}$$
$$\leq \left[\max\left\{\begin{array}{c} d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right), \\ \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} \end{array}\right\}\right]^k,$$

where $k \in \left[\frac{1}{6},1\right)$. Thus all the conditions of Theorem 9 (or Corollary 3) are satisfied. Moreover, $x = 0$ is a fixed point of T.

Similarly, we can prove the following theorem.

Theorem 10.*Let A and B be two closed subsets of a complete generalized metric space (X,d) such that $A \cap B \neq \emptyset$, $T : A \cup B \longrightarrow A \cup B$ a cyclic mapping and $\theta \in \Theta$. If for all $x \in A$ and $y \in B$, we have $\theta\left(d\left(Tx,Ty\right)\right) \leq \left[\theta\left(d(x,y)\right)\right]^k$, where $k \in (0,1)$. Then T has a unique fixed point in $A \cap B$.*

Corollary 4.*Let A and B be two closed subsets of a complete metric space (X,d) such that $A \cap B \neq \emptyset$, $T : A \cup B \longrightarrow A \cup B$ a cyclic mapping and $\theta \in \Theta$. If for all $x \in A$ and $y \in B$, we have*

$$\theta\left(d\left(Tx,Ty\right)\right) \leq \left[\theta\left(d(x,y)\right)\right]^k,$$

where $k \in (0,1)$. Then T has a unique fixed point in $A \cap B$.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] S. Alizadeh, F. Moradlou, P. Salimi, Some fixed point results for (α,β)-(ψ,ϕ)-contractive mappings, Filomat 28(3), 635-647(2014).
[2] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math.(Debr.) 57, 31-37 (2000).
[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux equations itegrales, Fund. Math. 3 (1922) 133–181.
[4] P. Das, A fixed point theorem on a class of generalized metric spaces, Korean J. Math. Sci. 9, 29-33 (2002).
[5] W.A. Kirk, N. Shahzad, Generalized metrics and Caristi's theorem, Fixed Point Theory Appl. 2013, Article ID 129 (2013).
[6] W. A. Kirk, P. S. Srinavasan and P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, Fixed Point Theory. 4 (2003) 79–89.
[7] H. Lakzian, B. Samet, Fixed points for (ψ,ϕ)-weakly contractive mappings in generalized metric spaces, Appl. Math. Lett. 25(5), 902-906 (2012).
[8] B. Samet, Discussion on: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, by A. Branciari. Publ. Math. (Debr.) 76(4), 493-494 (2010).
[9] M. Turinici, Functional contractions in local Branciari metric spaces (2012). arXiv:1208.4610v1 [math.GN].
[10] E. Karapınar, Fixed points results for alpha-admissible mapping of integral type on generalized metric spaces, Abstr. Appl. Anal. 2014, Article ID 141409 (2014).
[11] E. Karapınar, Discussion on contractions on generalized metric spaces, Abstr. Appl. Anal. 2014, Article ID 962784(2014).
[12] V. La Rosa, P. Vetro, Common fixed points for α-ψ-ϕ-contractions in generalized metric spaces, Nonlinear Anal., Model. Control 19(1), 43-54 (2014).
[13] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl. 2014, Article ID38(2014).
[14] M. Jleli, E. Karapinar, B. Samet, Further generalizations of the Banach contraction principle, J. Inequal. Appl. 2014, 2014:439.
[15] M. Jleli, B. Samet, The Kannan's fixed point theorem in a cone rectangular metric space, J. Nonlinear Sci. Appl. 2(3), 161-197(2009).
[16] M. Geraghty, On contractive mappings. Proceedings of American Mathematical Society 40 (1973) 604-608.
[17] M. Păcurar and I.A. Rus, Fixed point theory for cyclic φ-contractions, Nonlinear Anal. 72 (3-4) (2010) 1181–1187.
[18] G. Petruşel, Cyclic representations and periodic points, Studia Univ. Babes-Bolyai Math. 50 (2005) 107–112.
[19] Sh. Rezapour, M. Derafshpour and N. Shahzad, Best proximity point of cyclic φ-contractions in ordered metric spaces, Topol. Methods Nonlinear Anal. 37 (2011) 193–202.
[20] I. A. Rus, Cyclic representations and fixed points, Ann.Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity. 3 (2005) 171–178.
[21] W. Sintunavarat and P. Kumam, Common fixed point theorem for cyclic generalized multi-valued contraction mappings, Appl. Math. Lett. 25 (2012) 1849–1855.
[22] RP. Agarwal, MA. Alghamdi, N. Shahzad, Fixed point theory for cyclic generalized contractions in partial metric spaces, Fixed Point Theory Appl. 2012, 40 (2012).
[23] S.B. Nadler Jr, Multivalued contraction mappings. Pacific J. Math. 30 (1969) 475-487.
[24] H. Aydi, C. Vetro, W. Sintunavarat, P. Kumam, Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces, Fixed Point Theory Appl. 2012, 124 (2012).
[25] E. Karapınar, Fixed point theory for cyclic weak φ-contraction, Appl. Math. Lett. 24(6), 822-825 (2011).
[26] E. Karapınar, K. Sadaranagni, Fixed point theory for cyclic $(\varphi$-$\psi)$-contractions, Fixed Point Theory Appl. 2011, 69 (2011).
[27] HK. Nashine, W. Sintunavarat, P. Kumam, Cyclic generalized contractions and fixed point results with applications to an integral equation, Fixed Point Theory Appl. 2012, 217 (2012).
[28] MA. Petric, Some results concerning cyclical contractive mappings, Gen. Math. 18(4), 213-226 (2010).

[29] E. Rakotch, A note on contractive mappings, Proceedings of the American Mathematical Society 13 (1962) 459-465.
[30] M. Arshad, E. Ameer, E. Karapinar, Generalized contractions with triangular α-orbital admissible mapping on Branciari metric spaces, Journal of Inequalities and Applications 2016, 2016:63
[31] Z. Mustafa, M. Arshad, S.U,Khan, J. Ahmad, M. Jaradat, Common fixed point for multivalued mappings in G-metric spaces with applications, J. Nonlinear Sci. Appl., 10 (2017), 2550–2564
[32] S. U. Khan, M. Arshad, Fixed points of Multi valued mappings in Dualistic partial metric spaces, Bull. Of. Math. Anal. And appl. Volume 8, Issue 3(2016), Pages 49-58.

Eskandar Ameer is a PhD Scholar in Pure Mathematics. His research interests are in the areas of Functional Analysis, Real Analysis and Complex Analysis. He has published research articles in reputed international journals of mathematics. He is referee and editor of mathematical journals.