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On the Solution and Periodic Nature of Higher-order Difference Equation

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Abstract: The main objective of this paper is to study the positive solution and the periodic nature of the following difference equation

$$x_{n+1} = \delta x_n + \frac{ax_{n-l} + bx_{n-k} + cx_{n-s}}{\alpha x_{n-l} + \beta x_{n-k} + \gamma x_{n-s}}, \quad n = 0, 1, \ldots,$$

where the initial conditions $x_{-r},\ x_{-r+1},\ x_{-r+2}, \ldots,\ x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s\}$ is nonnegative integer and $\delta,\ \alpha,\ \beta,\ \gamma,\ a,\ b,\ c$ are positive constants.

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1 Introduction

The main objective of this paper is to investigate the periodic nature and global attractivity of the solutions of the difference equation

$$x_{n+1} = \delta x_n + \frac{ax_{n-l} + bx_{n-k} + cx_{n-s}}{\alpha x_{n-l} + \beta x_{n-k} + \gamma x_{n-s}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, \ldots,\ x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s\}$ is nonnegative integer and $\delta, a, b, c, \alpha, \beta, \gamma$ are constants.

Difference equations emerge as a natural representation of discovered evolution phenomin because most analysis of time evolving variables are discrete. Moreover several results in the theory of difference equations have been obtained as discrete analogues and as numerical solutions of differential equations. This is notably true in the case of Liapunov theory of stability. Furthermore, it has applications in biology, ecology, economy, physics, and so on. So, recently there has been an increasing interest in the study of qualitative analysis of scalar rational difference equations and rational system of difference equations. Although difference equations look simple in form, it is quite difficult to understand thoroughly the behaviors of their solutions because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equation come from the results of rational difference equations See [1]-[10] and the references cited therein.

In [11] M.R.S. Kulenovic investigated the global stability, periodic nature and gave the solution of the following non-linear difference equation.

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{A + Bx_{n-1}}. \tag{1.2}$$

M. Aloqeili [12] studied the stability properties and semi-cyclical behavior of the difference equation.

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}. \tag{1.3}$$

Elabbasy et al. [13] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l} x_{n-k}}{bx_{n-p} + cx_{n-q}}. \tag{1.4}$$

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Yalçınkaya [28] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}. \quad (1.5)$$

For some related work see [15]-[25].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \quad (1.6)$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.7) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}). \quad (1.7)$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.7), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.*(Periodicity)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 2.*(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq.(1.7) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta, \quad (1.8)$$

we have

$$|x_n - \overline{x}| < \varepsilon \quad \text{for all} \quad n \geq -k. \quad (1.9)$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.7) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(1.7) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma, \quad (1.10)$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}. \quad (1.11)$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.7) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}. \quad (1.12)$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.7) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.7).

(v) The equilibrium point $\overline{x}$ of Eq.(1.7) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(1.7) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \quad (1.13)$$

Theorem A [14] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1, \quad (1.14)$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, 2, ..., \quad (1.15)$$

*Remark.*Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ..., \quad (4)$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, ...x_{n-k}) \qquad n = 0, 1, 2..., \quad (1.16)$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [15] : Let $[\alpha, \beta]$ be an interval of real numbers and assume that

$$g : [\alpha, \beta]^{k+1} \to [\alpha, \beta],$$

is a continuous function satisfying the following conditions :

(a) For each integer i with $1 \leq i \leq k+1$;the funtion $g(z_1, z_1, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_1, ..., z_{i-1}, z_{i+1}..., z_{k+1}$

(b) If $(\Phi, \Psi) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system

$$\Psi = g(\Psi_1, \Psi_1, ..., \Psi_{k+1}) \ \ and \ \ \Phi = g(\Phi_1, \Phi_1, ..., \Phi_{k+1}), \quad (1.17)$$

then $\Phi = \Psi$,where for each $i = 1, 2, ...k+1$, we set

$$\Phi_i = \begin{Bmatrix} \Phi & \text{if } g \text{ is non-decreasing in } z_i \\ \Psi & \text{if } g \text{ is non-inecreasing in } z_i \end{Bmatrix} \ \ and \ \ \Psi_i = \begin{Bmatrix} \Psi & \text{if } g \text{ is non-decreasing in } z_i \\ \Phi & \text{if } g \text{ is non-inecreasing in } z_i \end{Bmatrix}$$

Then Eq.(1.17) has a unique equilibrium $\overline{x} \in [\alpha, \beta]$ and every solution of Eq.(1.17) converges to $\overline{x}$.

In this article we proceed as follows. In section 2 , we investigated that when,

$$(\alpha + \beta + \gamma)(a + b + c) > \max \begin{Bmatrix} |2\alpha(b+c) - 2a(\beta+\gamma)|, |2\gamma(a+b) - 2c(\alpha+\beta)|, \\ |2\beta(a+c) - 2b(\alpha+\gamma)| \end{Bmatrix}$$

2 Local Stability of the Equilibrium Point of Eq.(1.1)

In this section we study the local stability character of the equilibrium point of.(1.1). The equilibrium points of Eq.(1.1) are given by the relation

$$\overline{x} = \delta\overline{x} + \frac{a\overline{x} + b\overline{x} + c\overline{x}}{\alpha\overline{x} + \beta\overline{x} + \gamma\overline{x}}. \tag{2.1}$$

If $a < 1$, then the equilibrium points of (1.1) is given by

$$\overline{x} = \frac{a+b+c}{(\alpha+\beta+\gamma)(1-\delta)}. \tag{2.2}$$

Let $f : (0,\infty)^4 \longrightarrow (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v,w,t) = \delta u + \frac{av+bw+ct}{\alpha v+\beta w+\gamma t}. \tag{2.3}$$

Therefore at $\overline{x} = \dfrac{a+b+c}{(\alpha+\beta+\gamma)(1-\delta)}$

$$\left\{\frac{\partial f}{\partial u}\right\}_{\overline{x}} = \delta = -\eta_3,$$

$$\left\{\frac{\partial f}{\partial v}\right\}_{\overline{x}} = \frac{(a\beta-b\alpha)+(a\beta-c\alpha)}{(\alpha+\beta+\gamma)^2\overline{x}} = \frac{[(a\beta-b\alpha)+(a\gamma-c\alpha)](1-\delta)}{(\alpha+\beta+\gamma)(a+b+c)} = -\eta_2,$$

$$\left\{\frac{\partial f}{\partial w}\right\}_{\overline{x}} = -\frac{(a\beta-b\alpha)+(b\gamma-c\beta)}{(\alpha+\beta+\gamma)^2\overline{x}} = \frac{[-(a\beta-b\alpha)+(b\gamma-c\beta)](1-\delta)}{(\alpha+\beta+\gamma)(a+b+c)} = -\eta_1, \tag{2.4}$$

$$\left\{\frac{\partial f}{\partial t}\right\}_{\overline{x}} = -\frac{(a\gamma-c\alpha)-(b\gamma-c\beta)}{(\alpha+\beta+\gamma)^2\overline{x}} = \frac{[-(a\gamma-c\alpha)-(b\gamma-c\beta)](1-\delta)}{(\alpha+\beta+\gamma)(a+b+c)} = -\eta_0.$$

Then the linearized equation of Eq.(1.1) about $\overline{x}$ is

$$y_{n+1} + \eta_3 y_n + \eta_2 y_{n-l} + \eta_1 y_{n-k} + \eta_0 y_{n-s} = 0.$$

whose characteristic equation is

$$\lambda^4 + \eta_3\lambda^3 + \eta_2\lambda^2 + \eta_1\lambda + \eta_0 = 0. \tag{2.5}$$

Theorem 1.*Assume that*

$$(\alpha+\beta+\gamma)(a+b+c) > \max\left\{ |2\alpha(b+c)-2a(\beta+\gamma)|, |2\gamma(a+b)-2c(\alpha+\beta)|, |2\beta(a+c)-2b(\alpha+\gamma)| \right\}$$

Then the positive equilibrium point $\overline{x} = \dfrac{a+b+c}{(\alpha+\beta+\gamma)(1-\delta)}$ *of* Eq.(1.1) is locally asymptotically stable.

*Proof.*It follows by Theorem A that, (2.5) is asymptotically stable if all roots of (2.5) lie inside the open disc $|\lambda| < 1$,that is, if

$$|\delta| + \left|\frac{[(aB-bA)+(aC-cA)](1-\delta)}{(A+B+C)(a+b+c)}\right| + \left|\frac{[-(aB-bA)+(bC-cB)](1-\delta)}{(A+B+C)(a+b+c)}\right| + \left|\frac{[-(aC-cA)-(bC-cB)](1-\delta)}{(A+B+C)(a+b+c)}\right| < 1, \tag{2.6}$$

and so

$$|[(aB-bA)+(aC-cA)](1-\delta)| + |[-(aB-bA)+(bC-cA)](1-\delta)| . + |[-(aC-cA)-(bC-cB)](1-\delta)| < [(A+B+C)(a+b+c)](1-\delta), \tag{2.7}$$

dividing numerator and denominator by $(1-\delta)$gives,

$$|(aB-bA)+(aC-cA)| + |-(aB-bA)+(bC-cB)| + |-(aC-cA)-(bC-cB)| < (A+B+C)(a+b+c) \tag{2.8}$$

Suppose that

$$\begin{aligned} \phi_1 &= (aB-bA)+(aC-cA) \\ \phi_2 &= -(aB-bA)+(bC-cB) \\ \phi_3 &= -(aC-cA)-(bC-cB). \end{aligned} \tag{2.9}$$

We consider the following cases.

(1) $\phi_1 > 0, \phi_2 > 0$,and $\phi_3 > 0$ In this case we see from Eq.(2.8) that,

$$(a\beta-b\alpha)+(a\gamma-c\alpha)-(a\beta-b\alpha)+(b\gamma-c\beta) - (a\gamma-c\alpha)-(b\gamma-c\beta) < (a+b+c)(\alpha+\beta+\gamma) \tag{2.10}$$

$$\Leftrightarrow$$

$$(a+b+c)(\alpha+\beta+\gamma) > 0, \tag{2.11}$$

That is always true.

(2) $\phi_1 > 0, \phi_2 > 0$,and $\phi_3 < 0$.It follows from (2.8) that

$$(a\beta-b\alpha)+(a\gamma-c\alpha)-(a\beta-b\alpha)+(b\gamma-c\beta) + (a\gamma-c\alpha)+(b\gamma-c\beta) < (a+b+c)(\alpha+\beta+\gamma) \tag{2.12}$$

$$\Leftrightarrow$$

$$2\gamma(a+b)-2c(\alpha+\beta) < (a+b+c)(\alpha+\beta+\gamma). \tag{2.13}$$

So, is satisfied by assumption.

(3) $\phi_1 > 0, \phi_2 < 0, \phi_3 > 0$ we see from (2.8) that

$$(a\beta-b\alpha)+(a\gamma-c\alpha)+(a\beta-b\alpha)-(b\gamma-c\beta) - (a\gamma-c\alpha)-(b\gamma-c\beta) < (a+b+c)(\alpha+\beta+\gamma) \tag{2.14}$$

$$\Leftrightarrow$$

$$2\beta(a+c)-2b(\alpha+\gamma) < (a+b+c)(\alpha+\beta+\gamma). \tag{2.15}$$

That is satisfied by assumption.
(4) $\phi_1 > 0, \phi_2 < 0, \phi_3 < 0$ we see from (2.8) that

$$(a\beta - b\alpha) + (a\gamma - c\alpha) + (a\beta - b\alpha) - (b\gamma - c\beta) + (a\gamma - c\alpha) + (b\gamma - c\beta) < (a+b+c)(\alpha+\beta+\gamma) \quad (2.16)$$

$\Leftrightarrow$

$$2a(\beta+\gamma) - 2\alpha(b+c) < (a+b+c)(\alpha+\beta+\gamma). \quad (2.17)$$

That is satisfied by assumption.

(5) $\phi_1 < 0, \phi_2 > 0, \phi_3 > 0$ we see from (2.8) that

$$-(a\beta - b\alpha) - (a\gamma - c\alpha) - (a\beta - b\alpha) + (b\gamma - c\beta) - (a\gamma - c\alpha) - (b\gamma - c\beta) < (a+b+c)(\alpha+\beta+\gamma) \quad (2.18)$$

$\Leftrightarrow$

$$2\alpha(b+c) - 2a(\beta+\gamma) < (a+b+c)(\alpha+\beta+\gamma). \quad (2.19)$$

That is satisfied by assumption.

(6) $\phi_1 < 0, \phi_2 > 0, \phi_3 < 0$ we see from (2.8) that

$$-(a\beta - b\alpha) - (a\gamma - c\alpha) - (a\beta - b\alpha) + (b\gamma - c\beta) + (a\gamma - c\alpha) + (b\gamma - c\beta) < (a+b+c)(\alpha+\beta+\gamma) \quad (2.20)$$

$\Leftrightarrow$

$$2b(\alpha+\gamma) - 2\beta(a+c) < (a+b+c)(\alpha+\beta+\gamma). \quad (2.21)$$

That is satisfied by assumption.

(7) $\phi_1 < 0, \phi_2 < 0, \phi_3 > 0$ we see from (2.8) that

$$-(a\beta - b\alpha) - (a\gamma - c\alpha) + (a\beta - b\alpha) - (b\gamma - c\beta) - (a\gamma - c\alpha) - (b\gamma - c\beta) < (a+b+c)(\alpha+\beta+\gamma) \quad (2.22)$$

$\Leftrightarrow$

$$2c(\alpha+\beta) - 2\gamma(a+b) < (a+b+c)(\alpha+\beta+\gamma). \quad (2.23)$$

That is satisfied by assumption
(8) $\phi_1 < 0, \phi_2 < 0, \phi_3 < 0$ we see from (2.8) that

$$-(a\beta - b\alpha) - (a\gamma - c\alpha) + (a\beta - b\alpha) - (b\gamma - c\beta) + (a\gamma - c\alpha) + (b\gamma - c\beta) < (a+b+c)(\alpha+\beta+\gamma) \quad (2.24)$$

$\Leftrightarrow$

$$(a+b+c)(\alpha+\beta+\gamma) > 0. \quad (2.25)$$

which is always true. This completes the proof.

3 Existence of Periodic Solutions

This section deals with the study of existence of periodic solutions of Eq.(1.1).

Theorem 2. *Eq.(1.1) has positive prime period two solutions* $\Leftrightarrow$

$$(a-b+c)(\alpha-\beta+\gamma)(1+\delta) + 4(\delta\beta(a+c) + b(\alpha+\gamma)) > 0 \\ , \alpha+\gamma > \beta, \ a+c > b \ , l-odd\, k, s, -even. \quad (3.1)$$

Proof. Let us suppose that there exists a prime period two solution

$$..., p, q, p, q, ..., \quad (3.2)$$

of Eq.(1.1). We will show that Condition (3.1) satisfied. We get from Eq. (1.1) (when $l-$odd $k, s, -$even) that,

$$p = \delta q + \frac{ap + (b+c)q}{\alpha p + (\beta+\gamma)q} = \delta q + \frac{ap + dq}{\alpha p + \mu q}, \quad (3.3)$$
$$q = \delta q + \frac{aq + (b+c)p}{\alpha q + (\beta+\gamma)p} = \delta p + \frac{aq + dp}{\alpha q + \mu p}.$$

where $d = b + c, \ \mu = \beta + \gamma$
Then

$$\alpha p^2 + \mu pq = \delta\alpha pq + \delta\mu q^2 + ap + dq, \quad (3.4)$$

and

$$\alpha q^2 + \mu pq = \delta\alpha pq + \delta\mu p^2 + aq + dp. \quad (3.5)$$

Subtracting Eq.(3.4) from Eq. (3.5) we get,

$$\alpha(p^2 - q^2) = \delta\mu(q^2 - p^2) + a(p - q) + d(q - p).$$

Since $p \neq q$, we see that,

$$p+q=\frac{(a-d)}{(\alpha+\delta d)}. \tag{3.6}$$

Now, adding Eq.(3.4) and Eq.(3.5) we get,

$$\alpha(p^2+q^2)+2\mu pq=2\delta\alpha pq+\delta\mu(p^2+q^2)+(a+d)(p+q) \tag{3.7}$$

or

$$(\alpha-\delta\mu)(p^2-q^2)+2(\mu-\delta\alpha)pq=(a+d)(p+q) \tag{3.8}$$

It follows from Eq.(3.6),Eq.(3.8) and the relation $p^2+q^2=(p+q)^2-2pq$ for all $p,q\in R$, that

$$pq=\frac{(\alpha d+a\delta\mu)(a-d)}{(\mu-\alpha)(1+\delta)(\alpha+\delta\mu)^2}. \tag{3.9}$$

Now,it is clear from Eq.(3.6) and Eq.(3.9) that p and q are the two positive distinct roots of the quadratic equation

$$y^2-\left(\frac{(a-d)}{(\alpha+\delta d)}\right)y+\left(\frac{(\alpha d+a\delta\mu)(a-d)}{(\mu-\alpha)^2(1+\delta)(\alpha+\delta\mu)}\right)=0, \tag{3.10}$$

$$(\alpha+\delta d)y^2-(a-d)y+\left(\frac{(\alpha d+a\delta\mu)(a-d)}{(\mu-\alpha)(1+\delta)(\alpha+\delta\mu)}\right)=0, \tag{3.10}$$

and so

$$(a-d)^2-\frac{4(\alpha d+a\delta\mu)(a-d)}{(\alpha-\mu)(1+\delta)} \tag{3.11}$$

thus

$$(a-d)(\alpha-\mu)(1+\delta)+4(\alpha d+a\delta\mu)>0. \tag{3.12}$$

So, the inequality (3.1) satisfied.

Again suppose that inequality (3.1) holds. We will prove that Eq.(1.1) has a prime period two solutions. suppose that

$$p=\frac{a-d+\xi}{2(\alpha+\delta\mu)}, \qquad q=\frac{a-d-\xi}{2(\alpha+\delta\mu)}. \tag{3.13}$$

where $\xi=\sqrt{(a-d)^2-\frac{4(\alpha d+a\delta\mu)(a-d)}{(\mu-\alpha)(1+\delta)}}$.

From inequality (3.1)

$$(a-d)(\alpha-\mu)(1+\delta)+4(\alpha d+a\delta\mu)>0, \tag{3.14}$$

which is equivalent to

$$(a-d)^2>\frac{4(\alpha d+a\delta\mu)(a-d)}{(\mu-\alpha)(1+\delta)}. \tag{3.15}$$

Therefore p and q are distinct real numbers.
Set

$$x_{-l}=p,\ x_{-l+1}=q,\ ,x_{-k}=q,\ x_{-k+1}=p,\quad x_{-s}=q,\ x_{-s+1}=q,\ \text{and}\ x_0=q. \tag{3.16}$$

We will prove that,

$$x_1=x_{-1}=p \quad\text{and}\quad x_2=x_0=q. \tag{3.17}$$

From Eq.(1.1) that

$$x_1=\delta x_0+\frac{ax_{-l}+bx_{-k}+cx_{-s}}{\alpha x_{-l}+\beta x_{-k}+\gamma x_{-s}}=\delta q+\frac{ap+bq+cq}{\alpha p+\beta q+\gamma q}=\delta q+\frac{ap+dq}{\alpha p+\mu q} \tag{3.18}$$

$$=\delta\left(\frac{a-d-\xi}{2(\alpha+\delta\mu)}\right)+\frac{a\left(\frac{a-d+\xi}{2(\alpha+\delta\mu)}\right)+d\left(\frac{a-d-\xi}{2(\alpha+\delta\mu)}\right)}{\alpha\left(\frac{a-d+\xi}{2(\alpha+\delta\mu)}\right)+\mu\left(\frac{a-d-\xi}{2(\alpha+\delta\mu)}\right)}, \tag{3.19}$$

dividing the denominator and numerator by $2(\alpha+\delta\mu)$ gives

$$x_1=\frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)}+\frac{(a-d)\{(a+d)+\xi\}}{(\alpha+\mu)(a-d)+(\alpha-\mu)\xi}. \tag{3.20}$$

Multiplying the denominator and numerator by $\{(\alpha+\mu)(a-d)-(\alpha-\mu)\xi\}$we get,

$$\begin{aligned} x_1&=\frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)} + \frac{(a-d)[(a+d)+\xi][(\alpha+\mu)(a-d)-(\alpha-\mu)\xi]}{[(\alpha+\mu)(a-d)+(\alpha-\mu)\xi][(\alpha+\mu)(a-d)-(\alpha-\mu)\xi]},\\ &=\frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)} + \frac{(a-d)\{(\alpha+\mu)(a^2-d^2)+\xi[(\alpha+\mu)(a-d)-(\alpha-\mu)(a+d)]-(\alpha-\mu)\lambda\}}{(\alpha+\mu)^2(a-d)^2-(\alpha-\mu^2)\xi^2},\\ &=\frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)}+\frac{(a-d)\{(\alpha+\mu)(a^2-d^2)+2\xi(a\mu-\alpha d)-\chi\}}{(\alpha+\mu)^2(a-d)^2-(\alpha-\mu^2)\lambda},\\ &=\frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)} + \frac{(a-d)\{2(a-d)[\alpha d+\mu a-2(\delta a\mu+\alpha d)/(1+\delta)]+2\xi(a\mu-\alpha d)\}}{4(a-d)[\alpha\mu(a-d)+(\alpha-\mu)(a\delta\mu+\alpha d)/(1+\delta)} \end{aligned} \tag{3.21}$$

where $\chi=(\alpha-d)(a-d)^2+\frac{4(\alpha d+a\delta\mu)(a-d)}{(1+\delta)}$ and $\lambda=(a-d)^2-4(\alpha d+a\delta\mu)(a-d)/(\mu-\alpha)(1+\delta)$

Multiply the denominator and numerator by $(1+\delta)$,we get

$$x_1=\frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)}+\frac{(a-d)[(\alpha d+\mu a)(1+\delta)-2(a\delta\mu+\alpha d)]+\xi(1+\delta)(a\mu-\alpha d)}{(a-d)(1+\delta)(a-d)[\alpha^2+\mu^2+2\alpha\mu-\alpha^2-\mu^2+2\alpha\mu]-4(a-d)(\alpha d+a\delta\mu)}, \tag{3.22}$$

$$\begin{aligned} x_1 &= \frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)}+ \\ &\quad \frac{2(a-d)(\alpha d-a\mu)\{(a-d)(\delta-1)-\xi(1+\delta)\}}{4(a-d)[\alpha\mu(1+\delta)(a-d)-(a\delta\mu+\alpha d)(\alpha-\mu)]} \\ &= \frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)}+\frac{[(a-d)(\delta-1)-\xi(1+\delta)](\alpha d-a\mu)}{2[\alpha\mu(a-d+a\delta-\delta d)-(\alpha^2 d-\alpha d\mu+a\delta\alpha\mu-a\delta\mu^2)]} \\ &= \frac{\delta a-\delta d-\delta\xi}{2(\alpha+\delta\mu)}+\frac{[(a-d)(\delta-1)-\xi(1+\delta)](\alpha d-a\mu)}{2(\alpha+\mu\delta)(a\mu-\alpha d)}. \end{aligned} \tag{3.23}$$

Dividing numerator and denominator by $(\alpha d-a\mu)$ we get,

$$x_1=\frac{\delta a-\delta d-\delta\xi+(a-d)(1-\delta)+\xi(1+\delta)}{2(\alpha+\mu\delta)}=\frac{a-d+\xi}{2(\alpha+\mu\delta)}=p \tag{3.24}$$

Similarly we can prove that,

$$x_2=q. \tag{3.25}$$

Then it follows by induction that

$$x_{2n}=q \quad \text{and} \quad x_{2n+1}=p \quad \text{for all} \quad n\geq -1. \tag{3.26}$$

Thus Eq.(1.1) has the positive prime period two solution

$$...,p,q,p,q,...,$$

where p and q are the distinct roots of the quadratic equation Eq.(3.10) and the proof is complete.

The following Theorems can be proved similarly.

Theorem 3. *Eq.(1.1) has a prime period two solutions* $\Leftrightarrow$

$$(b-a-c)(\beta-\alpha-\gamma)(1-\delta)+4(\beta(a+c)+\delta b(\alpha+\gamma))>0, \quad k-odd\ ,l,s-even. \tag{3.27}$$

Theorem 4. *Eq.(1.1) has a prime period two solutions* $\Leftrightarrow$

$$(c-a-b)(\gamma-\alpha-\beta)(1-\delta)+4(\gamma(a+b)+\delta c(\alpha+\beta))>0 \quad ,\ l,k-even\ ,s-odd. \tag{3.28}$$

Theorem 5. *Eq.(1.1) has a prime period two solutions* $\Leftrightarrow$

$$(b+c-a)(\beta+\gamma-\alpha)(1-\delta)+4(a(\beta+\gamma)+\delta\alpha(b+c)>0 \quad ,\ k,s-odd,l-even. \tag{3.29}$$

Theorem 6. *Eq.(1.1) has a prime period two solutions* $\Leftrightarrow$

$$(a+b-c)(\alpha+\beta-\gamma)(1-\delta)+4(c(\alpha+\beta)+\delta\gamma(a+b))>0 \quad ,\ l,t-even,s,k-odd. \tag{3.30}$$

Theorem 7. *Eq.(1.1) has a prime period two solutions* $\Leftrightarrow$

$$(a+c-b)(\alpha+\gamma-\beta)(1-\delta)+4(b(\alpha+\gamma)+\delta\beta(a+c))>0 \quad ,\ k-even,s,l-odd. \tag{3.31}$$

4 Global Attractivity of the Equilibrium Point of Eq.(1.1)

In this section we investigate the global attractivity character of solutions of Eq.(1.1).

Theorem 8. *The equilibrium point $\overline{x}$ of Eq.(1.1) is global attractor.of Eq.(1.1) $\Leftrightarrow$ one of the following conditions holds:*

$$\begin{aligned} &(1)\frac{a}{\alpha}\geq\frac{b}{\beta}\geq\frac{c}{\gamma}, \quad c\geq a+b \\ &(2)\frac{a}{\alpha}\geq\frac{c}{\gamma}\geq\frac{b}{\beta}, \quad b\geq a+c \\ &(3)\frac{b}{\beta}\geq\frac{a}{\alpha}\geq\frac{c}{\gamma}, \quad c\geq a+b \\ &(4)\frac{b}{\beta}\geq\frac{c}{\gamma}\geq\frac{a}{\alpha}, \quad a\geq b+c \\ &(5)\frac{c}{\gamma}\geq\frac{b}{\beta}\geq\frac{a}{\alpha}, \quad a\geq b+c \\ &(6)\frac{c}{\gamma}\geq\frac{a}{\alpha}\geq\frac{b}{\beta}, \quad b\geq a+c \end{aligned} \tag{4.1}$$

Proof. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq(1.1) also let f be a function defined by Eq.(2.3) Assume that Eq.(4.1) is true then, it is easy to see that the function $f(u,v,w,t)$ is non-decreasing in u,v.and non increasing in t. But not clear what is happening with w so we consider two cases.

Case 1 Assume that $f(u,v,w,t)$ is nondecreasing in w. Suppose that (Φ,Ψ) is a solution of the system

$$\Phi=f(\Phi,\Phi,\Psi,\Phi) \quad and \quad \Psi=f(\Psi,\Psi,\Phi,\Psi). \tag{4.2}$$

We see from Eq.(1.1), that

$$\Phi=\delta\Phi+\frac{a\Phi+b\Phi+c\Psi}{\alpha\Phi+\beta\Phi+\gamma\Psi}, \quad \Psi=\delta\Psi+\frac{a\Psi+b\Psi+c\Phi}{\alpha\Psi+\beta\Psi+\gamma\Phi}, \tag{4.3}$$

So,

$$\Phi(1-\delta)=\frac{a\Phi+b\Phi+c\Psi}{\alpha\Phi+\beta\Phi+\gamma\Psi}, \quad \Psi(1-\delta)=\frac{a\Psi+b\Psi+c\Phi}{\alpha\Psi+\beta\Psi+\gamma\Phi}, \tag{4.4}$$

Then,

$$(\alpha+\beta)(1-\delta)\Psi^2+\gamma(1-\delta)\Phi\Psi = (a+b)\Psi+c\Phi,$$
$$(\alpha+\beta)(1-\delta)\Phi^2+\gamma(1-\delta)\Phi\Psi = (a+b)\Phi+c\Psi \quad (4.5)$$

by subtracting, we get

$$(\Psi-\Phi)\{(\alpha+\gamma)(1-\delta)(\Phi+\Psi)+(c-a-b)\}=0 \quad (4.6)$$

under condition $c \geq a+b$, and $\delta < 1$,Thus

$$\Psi = \Phi.$$

From Theorem B it follows that $\overline{x}$ is a global attractor of Eq.(1.1) and hence, the proof is completed.

Case 2 Assume that $f(u,v,w,t)$ is nonincreasing in w. Suppose that (Φ,Ψ) is a solution of the system

$$\Phi = f(\Phi,\Phi,\Psi,\Psi) \quad and \quad \Psi = f(\Psi,\Psi,\Phi,\Phi). \quad (4.7)$$

We see from Eq.(1.1), that

$$\Phi = \delta\Phi+\frac{a\Phi+b\Psi+c\Psi}{\alpha\Phi+\beta\Psi+\gamma\Psi}, \quad \Psi = \delta\Psi+\frac{a\Psi+b\Phi+c\Phi}{\alpha\Psi+\beta\Phi+\gamma\Phi}, \quad (4.8)$$

So,

$$\Phi(1-\delta) = \frac{a\Phi+b\Psi+c\Psi}{\alpha\Phi+\beta\Psi+\gamma\Psi}, \quad \Psi(1-\delta) = \frac{a\Psi+b\Phi+c\Phi}{\alpha\Psi+\beta s+\gamma\Phi}, \quad (4.9)$$

Then,

$$\alpha(1-\delta)\Phi^2+(\beta+\gamma)(1-\delta)\Phi\Psi = (b+c)\Psi+a\Phi,$$
$$\alpha(1-\delta)\Psi^2+(\beta+\gamma)(1-\delta)\Phi\Psi = (b+c)\Phi+a\Psi \quad (4.10)$$

subtracting, we get

$$(\Psi-\Phi)\{\alpha(1-\delta)(\Phi+\Psi)+(a-b-c)\}=0 \quad (4.11)$$

under condition $a \geq b+c$, and $\delta < 1$,Thus

$$\Psi = \Phi.$$

From Theorem B it follows that $\overline{x}$ is a global attractor of Eq.(1.1) and hence, the proof is completed.

5 Boundedness of Solutions of (1.1)

In this section we study the boundedness of solution of Eq.(1.1)

$$x_{n+1} = \delta x_n+\frac{ax_{n-l}+bx_{n-k}+cx_{n-s}}{\alpha x_{n-l}+\beta x_{n-k}+\gamma x_{n-s}} \leq \delta x_n+\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} \quad ,\text{for all } n \geq 1. \quad (5.1)$$

By comparison, one can see that

$$\lim_{n\to\infty}\sup x_n \leq \frac{a\beta\gamma+b\alpha\gamma+c\alpha\beta}{(1-\delta)\alpha\beta\gamma} = M \quad (5.2)$$

So, the solution is bounded.
Now, we will show that $\exists\ m > 0$ such that

$$x_n \geq m, \text{for all } n \geq 1$$

Transformation,

$$x_n = \frac{1}{y_n} \quad (5.3)$$

will change Eq.(1.1) to the following form

$$y_{n+1} \leq \frac{\alpha bc+\beta ac+\gamma ab}{abc} = T \quad (5.4)$$

So, we obtain

$$x_n = \frac{1}{y_n} \geq \frac{1}{T} = \frac{abc}{\alpha bc+\beta ac+\gamma ab} = m, \text{ for all } n \geq 1 \quad (5.5)$$

from Eq.(5.2) and Eq.(5.5) we see that,

$$m \leq x_n \leq M, \quad \text{for all } n \geq 1 \quad (5.6)$$

Hence, every solution of Eq.(1.1) is bounded and persists.

6 Numerical Examples

In order to verify our results of this paper, we consider some numerical examples as follows.

Example 1. We suppose, $l=1,\ k=2,\ s=3,\ x_{-3}=2,\ x_{-2}=3,\ x_{-1}=8,\ x_0=5, \delta=0.6,\ a=7,\ b=3,\ c=9,\ \alpha=3.8,\ \beta=0.2,\ \gamma=1$**.** **(**See Fig. 1)

Example 2. (See Fig. 2), since $l=2,\ k=3, s=4,\ x_{-4}=0.2,\ x_{-3}=4.1,\ x_{-2}=9,\ x_{-1}=10,\ x_0=0.1,\ \delta=0.5,\ a=0.9,\ b=2,\ c=7,\ \alpha=4.2,\ \beta=0.2,\ \gamma=2.$

Example 3. See Fig. 3, since $l=1,\ k=2, s=3,\ x_{-3}=12,\ x_{-2}=7,\ x_{-1}=8,\ x_0=3,\ a=0.1,\ b=0.2,\ c=0.5,\ \alpha=0.6,\ \beta=0.2, \gamma=0.3$

Example 4. Fig. 4. shows the solutions when $\delta=0.1,\ a=0.9,\ b=0.1, c=0.03,\ \alpha=2, \beta=0.2, \gamma=0.1.2,\ l=2,\ k=4, s=6,\ x_{-6}=q, x_{-5}=p, x_{-4}=q\ x_{-3}=p,\ x_{-2}=q,\ x_{-1}=p,\ x_0=q.$

Since

$$\left(p,q = \frac{(a-d)\pm\sqrt{(a-d)^2-\dfrac{4(\alpha d+a\delta\mu)(a-d)}{(\mu-\alpha)(1+\delta)}}}{2(\alpha+\delta\mu)} \right)$$

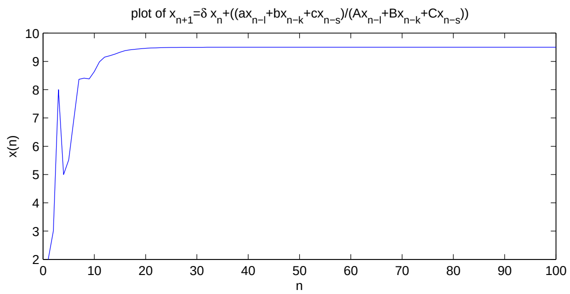

Fig. 1: This figure shows the behavior of solution of Eq.(1.1) when, $l=1,\ k=2,\ s=3,\ x_{-3}=2,\ x_{-2}=3,\ x_{-1}=8,\ x_0=5, \delta=0.6,\ a=7,\ b=3,\ c=9,\ A=3.8,\ B=0.2,\ C=1$**.**

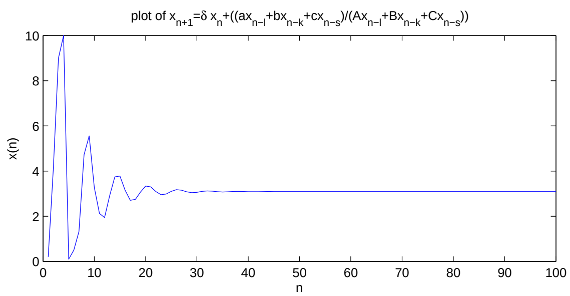

Fig. 2: This figure shows the converging behavior towards the equilibrium.

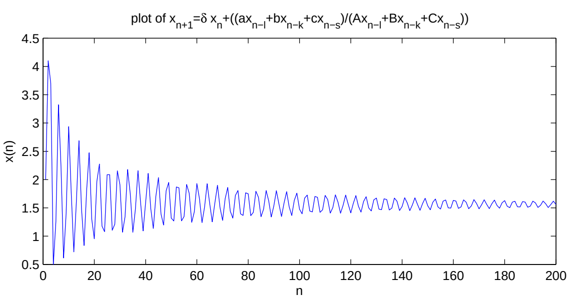

Fig. 3: This figure shows the asymptotic behavior towards the equilibrium when $l=1,\ k=2, s=3,\ x_{-3}=12,\ x_{-2}=7,\ x_{-1}=8,\ x_0=3,\ a=0.1,\ b=0.2,\ c=0.5,\ \alpha=0.6,\ \beta=0.2,\ \gamma=0.3$

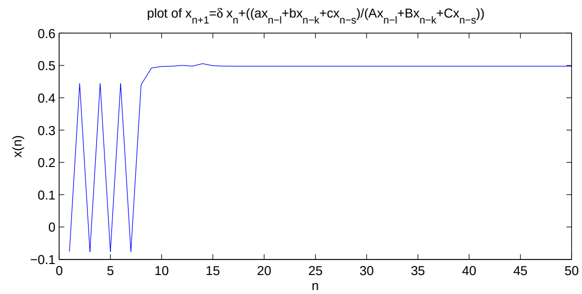

Fig. 4: This figure shows the periodic solution of Eq. (1.1) when $x_{-1}=p,\ x_0=q$

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