Generalized Incomplete 2D Hermite Polynomials and Their Generating Relations

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Abstract: The object of this paper is to derive generating relations for the generalized Incomplete 2D Hermite polynomials \( \phi_{m,n}(x, y; \tau) \) by giving suitable interpretations to the indices (m) and (n) through Weisner’s method.

Keywords: Generalized incomplete 2D Hermite Polynomials, Generating relations.

1. Introduction

The generalized incomplete 2D Hermite polynomials discussed in the present paper are characterized by two indices, two variables and one parameter.

These polynomials are defined [2, Eq.(12)]

\[ \phi_{m,n}(x, y; \tau) = \frac{1}{\sqrt{m!n!}} \left( \frac{\tau}{2} \right)^{\frac{m+n}{2}} e^{\frac{x^2}{2\tau}} \]  

(1.1)

where the Incomplete 2D Hermite polynomials \( h_{m,n}(x, y, \tau) \) are explicitly provided by the series [2, Eq.(1a)]

\[ h_{m,n}(x, y, \tau) = m!n! \sum_{r=0}^{\min(m, n)} \frac{\tau^r}{r!} \frac{x^{m-r} y^{n-r}}{(m-n)! (n-r)!} \]  

(1.2)

These polynomials defined by (1.1) satisfy the following simultaneous partial differential equations

\[ \tau \frac{\partial^2 w}{\partial y^2} - \frac{x^2 w}{4\tau} - mw = 0 \]  

(1.3)

and

\[ \tau \frac{\partial^2 w}{\partial x^2} - \frac{y^2 w}{4\tau} - nw = 0 \]  

(1.4)

Dattoli et al. [2, 3, 4] introduced and discussed a theory of Incomplete 2D Hermite polynomials. They discussed the properties of a new family of multi index Lucas type polynomials, which are often encountered in problems of intracavity photon statistics. They develop an approach based on the integral representation method and show that this class of polynomials can be derived from recently introduced multi-index Hermite like polynomials.

Wunnsche [7] introduced Hermite 2D polynomials and discussed their properties and their explicit representations. Recently, Subuhi et al. [5, 6] derived some generating relations involving Hermite 2D and some implicit summations formulae for Incomplete 2D Hermite polynomials by using different analytical means on their respective generating functions.

In this paper, we have obtained new generating relations for the generalized Incomplete 2D Hermite Polynomials by constructing a Lie algebra with the help of Weisner’s [1] method by giving suitable interpretations to the indices (m) and (n) of the polynomials under consideration. The principal interest in the given results lies in the fact that a number of special cases listed in section 3 would yield many new results of the theory of special functions.
2. **Group-theoretic method**

Replacing \( n \) by \( p \frac{\partial}{\partial p} \) and \( m \) by \( s \frac{\partial}{\partial s} \) in (1.3) and (1.4) respectively, we get

\[
\tau \frac{\partial^2 u}{\partial y^2} - s \frac{\partial u}{\partial s} - \frac{x^2 u}{4\tau} = 0, \tag{2.1}
\]

\[
\tau \frac{\partial^2 u}{\partial x^2} - p \frac{\partial u}{\partial p} - \frac{y^2 u}{4\tau} = 0. \tag{2.2}
\]

We see that \( u(x, y, p, s; \tau) = \phi_{m,n}(x, y; \tau) p^n s^m \) is a solution of (2.1) and (2.2), since \( \phi_{m,n}(x, y, \tau) \) is a solution of (1.3) and (1.4).

We first consider the following first order linear differential operators

\[
A_i = p \frac{\partial}{\partial p}, \quad A_2 = s \frac{\partial}{\partial s}, \quad A_3 = s^{-1} \left(2 \sqrt{2} \left(\frac{x}{2} - \tau \frac{\partial}{\partial y}\right)\right),
\]

\[
A_4 = s \left(\frac{1}{\sqrt{2}} \left(\frac{x}{2} + \tau \frac{\partial}{\partial y}\right)\right), \quad A_5 = p^{-1} \left(\frac{-2y}{\sqrt{2}} \left(\frac{y}{2} - \tau \frac{\partial}{\partial x}\right)\right), \quad A_6 = p \left(\frac{1}{\sqrt{2}} \left(\frac{y}{2} + \tau \frac{\partial}{\partial x}\right)\right)
\]

such that

\[
A_i[\phi_{m,n}(x, y; \tau) p^n s^m] = n\phi_{m,n}(x, y; \tau) p^n s^m, \quad A_2[\phi_{m,n}(x, y; \tau) p^n s^m] = m\phi_{m,n}(x, y; \tau) p^n s^m,
\]

\[
A_3[\phi_{m,n}(x, y; \tau) p^n s^m] = \sqrt{m}\phi_{m,1,n}(x, y; \tau) p^n s^{m-1}, \quad A_4[\phi_{m,n}(x, y; \tau) p^n s^m] = m^{1/2}\phi_{m,1,n}(x, y; \tau) p^n s^{m-1},
\]

\[
A_5[\phi_{m,n}(x, y; \tau) p^n s^m] = \sqrt{n}\phi_{m,n-1}(x, y; \tau) p^n s^{m+1}, \quad A_6[\phi_{m,n}(x, y; \tau) p^n s^m] = m^{1/2}\phi_{m,1,n}(x, y; \tau) p^n s^{m+1},
\]

where the operators \( \{A_i : i = 1,2,3,4,5,6\} \) satisfy the following commutation relations

\[
[A_1, A_2] = 0, \quad [A_2, A_3] = -A_3, \quad [A_1, A_3] = 0, \quad [A_3, A_4] = 0, \quad [A_4, A_5] = 0, \quad [A_5, A_6] = 0, \quad [A_1, A_5] = 0, \quad [A_2, A_6] = 0, \quad [A_3, A_6] = 0, \quad [A_4, A_5] = 0
\]

\[
[A_1, A_3] = -A_3, \quad [A_2, A_6] = 0, \quad [A_1, A_6] = A_6
\]

where \( [A, B] = AB - BA \).

The above commutation relations show that the set of operators \( \{A_i : i = 1,2,3,4,5,6\} \) generate a Lie-algebra \( \lambda \) and the sets of operators \( \{A_1, A_4, A_6\} \) and \( \{A_2, A_3, A_4\} \) form a subalgebras of \( \lambda \).

It is clear that the differential operators

\[
L_1 = \tau \frac{\partial^2}{\partial y^2} - s \frac{\partial}{\partial s} - \frac{x^2}{4\tau}, \quad L_2 = \tau \frac{\partial^2}{\partial x^2} - p \frac{\partial}{\partial p} - \frac{y^2}{4\tau},
\]

which can be expressed as:

\[
L_1 = A_1 A_4 - m, \quad L_2 = A_3 A_6 - n
\]

commutes with \( \{A_i : i = 1,2,3,4,5,6\} \) that is

\[
[L_1, A_i] = 0 , i=1,2,3,4,5,6, \quad [L_2, A_i] = 0 , i=1,2,3,4,5,6 \tag{2.3}
\]

The extended form of the groups generated by \( \{A_i : i = 1,2,3,4,5,6\} \) are as follows:

\[
e^{a_1 A_1} f(x, y, p, s; \tau) = f(x, y, pe^{a_1}, s; \tau), \quad e^{a_2 A_2} f(x, y, p, s; \tau) = f(x, y, p, se^{a_2}; \tau), \]

\[
e^{a_1 A_3} f(x, y, p, s; \tau) = \exp \left( \frac{-a_1 y}{s \sqrt{2\tau}} \right) f(x, y + \frac{a_1 s \tau}{2\sqrt{2\tau}}, p, s; \tau), \quad e^{a_3 A_5} f(x, y, p, s; \tau) = \exp \left( \frac{a_3 x}{s \sqrt{2\tau}} \right) f(x + \frac{a_3 \sqrt{2\tau}}{s}, p, s; \tau),
\]

\[
e^{a_4 A_6} f(x, y, p, s; \tau) = \exp \left( \frac{-a_4 y}{p \sqrt{2\tau}} \right) f(x, y, p + \frac{a_4 \sqrt{2\tau}}{s}, s; \tau), \quad e^{a_5 A_4} f(x, y, p, s; \tau) = \exp \left( \frac{a_5 x}{s \sqrt{2\tau}} \right) f(x + \frac{a_5 \sqrt{2\tau}}{s}, y, p, s; \tau),
\]

\[
e^{a_3 A_7} f(x, y, p, s; \tau) = \exp \left( \frac{a_6 y}{p \sqrt{2\tau}} \right) f(x + \frac{a_5 \sqrt{2\tau}}{s}, y, p, s; \tau)
\]
where \( f(x, y, p, s; \tau) \) is an arbitrary function. Then we have

\[
e^{a_{i}A_{i}}e^{a_{j}A_{j}}e^{a_{i}A_{i}}e^{a_{j}A_{j}}f(x, y, p, s; \tau) = \exp \left( \frac{a_{i}py}{2\sqrt{2}\tau} \right) \exp \left( -\frac{a_{j}y}{p\sqrt{2}\tau} \right) \exp \left( \frac{a_{i}sx}{2\sqrt{2}\tau} \right) \exp \left( -\frac{a_{j}x}{s\sqrt{2}\tau} \right) \sum_{i=0}^{2} \sum_{k=0}^{l} \frac{a_{i}k}{l! \sqrt{(n+1)(n+2)\ldots(n+k)}} \sqrt{(n+k)(n+k-l-1)\ldots(n+l-1)} \phi_{m,n+l-k}(x, y; \tau) p^{n-k}s^{m}
\]

\[
(3.2)
\]

Combining the above two relations (3.1) and (3.2), we get

\[
\exp \left( -\frac{ay}{p\sqrt{2}\tau} \right) \exp \left( \frac{bpy}{2\sqrt{2}\tau} \right) \phi_{m,n}(x + a\sqrt{2}\tau + bp\tau; y; \tau) = \sum_{i=0}^{2} \sum_{k=0}^{l} \frac{b_{i}k}{l! \sqrt{(n+1)(n+2)\ldots(n+k)}} \sqrt{(n+k)(n+k-l-1)\ldots(n+l-1)} \phi_{m,n+l-k}(x, y; \tau)
\]

where \(|b| < \infty \) and \(|d| < \infty\). If we put \(a = 0, p = 1\) in equation (3.3), we get

\[
\exp \left( \frac{by}{2\sqrt{2}\tau} \right) \phi_{m,n}(x + b\tau; y; \tau) = \sum_{i=0}^{2} \sum_{k=0}^{l} \frac{b_{i}k}{l! \sqrt{(n+1)(n+2)\ldots(n+k)}} \phi_{m,n+l-k}(x, y; \tau)
\]

where \(|b| < \infty\). If we put \(b = 0, p = 1\) in equation (3.3), we get

\[
\exp \left( -\frac{cy}{s\sqrt{2}\tau} \right) \exp \left( \frac{dxy}{2\sqrt{2}\tau} \right) \phi_{m,n}(x + c\sqrt{2}\tau + ds\tau; y; \tau) = \sum_{i=0}^{2} \sum_{k=0}^{l} \frac{d_{i}k}{l! \sqrt{(n+1)(n+2)\ldots(n+k)}} \sqrt{(n+k)(n+k-l-1)\ldots(n+l-1)} \phi_{m,n+l-k}(x, y; \tau)
\]

where \(|d| < \infty\). Again by setting \(\{a_{i} = 0 : i = 1,2,3,4; a_{5} = a, a_{6} = b\}\) and writing \(f(x, y, p, s; \tau) = \phi_{m,n}(x, y; \tau) p^{n}s^{m}\) in (2.4), we get

\[
e^{a_{i}A_{i}}e^{a_{j}A_{j}}(\phi_{m,n}(x, y; \tau)p^{n}s^{m}) = \exp \left( -\frac{ay}{p\sqrt{2}\tau} \right) \exp \left( \frac{bpy}{2\sqrt{2}\tau} \right) \phi_{m,n}(x + a\sqrt{2}\tau + bp\tau; y; \tau) p^{n}s^{m}
\]

\[
(3.1)
\]

but
\[ e^{dl_k} e^{cl_k} (\phi_{m,n}(x,y;\tau) p^m s^n) = \]
\[ \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} c^i d^k \frac{1}{\sqrt{(m+1)(m+2)\ldots(m+k)}} \]
\[ \frac{1}{\sqrt{(m+k)(m+k-1)\ldots(m+k-l+1)}} \phi_{m+k-l}(x,y;\tau) p^m s^{m+k-l} \]
Combining the above two relations (3.6) and (3.7), we get
\[ \exp \left( -cy \frac{s}{\sqrt{2\tau}} \right) \exp \left( - \frac{dsx}{2\sqrt{2\tau}} \right) \phi_{m,n}(x,y) \]
\[ + c \frac{\sqrt{2\tau}}{s} \frac{dsx}{\sqrt{2\tau}} \phi_{m,n}(x,y) = \]
\[ \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} c^i d^k \frac{1}{\sqrt{(m+1)(m+2)\ldots(m+k)}} \]
\[ \frac{1}{\sqrt{(m+k)(m+k-1)\ldots(m+k-l+1)}} \phi_{m+k-l}(x,y;\tau) s^{k-l} \]
where \( |c| < \infty, |d| < \infty \).

If we put \( c = 0, s = 1 \) in equation (3.8), we get
\[ \exp \left( \frac{dx}{2\sqrt{2\tau}} \right) \phi_{m,n}(x,y) + \frac{d\tau}{\sqrt{2\tau}}(x,y;\tau) = \]
\[ \sum_{k=0}^{\infty} d^k \frac{1}{\sqrt{(m+1)(m+2)\ldots(m+k)}} \phi_{m+k,n}(x,y;\tau) \]
where \( |d| < \infty \). If we put \( d = 0, s = 1 \) in equation (3.8), we get
\[ \exp \left( -cy \frac{\sqrt{2\tau}}{\sqrt{2\tau}} \right) \phi_{m,n}(x,y) = \]
\[ \sum_{i=0}^{\infty} c^i d^k \frac{1}{\sqrt{(m-l)(m-2)\ldots(m-l+1)}} \phi_{m-l,n}(x,y;\tau) \]
where \( |c| < \infty \).

4. Conclusion

We have seen that Weisner's group theoretic method is a powerful tool in getting generating functions. It is also interesting to define a new function which forms a generalization for the generalized incomplete 2D Hermite Polynomials under consideration and then by using Lie theoretic technique, we can obtain generating functions. We will deal with this aspect in the subsequent communication.

References


M.J.S. Shahwan received his Ph.D. degree in Mathematics from Rajasthan University, Jaipur, India in 1996. He is currently an Associate Professor at College of Science, Department of Mathematics, University of Bahrain. His research interest is in Lie theory and Special functions. He has more than 17 research publications.