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Numerical Solution of the Time Dependent Emden-Fowler Equations with Boundary Conditions using Modified Decomposition Method

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Abstract: We propose a new modification to Adomian decomposition method for numerical treatment of the time-dependent Emden-Fowler-types equations with the Neumann and Dirichlet boundary conditions. In new modified method, we use all the boundary conditions to derive an integral equation before establishing the recursive scheme. The new modified decomposition method (MDM) will be used without unknown constants while computing the successive solution components. Unlike the recursive schemes that result from using the ADM, the new MDM avoids solving a sequence of nonlinear algebraic or transcendental equations for the derivation of unknown constants. Moreover, the proposed technique is reliable enough to overcome the difficulty of the singular point at $x = 0$. Five illustrative examples are examined to demonstrate the accuracy and applicability of the proposed method.

Keywords: Emden-Fowler Equations, Heat-type equations, Wave-type equations, Adomian decomposition method, Adomian Polynomials, Singular Behavior

1 Introduction

It is well known that the time-dependent Emden-Fowler equations can describe either heat diffusion or wave type equation. Many problems in the literature of the diffusion of heat perpendicular to the surfaces of parallel planes are modeled by the heat equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\alpha}{x}\frac{\partial u(x,t)}{\partial x} + af(x,t)g(u) + h(x,t) = \frac{\partial u(x,t)}{\partial t}, \quad 0 < x < l. \tag{1}$$

Here, $f(x,t)g(u) + h(x,t)$ is the nonlinear heat source, $u(x,t)$ is the temperature, and t is the dimensionless time variable. For the steady-state case, and for $h(x,t) = 0$, Eq. (1) is the Emden-Fowler equation [1] given by

$$u_{xx} + \frac{\alpha}{x}u_x + af(x)g(u) = 0. \tag{2}$$

For $f(x) = 1$ and $g(u) = u^n$, this equation is known as the standard Lane-Emden equation of the first kind, whereas the second kind is obtained when $g(u) = e^u$. It is well-known that the Lane–Emden equation is used in modelling a thermal explosion in either an infinite cylinder $(\alpha = 1)$ or a sphere $(\alpha = 2)$, where α is the shape factor of the equation. In [2], Harley and Momoniat studied this problem, where approximate first integrals were obtained and employed to study the qualitative features of the solutions. For more information on recently works on Lane-Emden equations, see details [2, 3, 4, 5].

However, the time-dependent Emden-Fowler equation of the wave type, with singular behavior, is of the form:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\alpha}{x}\frac{\partial u(x,t)}{\partial x} + af(x,t)g(u) + h(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < l. \tag{3}$$

Here, $f(x,t)g(u) + h(x,t)$ is the nonlinear source, t is the dimensionless time variable, and $u(x,t)$ is the displacement of the wave at the position x and at time t.

In this work, we will concern ourselves on studying the heat type and the wave type of the Emden-Fowler-types equations (1) and (3) with the

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Neumann and Dirichlet boundary conditions:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=0, \qquad u(x,t)|_{x=l}=g(t). \tag{4}$$

Recently, many researchers [6,7,8,9,10] have shown interest to the study of ADM for different scientific models. According to Wazwaz [11], we define the partial differential singular operator $L_{xx}:=x^{-\alpha}\frac{\partial}{\partial x}\left(x^{\alpha}\frac{\partial}{\partial x}\right)$, then Eq. (1) can be rewritten as

$$L_{xx}u=u_t-af(x,t)g(u)-h(x,t). \tag{5}$$

Let us formally define the left-inverse integral operator [11]

$$L_{xx}^{-1}=\int_0^x\frac{1}{s^{\alpha}}\int_0^s x^{\alpha}[\cdot]dxds.$$

Operating with L_{xx}^{-1} on both the sides of (5) yields

$$u(x,t)=c(t)+L_{xx}^{-1}[u_t-af(x,t)g(u)-h(x,t)], \quad c(t)=u(0,t). \tag{6}$$

The ADM gives the solution $u(x,t)$ by an finite series of components

$$u(x,t)=\sum_{j=0}^{\infty}u_j(x,t), \tag{7}$$

and the nonlinear function $g(u)$ by an infinite series of Adomial polynomials [7]

$$g(u)=\sum_{j=0}^{\infty}A_j, \tag{8}$$

where

$$A_j=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[g\left(\sum_{j=0}^{\infty}y_j\lambda^j\right)\right]_{\lambda=0}, \quad j=0,1,2,\ldots$$

Substituting the series from (7) and (8) into (6) we obtain

$$\sum_{j=0}^{\infty}u_j(x,t)=c(t)+L_{xx}^{-1}\left[\sum_{j=0}^{\infty}\frac{\partial u_j}{\partial t}-af(x,t)\sum_{j=0}^{\infty}A_j-h(x,t)\right].$$

Identifying the zeroth component $u_0=c(t)$, the ADM admits the recursive scheme:

$$\left.\begin{aligned}&u_0(x,t)=c,\\&u_j(x,t)=L_{xx}^{-1}\left[\frac{\partial u_j}{\partial t}-af(x,t)A_j-h(x,t)\right], \quad j=1,2,\ldots\end{aligned}\right\} \tag{9}$$

We note that the above scheme depends on $c(t)$, where in order to determine it, we have to impose the boundary conditions. This in turn leads in general to a sequence of nonlinear (transcendental) of equations. It is obvious that for solving such a system for $c(t)$, a huge size of computational work is needed.

To the best of our knowledge, no one has applied the ADM to solve the time-dependent Emden-Fowler-types equations with the Neumann and Dirichlet boundary conditions of the form (1), (3) and (4). However, the time-dependent Emden-Fowler-types equations with initial conditions were studied (for details see [11,12,13]). Note the convergence of Adomian decomposition method was established by many authors (for details see [14,15,16]). For more information on recent works on ADM for differential euqtions, see details [17,18,19,20,21,22].

In this work, we aim to develop a modified decomposition method (MDM), to study the series solution of the Emden-Fowler-types equations (1), (3) and (4). The presence of singularity at $x=0$, as well as strong nonlinearity, such problems pose difficulties in obtaining their solutions. The proposed method, that will be presented later, is based on the ADM. However, in the proposed scheme, we will use all the boundary conditions to derive an integral equation before establishing the recursive scheme for the solution of the considered problems. Thus, we develop MDM without any unknown constant while computing the successive solution components. Unlike most of earlier recursive schemes which use ADM, the MDM avoids solving a sequence of nonlinear algebraic or transcendental equations for unknown constant. We will examine five numerical examples to show the reliability and efficiency of the proposed method.

2 The modified decomposition method

2.1 Emden-Fowler heat-type equation

In order to overcome the singular behavior at $x=0$, we rewrite the Emden-Fowler equation (1) as follows:

$$\frac{\partial}{\partial x}\left[x^{\alpha}\frac{\partial u}{\partial x}\right]=x^{\alpha}\left\{\frac{\partial u}{\partial t}-af(x,t)g(u)-h(x,t)\right\}, \; 0<x<l, \tag{10}$$

with the Neumann and Dirichlet boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0, \qquad u(x,t)|_{x=l}=g(t), \quad 0<t\leq T. \tag{11}$$

Integrating both side of Eq.(10) w.r.t. x partially from 0 to x, and then dividing both sides of the above equation by x^{α} and using the boundary condition $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$, we obtain the Volterra integro-partial-differential equation

$$\frac{\partial u}{\partial x}=\frac{1}{x^{\alpha}}\int_0^x\xi^{\alpha}\left\{\frac{\partial u}{\partial t}-af(\xi,t)g(u)-h(\xi,t)\right\}d\xi. \tag{12}$$

We again integrate Eq. (12) w.r.t. x partially from x to l and using $u(x,t)|_{x=l}=g(t)$, we obtain the integro-differential equation

$$u(x,t) = g(t) - \int_x^l \frac{1}{s^\alpha} \int_0^s \xi^\alpha \left\{ \frac{\partial u}{\partial t} - af(\xi,t)g(u) - h(\xi,t) \right\} d\xi ds. \quad (13)$$

We then seek the solution of Eq. (10) in the form of the decomposition series

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t), \quad (14)$$

and the nonlinear term $g(u)$ by an infinite series of Adomial polynomials

$$g(u) = \sum_{j=0}^{\infty} A_j. \quad (15)$$

Substituting Eqs. (14) and (15) into (13) we obtain

$$\sum_{j=0}^{\infty} u_j(x,t) = g(t) - \int_x^l \frac{1}{s^\alpha} \int_0^s \xi^\alpha \left\{ \sum_{j=0}^{\infty} \left[\frac{\partial u_j}{\partial t} \right] - af(\xi,t) \left[\sum_{j=0}^{\infty} A_j \right] - h(\xi,t) \right\} d\xi ds. \quad (16)$$

This in turn leads to the following recursive scheme

$$\left.\begin{aligned} u_0 &= g(t), \\ u_j &= -\int_x^l \frac{1}{s^\alpha} \int_0^s \xi^\alpha \left\{ \frac{\partial u_{j-1}}{\partial t} - af(\xi,t)A_{j-1} - h(\xi,t) \right\} d\xi ds, \quad j = 1,2... \end{aligned}\right\} \quad (17)$$

Then, the approximate series solution as $\psi_n(x,t) = \sum_{j=0}^{n} u_j(x,t)$.

2.2 Emden-Fowler wave-type equation

Following the analysis presented earlier, and to overcome the singular behavior at $x = 0$, we rewrite the Emden-Fowler wave-type equation (3) as

$$\frac{\partial}{\partial x}\left[x^\alpha \frac{\partial u}{\partial x} \right] = x^\alpha \left\{ \frac{\partial^2 u}{\partial t^2} - af(x,t)g(u) - h(x,t) \right\}, \ 0 < x < l \quad (18)$$

with the Neumann and Dirichlet boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(x,t)|_{x=l} = g(t), \quad 0 < t \le T. \quad (19)$$

Proceeding as before, we obtain the modified decomposition method for this case as

$$\left.\begin{aligned} u_0 &= g(t), \\ u_j &= -\int_x^l \frac{1}{s^\alpha} \int_0^s \xi^\alpha \left\{ \frac{\partial^2 u_{j-1}}{\partial t^2} - af(\xi,t)A_{j-1} - h(\xi,t) \right\} d\xi ds, \quad j = 1,2,3... \end{aligned}\right\} \quad (20)$$

Then, the approximate series solution is $\psi_n(x,t) = \sum_{j=0}^{n} u_j(x,t)$.

3 Numerical Results

In this section we examine some different models of time-dependent Emden-Fowler heat-type as well as wave-type equations. All the results are calculated using the symbolic software Mathematica. To show the accuracy of the MDM, the maximum error is defined as:

$$E_n = \max |u(x,t) - \psi_n(x,t)|, \quad n = 1,2,\dots \quad (21)$$

where $u(x,t)$ is the analytical solutions of the considered models and $\psi_n(x,t)$ is the approximate solutions.

3.1 Emden-Fowler heat-type equation

Firstly, we consider some models of heat-type equations with singular behavior at $x = 0$.

*Example 1.*Consider the following linear time-dependent Emden-Fowler heat-type equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{5}{x}\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t} + \left(12t^2 - 2tx^2 + 4t^4x^2\right) u(x,t), \quad (22)$$

with the boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(x,t)|_{x=1} = e^{t^2}, \quad 0 < t \le T, \quad (23)$$

The analytical solution of the problem is $u(x,t) = e^{x^2 t^2}$.

According to the proposed MDM (17), the problem (22)-(23) can be written as:

$$\left.\begin{aligned} u_0(1,t) &= e^{t^2}, \\ u_j(x,t) &= -\int_x^1 \frac{1}{s^5} \int_0^s \xi^5 \left\{ \frac{\partial u_{j-1}}{\partial t} + \left(12t^2 - 2t\xi^2 + 4t^4\xi^2\right) u_{j-1} \right\} d\xi ds, \quad j = 1,2... \end{aligned}\right\} \quad (24)$$

Hence, the n-terms truncated series solution is obtained as $\psi_n(x,t) = \sum_{j=0}^{n} u_j(x,t)$.

In order to obtain the maximum error $E_n = \max|u - \psi_n|$, we use the Mathematica Command '**NMaximize**'. Then, the numerical results of error E_n, $n = 1,2,3,4,5,6$ are given in Table 1 (with time $t = 0.5, 1$). From Table 1, it can be concluded that, the error decreases monotonically with the increase of the integer n.

Table 1: The maximum absolute error E_n of Example 1

	E_1	E_2	E_3	E_4	E_5	E_6
$t = 0.5$	0.113889	0.0410658	0.0162361	0.00703156	0.00330148	0.00168682
$t = 1$	0.940007	0.3358410	0.1598150	0.08298140	0.04493210	0.02494430

*Example 2.*Consider the following nonlinear time-dependent Emden-Fowler heat-type equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{5}{x}\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t} + f(x,t)e^{u(x,t)} + h(x,t)e^{\frac{1}{2}u(x,t)}, \quad (25)$$

with the boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \ u(x,t)|_{x=1} = -2\ln(1+t), \ \ 0 < t \leq T, \quad (26)$$

where $f(x,t) = (24t + 16t^2x^2)$, and $h(x,t) = 2x^2$. The analytical solution of the problem is $u(x,t) = -2\ln(1+tx^2)$.

According to the proposed MDM (17), the problem (25)-(26) can be written as

$$\left.\begin{aligned} u_0(1,t) &= -2\ln(1+t), \\ u_j(x,t) &= -\int_x^1 \frac{1}{s^5}\int_0^s \xi^5\left\{\frac{\partial u_{j-1}}{\partial t} + \right. \\ &\left. f(\xi,t)A_{j-1} + h(\xi,t)B_{j-1}\right\}d\xi ds, \ \ j = 1,2\ldots \end{aligned}\right\} \quad (27)$$

The Adomian polynomial for the nonlinear term e^u are given as

$$A_0 = e^{u_0}; \ \ A_1 = e^{u_0}u_1; \ \ A_2 = \frac{1}{2}e^{u_0}\left(u_1^2 + 2u_2\right)\ldots$$

and for the term $e^{\frac{u}{2}}$ are as

$$B_0 = e^{\frac{u_0}{2}}; \ B_1 = \frac{1}{2}e^{\frac{u_0}{2}}u_1; \ B_2 = \frac{1}{8}e^{\frac{u_0}{2}}\left(u_1^2 + 4u_2\right)\ldots$$

Hence, the n-terms truncated series solution is obtained as $\psi_n(x,t) = \sum_{j=0}^{n} u_j(x,t)$. In this case, the maximum error E_n, $n = 1,2,3,4,5,6$ is listed in Table 2 (with time $t = 0.5, 1$). From the table, we observe that the error decreases with an increases in n.

Table 2: The maximum absolute error E_n of Example 2

	E_1	E_2	E_3	E_4	E_5	E_6
$t = 0.5$	0.810930	0.241486	0.096205	0.0412265	0.0181383	0.0081151
$t = 1$	0.940007	0.335841	0.159815	0.0829814	0.0449321	0.0249443

*Example 3.*We next consider the following nonlinear time-dependent Emden-Fowler heat-type equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\alpha}{x}\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t} + f(x,t)e^{u(x,t)} \quad (28)$$

$$+ h(x,t)e^{2u(x,t)}, \quad (29)$$

with the boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \ \ u(x,t)|_{x=1} = \ln\left(\frac{1}{3+t^\beta}\right), \ \ 0 < t \leq T, \quad (30)$$

where $f(x,t) = t(tx)^{-2+\beta}\beta\left(x^2 - t(-1+\alpha+\beta)\right)$, $h(x,t) = t^2(tx)^{-2+2\beta}\beta^2$, α and β are physical parameters. The analytical solution of the problem is $u(x,t) = \ln(\frac{1}{3+(xt)^\beta})$.

According to the MDM (17), the problem (28)-(30) can be written as:

$$\left.\begin{aligned} u_0(1,t) &= \ln\left(\frac{1}{3+t^\beta}\right), \\ u_j(x,t) &= -\int_x^1 \frac{1}{s^\alpha}\int_0^s \xi^\alpha\left\{\frac{\partial u_{j-1}}{\partial t}\right. \\ &\left. + f(\xi,t)A_{j-1} + h(\xi,t)B_{j-1}\right\}d\xi ds, \ \ j = 1,2\ldots \end{aligned}\right\} \quad (31)$$

For the nonlinear terms e^u and e^{2u}, the Adomian polynomials are given by

$$A_0 = e^{u_0}; A_1 = e^{u_0}u_1; \ A_2 = \frac{1}{2}e^{u_0}\left(u_1^2 + 2u_2\right)\ldots$$

$$B_0 = e^{2u_0}; \ B_1 = 2e^{2u_0}u_1; \ B_2 = 2e^{2u_0}\left(u_1^2 + u_2\right)\ldots \quad (32)$$

Hence, the n-terms truncated series solution is obtained as $\psi_n(x,t) = \sum_{j=0}^{n} u_j(x,t)$. The maximum error E_n, $n = 1,2,3,4,5,6$ are listed in Table 3 (where $\alpha = 1, \beta = 2$) and Table 4 (where $\alpha = 2, \beta = 2$). In each cases, we find that the error decreases uniformly with an increases in t. From the same Tables, we observe that the E_n decreases when α increases from $\alpha = 1$ to $\alpha = 2$.

Table 3: The maximum absolute error E_n of Example 3 when $\alpha = 1, \beta = 2$

	E_1	E_2	E_3	E_4	E_5	E_6
$t = 0.5$	0.0930934	0.0530019	0.0102946	0.00936786	0.00156164	0.00076164
$t = 1$	0.0974429	0.0593728	0.0116003	0.00673511	0.00344730	0.00094733

Table 4: The maximum absolute error E_n of Example 3 when $\alpha = 2, \beta = 2$

	E_1	E_2	E_3	E_4	E_5	E_6
$t = 0.5$	0.0315944	0.0029070	0.0018675	0.00022654	0.00019022	0.00008022
$t = 1$	0.0815126	0.0046046	0.0032665	0.00080163	0.00050418	0.00009929

3.2 Emden-Fowler wave-type equation

Finally, we consider some models of wave-type equations with singular behavior at $x = 0$.

*Example 4.*Consider the following nonlinear time-dependent Emden-Fowler wave-type equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{2}{x}\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + f(x,t)e^{u(x,t)} \quad (33)$$

$$+h(x,t)e^{2u(x,t)}, \quad (34)$$

with the boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(x,t)|_{x=1} = \ln\left(\frac{1}{4+t}\right), \quad 0 < t \leq T, \quad (35)$$

where $f(x,t) = -6t$, and $h(x,t) = 4t^2x^2 - x^4$. The analytical solution of the problem is $u(x,t) = \ln(\frac{1}{4+x^2t})$.

According to the MDM (20), the problem (33)-(35) can be written as:

$$\left.\begin{array}{l} u_0(1,t) = \ln\left(\frac{1}{4+t}\right), \\ u_j(x,t) = -\int\limits_x^1 \frac{1}{s^2}\int\limits_0^s \xi^2\left\{\frac{\partial^2 u_{j-1}}{\partial t^2} + \right. \\ \qquad \left. f(\xi,t)A_{j-1} + h(\xi,t)B_{j-1}\right\} d\xi ds, \quad j = 1,2\ldots \end{array}\right\} \quad (36)$$

In this case, the Adomian polynomials for the nonlinear terms e^u and e^{2u} are given as in the equation (32). Hence, the n-terms truncated series solution is obtained as $\psi_n(x,t) = \sum_{j=0}^n u_j(x,t)$. In order to see the accuracy of proposed method, the maximum error E_n, $n = 1,2,3,4,5,6$ are listed in Table 5. Here, we observe that the maximum error decreases uniformly with an increases of n.

Table 5: The maximum absolute error E_n of Example 4

	E_1	E_2	E_3	E_4	E_5	E_6
$t = 0.5$	0.0161957	0.0036737	0.0009058	0.0008065	0.0002350	0.0001353
$t = 1$	0.0368578	0.0040331	0.0009966	0.0008882	0.0004039	0.0002039

*Example 5.*We finally study the following nonlinear time-dependent Emden-Fowler wave-type equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{3}{x}\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + f(x,t)e^{u(x,t)} \quad (37)$$

$$+h(x,t)e^{2u(x,t)}, \quad (38)$$

with the boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(x,t)|_{x=1} = \ln\left(\frac{1}{5+t}\right), \quad 0 < t \leq T, \quad (39)$$

where $f(x,t) = -8t$, and $h(x,t) = \left(4t^2x^2 - x^4\right)$. The analytical solution of the problem is $u(x,t) = \ln(\frac{1}{5+x^2t})$.

According to the MDM (20), the problem (37)-(39) can be written as:

$$\left.\begin{array}{l} u_0(x,t) = \ln\left(\frac{1}{5+t}\right), \\ u_j(x,t) = -\int\limits_x^1 \frac{1}{s^3}\int\limits_0^s \xi^3\left\{\frac{\partial^2 u_{j-1}}{\partial t^2} + \right. \\ \qquad \left. f(\xi,t)A_{j-1} + h(\xi,t)B_{j-1}\right\} d\xi ds, \quad j = 1,2\ldots \end{array}\right\} \quad (40)$$

In this case, the Adomian polynomials for the nonlinear terms e^u and e^{2u} are given as in the equation (23). Hence, the n-terms truncated series solution is obtained as $\psi_n(x,t) = \sum_{j=0}^n u_j(x,t)$. The maximum absolute error E_n, $n = 1,2,3,4,5,6$ are listed in Table 6 (t=0.5,1). In this case, we observe that the maximum error decreases with an increases of n.

Table 6: The maximum absolute error E_n of Example 5

	E_1	E_2	E_3	E_4	E_5	E_6
$t = 0.5$	0.0953102	0.0176375	0.0115035	0.00853207	0.00536796	0.00140959
$t = 1$	0.182322	0.0381574	0.0121871	0.00996439	0.00771276	0.00566078

4 Conclusion

We have investigated the time-dependent Emden-Fowler-types equations (1) and (3) with the Neumann and Dirichlet boundary conditions (4). We proposed a modified decomposition method, where we utilized all the boundary conditions to derive an integral equation before establishing the recursive scheme. Thus, we developed MDM without any unknown constant while computing the successive solution components. Unlike the most of earlier recursive schemes using ADM (see [23,24]), the MDM avoids solving a sequence of nonlinear algebraic or transcendental equations for unknown constant. This technique is reliable enough to overcome the difficulty of the singular point at $x = 0$. The proposed scheme was tested where convergence was emphasized for each model. Illustrative examples were investigated to confirm the applicability of the proposed method.

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