Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/090306

Half Logistic Inverted Weibull Distribution: Properties and Applications

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Received: 2 Nov. 2021, Revised: 7 May. 2022, Accepted: 23 Jun. 2022
Published online: 1 Sep. 2022

Abstract: Three parameters, half logistic inverted Weibull distribution is developed by compounding type I half logistic-G family with inverted Weibull distribution. The proposed model is unimodal and positively skewed whereas the hazard rate function is increasing and monotonically decreasing or inverted bathtub and reverse J shaped. Explicit expressions of reliability/ survival function, hazard rate function, revised hazard rate function, cumulative hazard rate function, quantile function, asymptotic behavior with mode, skewness, kurtosis, moments, residual life function, moment generating function, Rényi entropy, and q entropy, probability-weighted moment and order statistics are investigated of the proposed model. The value of the parameters with their confidence interval is obtained from the maximum likelihood estimation. To verify the finding of MLEs, we have done a simulation study and observed that MSEs for individual parameters fall to decrease as the size of the sample increases, emphasizing the effectiveness of the method of maximum likelihood estimation. Finally, two real data sets were used to validate the various theoretical findings. We observe that the HLIW distribution provides a better fit as compared to some other known distributions. Hence, the proposed model is an alternative model for survival, reliability, or lifetime time data analyses.

Keywords: Moments, Maximum likelihood estimation, Order statistics, Type I half logistic-G family, Weibull distribution

1 Introduction

In recent years, several methods of generating new distributions from classical distributions are theoretically appealing and applied in various areas. These flexible classical distributions have been widely used for modeling in different fields over the last decade. Lifetime analysis, finance, insurance, economics, engineering, and many other areas have been used, and an extended form of distribution also have been used, which are more adaptable fit in real scenarios [1,2]. The methods (or techniques) to generate a new flexible univariate continuous probability distribution(s), add one or more additional parameters to the baseline distribution. After the addition of one or more shape parameters to the baseline distribution, gives more flexibility, especially while studying the tail properties [3]. In literature, the most popular and valid distributions have been derived to analyze real lifetime data and modeling in different areas. Two parameters, exponentiated Weibull distribution have been extended from the Weibull distribution. This distribution is unimodal, increasing, and non-Weibull hazard shape [4]. Three parameters modified Weibull distribution has been generalization from Weibull distribution which is capable of modeling in bathtub-shaped hazard rate function [5]. A new two-parameter Weibull distribution has been generalization from the Weibull distribution, to form a flexible Weibull extension distribution[6]. Likewise, one most important probability distribution which is used in analyzing lifetime data is an exponentiated inverted Weibull distribution [7]. The exponentiated Weibull distribution developed from the generalization of the two-parameter Weibull distribution. This distribution is used for modeling in the non-monotone hazard rates function [8]. Similarly, an exponentiated modified Weibull extension distribution has four parameters with broader applications in lifetime data and reliability analysis; and three parameters, positively skewed and unimodal distribution formed an exponentiated generalized inverted exponential distribution [9,10]. Furthermore, adding one parameter to the flexible Weibull extension distribution formed a three-parameter exponential flexible Weibull extension

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distribution. This distribution ensures that the hazard function is either increased or bathtub shape [11]. A Lomax exponentiated Weibull distribution is a popular distribution that exhibits all major shapes of distribution such as symmetric, left-skewed, right-skewed, and bimodal [12]. The type II half logistic Kumaraswamy (TIIHLKw) distribution is an alternative distribution by introducing an additional parameter into the existing model in order to improve its ability to fit complex data sets [13]. Likewise, the type II half logistic Weibull distribution (TIIHLW) distribution, a three-parameter distribution has been inspired by the widespread use of the Weibull distribution in practice, as well as the fact that generalization allows for greater flexibility in analyzing positive real-life data [14]. In literature, not only distributions but also many well-known families of distributions have been derived. An exponentiated-G family of distribution has been derived as a monotonic hazard rate function with unimodal, bathtub shapes [15]. A new method was explored for deriving new distribution by adding a parameter to the Marshall-Olkin-G family of distributions [16]. The beta-G family has been proposed, which provides great flexibility in modeling, symmetric heavily-tailed distributions as well as skewed and bimodal distributions [17]. Likewise, some other families of distributions; generalized gamma-G family; generalized beta-G family; log- gamma-G family; exponentiated Kumaraswamy-G family; and exponentiated exponential Poisson-G family have been derived [18,19,20,21,22]. To continue the extension of the new family of distribution, the odd Lindley-G family; the exponentiated transmuted-G family; the Marshal-Olkin Kumaraswamy-G family; the odd log-logistic-G family; and the power Lindley-G family of distribution have been derived [1,23,24,25,26].

Let $f(x)$ be Probability Density Function (PDF) of random variable $X \in [a,b]; -\infty < \text{a} < \text{b} < \infty$ and $W[G(x)]$ be Cumulative Distribution Function (CDF), then satisfied the following properties.

(i)$W[G(x)] \in [a,b]$,

(ii)$W[G(x)]$ is differentiable and monotonically non-decreasing and

(iii)$W[G(x)] \to a$ as $x \to -\infty$, and $G(x)$ is the baseline CDF of type I half logistic-G family [27] of distribution

$$F(x) = \frac{1-\{1-G(x)\}^{\lambda}}{1+\{1-G(x)\}^{\lambda}}; \lambda > 0, \tag{1}$$

where,$\lambda > 0$ is an additional parameter. The PDF of such a family of distribution is

$$f(x) = \frac{2\lambda g(x)\{1-G(x)\}^{\lambda-1}}{\left[1+\{1-G(x)\}^{\lambda}\right]^2}; \lambda > 0. \tag{2}$$

The $g(x)$ is the PDF of the corresponding unimodal baseline distribution. The most popular baseline distribution is an inverted Weibull distribution which is used to analyze real-life data with some monotonic failure rates [7,28]. It is used in the reliability experiment of the life-testing data of any unit, system, or method that performs the duty without failure for a specified time period. It can be also used in modeling real-life data in many devices and variables such as relays, ball bearings, electron tubes, capacitors, germanium transistors, photo-conducting live cells, motors, automotive radiators, regulators, generators, turbine blades, textile fatigue, corrosion resistance, leakage of dry batteries, the return of products after shipment, the marketing life expectancy of drugs, number of turbine blades, corrosion resistance [28]. Therefore, the CDF of inverted Weibull distribution is

$$G(x) = e^{-\alpha x^{-\beta}}; x > 0, \alpha > 0, \beta > 0, \tag{3}$$

where, $\alpha > 0$ is scale parameters and $\beta > 0$ is the shape parameter. The PDF of the corresponding distribution is

$$g(x) = \frac{\alpha\beta e^{-\alpha x^{-\beta}}}{x^{\beta+1}}; x > 0, \alpha > 0, \beta > 0. \tag{4}$$

Our motivation, here is to simplify and extend the three-parameter half logistic inverted Weibull distribution to produce a more flexible model. The new model is referred as the type I half logistic-G family compounded with inverted Weibull distribution having $\alpha > 0$ is scale parameter, and $\beta > 0$ & $\lambda > 0$ are shape parameters. The proposed distribution is unimodal and can be used to reliability and survival data analysis. Its hazard rate function (HRF) has an inverted bathtub shape and a reverse J shape. As a result, the proposed distribution exhibits good statistical behavior. In section 2, we have to present CDF, PDF, reliability/ survival function, hazard rate function, reversed hazard function, and cumulative hazard rate function of this distribution. In section 3, we explore some important statistical properties such as quintile function and median; asymptotic behavior and mode; skewness and kurtosis; moments, residual life function, moment generating function; Rényi and q entropy; probability weighted moments, and order statistics. Similarly, in section 4, parameters are estimated by using Maximum Likelihood (ML), and in section 5, a simulation study is conducted to investigate the performance of the ML estimators. The significance of the new distribution is demonstrated by examining two real data sets: the COVID-19 test positive rate in Nepal and survival data sets. Finally, in section 6, we summarize the proposed distribution with further recommendations.

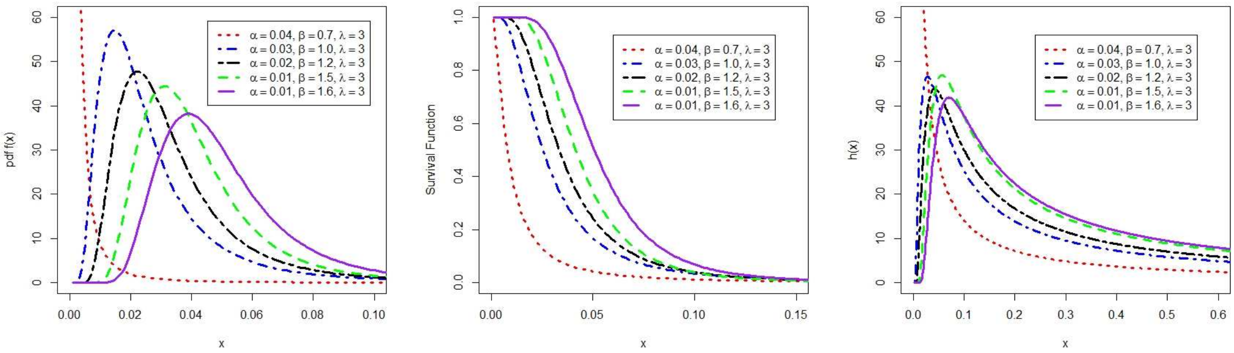

Fig. 1: Plot of density (left panel), survival function (center panel), and hazard rate function (right panel) for different values of α, β, and λ.

2 Model Analysis

For proposed model, the CDF of the equation (3) is compounded in equation (1), then the CDF of the half logistic inverted Weibull distribution is

$$F(x) = \frac{1 - \left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda}}{1 + \left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda}}; x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{5}$$

The PDF of the proposed model is obtained by substituting the values of (3) and (4) in equation (2), we get

$$f(x) = \frac{2\alpha\beta\lambda \left(e^{-\alpha x^{-\beta}}\right) \left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda - 1}}{x^{\beta+1} \left[1 + \left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda}\right]^{2}}; x > 0, \alpha > 0, \beta > 0, \lambda > 0, \tag{6}$$

where, $\alpha > 0$, is scale parameter and $\beta > 0$ & $\lambda > 0$ are shape parameters. The shape of the proposed model is unimodal and positively skewed [Fig 1 (left panel), Table 1].

Likewise, the survival function simply indicates the probability that the event of interest has not yet occurred by time x. Mathematically, it is defined as, $R(x) = 1 - F(x)$. Hence, the survival function of the proposed model is

$$R(x) = \frac{2\left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda}}{\left[1 + \left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda}\right]}; x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{7}$$

The shape of the reliability function for different values of the parameters of the proposed model is presented in figure [Fig 1 (center panel)].

Hazard rate function is a conditional density given that the event has not yet occurred before time x. It provides the important characteristic of the proposed distribution. Therefore, the hazard function of the proposed model is

$$h(x) = \frac{\alpha\beta\lambda}{x^{\beta+1} \left(e^{\alpha x^{-\beta}} - 1\right) \left[1 + \left\{1 - e^{-\alpha x^{-\beta}}\right\}^{\lambda}\right]}; x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{8}$$

Initially, the hazard rate function curve is increased until it reaches its maximum point, after that, it is monotonically decreased. It has an inverted bathtub shape and a reverse J shape [Fig. 1 (right panel)].

Similarly, the reverse-hazard function of the proposed model is

$$r(x)=\frac{2\alpha\beta\lambda e^{-\alpha x^{-\beta}}\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda-1}}{x^{\beta+1}\left[1-\left\{1-e^{-\alpha x^{-\beta}}\right\}^{2\lambda}\right]}. \tag{9}$$

Likewise,the cumulative hazard function is not a probability function but it measures the risk. The higher the value of the risk, the greater the risk of failure over time x. Mathematically, the cumulative hazard rate function is defined as

$$H(x)=-\ell n2-\lambda\ell n\left\{1-e^{-\alpha x^{-\beta}}\right\}+\ell n\left[1+\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda}\right]. \tag{10}$$

3 Statistical Properties

The proposed distribution has been derived from the generalized binomial series.

$$(1+z)^{-n}=\sum_{i=0}^{\infty}(-1)^i\binom{n+i-1}{i}z^i,\text{ and }(1-z)^n=\sum_{j=0}^{\infty}(-1)^j\binom{n}{j}z^j\ , \tag{11}$$

where, $|z|<1$, $n>0$. Now we have used the binomial theorem in equation (6), then PDF of the proposed model is;

$$f(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}n_{ij}x^{-\beta-1}\left(e^{-\alpha x^{-\beta}}\right)^{j+1}, \tag{12}$$

where,$n_{ij}=(-1)^{i+j}2\alpha\beta\lambda(i+1)\binom{\lambda(i+1)-1}{j}$.
Similarly,

$$[f(x)]^{\delta}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}n^*_{ij}x^{-\delta(\beta+1)}\left(e^{-\alpha x^{-\beta}}\right)^{\delta+j}, \tag{13}$$

where, $n^*_{ij}=(-1)^{i+j}(2\alpha\beta\lambda)^{\delta}\binom{2\delta+i-1}{i}\binom{\lambda(i+\delta)-\delta}{j}$.
Likewise,

$$[F(x)]^s=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{ijk}\left(e^{-\alpha x^{-\beta}}\right)^k, \tag{14}$$

where, $S_{ijk}=(-1)^{i+j+k}\binom{s+i-1}{i}\binom{s}{j}\binom{\lambda(i+j)}{k}$.

3.1 Quantile Function

Quantile function is used to study the theoretical aspects of the probability distribution. Mathematically, the quantile function is $Q(u)=F^{-1}(u)$. The corresponding quantile function of the proposed model is

$$Q(u)=\left[-\frac{1}{\alpha}\ell n\left\{1-\left(\frac{1-u}{1+u}\right)^{\frac{1}{\lambda}}\right\}\right]^{-\frac{1}{\beta}},\quad 0<u<1. \tag{15}$$

If we substitute $u=0.5$ in equation (15), then it becomes median of the proposed distribution, which is

$$\text{Median}=\left[-\frac{1}{\alpha}\ell n\left\{1-\left(\frac{1}{3}\right)^{\frac{1}{\lambda}}\right\}\right]^{-\frac{1}{\beta}}.$$

3.2 Asymptotic Behavior

To examine the asymptotic behavior, we have to check, $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x)$.

$$\lim_{x\to 0} \frac{2\alpha\beta\lambda e^{-\alpha x^{-\beta}}\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda-1}}{x^{\beta+1}\left[1+\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda}\right]^2} = \lim_{x\to\infty} \frac{2\alpha\beta\lambda e^{-\alpha x^{-\beta}}\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda-1}}{x^{\beta+1}\left[1+\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda}\right]^2} = 0.$$

The result shows that both limits are existing, hence the proposed distribution has conformed to unimodal distribution. Since, $f(x) > 0$ and $\frac{df(x)}{dx} = 0$ then equation (6) becomes

$$\frac{\alpha\beta}{x^{\beta+1}} - \frac{\alpha\beta\lambda(\lambda-1)}{x^{\beta+1}\left(e^{\alpha x^{-\beta}}-1\right)} - \frac{\beta+1}{x} - \frac{2\alpha\beta\lambda e^{-\alpha x^{-\beta}}\left\{1-e^{-\alpha x^{-\beta}}\right\}^{\lambda-1}}{x^{\beta+1}\left[1+\left\{1-e^{-\alpha x^{-\beta}}\right\}\right]} = 0. \tag{16}$$

The equation (16) is a non-linear equation which cannot solve analytically. This equation solves numerically by using the Newton-Raphson method.

3.3 Skewness and Kurtosis

The skewness and kurtosis are used in statistical analysis to measures the characteristics of a distribution. Bowley's skewness is based on quartiles [29], and it takes the form

$$S_k = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Moors' kurtosis[29,30] is based on octiles and it can be calculated as

$$K_u = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)};$$

where, Q (.) is the quantile function defined in equation (15).

The 100 random samples are generated from (15) at the initial value, $\alpha = 0.5$, $\beta = 1.0$ and $\lambda = 1.5$. After that, mean, median, mode, skewness and kurtosis have been calculated from sample data. Initially, the mean and median have increased and reached at maximum point, $\alpha = 0.6$, $\beta = 1.2$ and $\lambda = 3.5$ then values are gradually decreased. Furthermore, the relation between mean and median is mean > median. Similarly, modal value has maximum at, $\alpha = 0.7$, $\beta = 1.3$ and $\lambda = 4.0$. Likewise, we observed the different value of skewness and kurtosis. From the above finding, we concluded that proposed model is unimodal, positively skewed and non-normal (non-mesokurtic) [Fig. 1 (left panel), Table1].

Table 1: The mean, median, mode, skewness and kurtosis of proposed model

α	β	λ	Mean	Median	Mode	Skewness	Kurtosis
0.1	0.7	1.0	0.340010	0.198871	0.296642	3.679501	20.13055
0.2	0.8	1.5	0.508991	0.306783	0.483653	2.112162	8.176252
0.3	0.9	2.0	0.631953	0.437595	0.716345	0.955391	2.754898
0.4	1.0	2.5	0.710683	0.523659	0.994572	0.532455	1.747981
0.5	1.1	3.0	0.746695	0.697833	1.286998	0.201203	1.512289
0.6	1.2	3.5	0.751521	0.705575	1.300237	0.135954	1.426901
0.7	1.3	4.0	0.723447	0.666944	1.678890	0.215445	1.441377
0.8	1.4	4.5	0.693711	0.479678	1.678099	0.333205	1.472761
0.9	1.5	5.0	0.659935	0.430273	1.537579	0.437819	1.645351
1.0	1.6	5.5	0.623957	0.214007	1.296174	0.546914	1.675431

3.4 Moments

Moments are important for any statistical analysis, particularly in application areas. Let, $X \sim HLIW(\alpha,\beta,\lambda)$, then many important features and characteristics (measure of center tendency, dispersion, skewness and kurtosis) of proposed distribution can be obtained by using ordinary moments. The r^{th} raw moment of proposed distribution defined as

$$\mu_r' = E(X^r) = \int_o^\infty x^r f(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n_{ij}\int_0^\infty x^{r-\beta-1}\left(e^{-\alpha x^{-\beta}}\right)^{j+1}dx. \tag{17}$$

After the integration of (17), the r^{th}raw moment of the proposed model is

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{n_{ij}\Gamma\left(1-\frac{r}{\beta}\right)}{\beta\left[\alpha(j+1)\right]^{\left(1-\frac{r}{\beta}\right)}};\quad \beta > r.$$

Similarly, incomplete moments play an important role in measuring inequality. Hence, lower incomplete moments, say, $\varphi_s(t)$ is given by

$$\varphi_s(t) = \int_0^t x^s f(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n_{ij}\int_0^t x^{s-\beta-1}\left(e^{-\alpha x^{-\beta}}\right)^{j+1}dx. \tag{18}$$

Let, $\frac{1}{x^\beta} = y$ then equation (18) convert in the form of $\Gamma(s,t) = \int_t^\infty x^{s-1}e^{-x}dx$. Now, we apply this relation in equation (18), the lower incomplete moment is

$$\varphi_s(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{n_{ij}\Gamma\left[\left(1-\frac{s}{\beta}\right),\alpha(j+1)t^{-\beta}\right]}{\beta\left[\alpha(j+1)\right]^{\left(1-\frac{s}{\beta}\right)}};\quad \beta > s.$$

Likewise, conditional moment$\tau_s(t) = E\left(X^s/X > t\right) = 1/R(x)\int_t^\infty x^s f(x)dx$. We used the value of $f(x)$ from equation (12) and substitute,$1/x^\beta = y$ then conditional moment convert in the form of$\gamma(s,t) = \int_0^t x^{s-1}e^{-x}dx$.Therefore, the conditional moment becomes

$$\tau_s(t) = \frac{1}{R(x)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{n_{ij}\gamma\left[\left(1-\frac{s}{\beta}\right),\alpha(j+1)t^{-\beta}\right]}{\beta\left[\alpha(j+1)\right]^{\left(1-\frac{s}{\beta}\right)}};\quad \beta > s.$$

3.5 Probability Weighted Moment (PWM)

The weighted probability moment can be used to estimate the unknown parameter when maximum likelihood estimation is unavailable or difficult to apply. The probability weighted moment is superior to the existing moment for estimation of parameter. It can be compared with ordinary moment. It is defined as

$$\tau_{r,s} = E\left[X^r F(x)^s\right] = \int_{-\infty}^{\infty} x^r f(x)F(x)^s dx. \tag{19}$$

By substituting equations (12) *[where we use summation sign run from l and m instead of i and j in equation* (12)*]*, and (14) in equation (19) and integrating the equation (19), then we get the finding of PWM

$$\tau_{r,s} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} n_{lm}s_{ijk}\frac{\Gamma\left(1-\frac{r}{\beta}\right)}{\beta\left[\alpha(j+k+1)\right]^{\left(1-\frac{r}{\beta}\right)}};\beta > r.$$

3.6 Moment Generating Function (MGF)

The moment generating function of a random variable X provides the alternative route to analyze the characteristics of distribution. Its results can compare with directly the finding of PDF and CDF of X. The MGF is defined as

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(x^r). \tag{20}$$

After used the finding of (17) in equation (20), the MGF of proposed model is

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r n_{ij}\Gamma\left(1-\frac{r}{\beta}\right)}{r!\beta\left[\alpha(j+1)\right]^{\left(1-\frac{r}{\beta}\right)}};\quad \beta>r.$$

3.7 Residual life function

The residual life function is the expected remaining life $(X-t)^n$ given that the item survived at time t. The n^{th} moment of the residual life of X is defined as

$$m_n(t)=\frac{1}{R(t)}\int_t^{\infty}(x-t)^n f(x)dx. \tag{21}$$

We apply the expansion, $(x-t)^n=\sum_{r=0}^{n}(-1)^r\binom{n}{r}x^{n-r}t^r$into equation (21), we get

$$m_n(t)=\frac{1}{R(t)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}(-t)^r n_{ij}\binom{n}{r}\int_t^{\infty}x^{n-r-\beta-1}\left(e^{-\alpha x^{-\beta}}\right)^{j+1}dx. \tag{22}$$

Let, $\frac{1}{x^\beta}=y$ then equation (22) convert in the form of $\gamma(s,t)=\int_0^t x^{s-1}e^{-x}dx$. We apply this relation in (22), then n^{th} moment of the residual life function is

$$m_n(t)=\frac{1}{\beta R(t)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{n}(-t)^r\binom{n}{r}n_{ij}\frac{\gamma\left[\left(1-\frac{n-r}{\beta}\right),\alpha(j+1)t^{-\beta}\right]}{\left[\alpha(j+1)\right]^{\left(1-\frac{n-r}{\beta}\right)}};\beta>n-r.$$

Likewise, the mean residual life is the average outstanding life,$X-x$ given that the item has survived to time t. Thus, the expected additional lifetime given that a component has survived until time t is called the mean residual life. It is defined as

$$m(t)=E\left[X-x/X>t\right]=\frac{1}{R(t)}\int_t^{\infty}xf(x)dx-x. \tag{23}$$

We apply the value of $f(x)$ in equation (23) from (12), then it becomes

$$m(t)=E\left[X-x/X>t\right]=\frac{1}{R(t)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}n_{ij}\int_t^{\infty}x^{-\beta-1}\left(e^{-\alpha x^{-\beta}}\right)^{j+1}dx-x. \tag{24}$$

Let,$\frac{1}{x^\beta}=y$, then (24) convert in the form of $\gamma(s,t)=\int_0^t x^{s-1}e^{-x}dx$. Therefore, after solving the equation (24), we get the mean residual life function, which is

$$m(t)=\frac{1}{\beta R(t)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}n_{ij}\frac{\gamma\left[\left(1-\frac{1}{\beta}\right),\alpha(j+1)t^{-\beta}\right]}{\left[\alpha(j+1)\right]^{\left(1-\frac{1}{\beta}\right)}};\beta>1.$$

3.8 Rényi and q-entropies

The entropy of random variable X with density function $f(x)$is measured the variation of uncertainty or randomness of a system. A large value of entropy indicates greater uncertainty in the data. The theory of entropy has been used in many fields such as physics, engineering, economics, and other subjects [31]. It is used in statistics for hypotheses testing in parametric models and lifetime distribution [29]. The Rényi entropy is defined as

$$I_\delta(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}\left[f(x)\right]^\delta dx,\quad \delta>0\&\delta\neq 1. \tag{25}$$

By applying the relation (13) in equation (25) and integration of (25), then we get the result of Rényi entropy

$$I_\delta(X)=\frac{1}{1-\delta}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}n_{ij}^*\frac{\Gamma\left(\frac{\delta(\beta+1)-1}{\beta}\right)}{\beta\left[\alpha(\delta+j)\right]^{\left(\frac{\delta(\beta+1)-1}{\beta}\right)}}\right].$$

Similarly, the q-entropy is defined as

$$H_q(X) = \frac{1}{1-q}\log\left(1-\int_{-\infty}^{\infty}[f(x)]^q\,dx\right);\;\; q>0\&q\neq 1. \tag{26}$$

The q- entropy of proposed distribution is obtained by substitute the result of (25) in equation (26), only when replacing δ by q, we get the result of q- entropy

$$H_q(X) = \frac{1}{1-q}\log\left[1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n^*_{ij}\frac{\Gamma\left(\frac{q(\beta+1)-1}{\beta}\right)}{\beta\left[\alpha(q+j)\right]^{\left(\frac{q(\beta+1)-1}{\beta}\right)}}\right].$$

3.9 Order Statistics

Order statistics play a prominent role in real-life applications involving data related to life testing studies. It is required in many fields such as, climatology, engineering, industry and many others [1]. Let, $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ denotes the order statistics of random sample $X_1, X_2, \ldots, X_n$ from the proposed distribution with CDF and PDF of$X$. The PDF of r^{th} order statistics is $X_{(r)}$[32] is defined as

$$f_{X_{(r)}}(x_r) = \frac{f(x_{(r)})}{B(r,n-r+1)}\sum_{\upsilon=0}^{n-r}(-1)^{\upsilon}\binom{n-r}{\upsilon}\left[F(x_{(r)})\right]^{\upsilon+r-1}, \tag{27}$$

where, $B(.,.)$ is the beta function. We substitute the value of equation (12) and (14) in equation (27) and replacing s by$(\upsilon+r-1)$, then equation (27) becomes a PDF of r^{th} order statistics

$$f_{X_{(r)}}(x_{(r)}) = \frac{1}{B(r,n-r+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\upsilon=0}^{n-r}\eta^* x_{(r)}^{-(\beta+1)}\left(e^{-\alpha x_{(r)}^{-\beta}}\right)^{j+k+1}, \tag{28}$$

where, $\eta^* = (-1)^{\upsilon}\binom{n-r}{\upsilon} n_{ij}\, s_{ijk}$.

The k^{th} moment of r^{th} order statistics is calculated as

$$E(X^k_{(r)}) = \int_0^{\infty} x^k_{(r)}\; f_{X_{(r)}}(x_{(r)})\,dx_{(r)}. \tag{29}$$

Substitute the result of equation (28) in equation (29). After integration of equation (29), we get the result of the k^{th} moment of r^{th} order statistics is

$$E(X^k_{(r)}) = \frac{1}{B(r,n-r+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\upsilon=0}^{n-r}\eta^*\frac{\Gamma\left(1-\frac{k}{\beta}\right)}{\beta\left[\alpha(j+k+1)\right]^{\left(1-\frac{k}{\beta}\right)}};\beta>k.$$

4 Maximum Likelihood Estimation

The maximum likelihood estimation is a technique to estimate the unknown's parameters (α,β,λ) of proposed distribution based on $\underset{\sim}{x} = (x_1,\ldots,x_n)$, observed sample values. To calculate the unknown parameter, we maximize the likelihood function or equivalent to log-likelihood function $\ell(x;\alpha,\beta,\lambda)$. Now, log-likelihood function of the parameter vector $\ell(\underset{\sim}{\delta})$; where,$\underset{\sim}{\delta} = (\alpha,\beta,\lambda)^T$ or only ℓ is

$$\begin{aligned}\ell n(\ell) &= n\ell n(2\alpha\beta\lambda) - \alpha\Sigma_{i=1}^{n} x_i^{-\beta} + (\lambda-1)\Sigma_{i=1}^{n}\ell n\left(1-e^{-\alpha x_i^{-\beta}}\right)\\ &-(\beta+1)\Sigma_{i=1}^{n}\ell n(x_i) - 2\Sigma_{i=1}^{n}\ln\left[1+\left\{1-e^{-\alpha x_i^{-\beta}}\right\}^{\lambda}\right]\end{aligned} \tag{30}$$

We assume that the following standard regularity condition of likelihood function,$\ell(\underset{\sim}{\delta})$ are holds:

1.The random variable x is associated with the distribution and it does not depend on unknown parameters.
2.The parameter space of x, say Ψ is often & $\ell(\underset{\sim}{\delta})$ has a global maximum inΨ.
3.For all x; the fourth order likelihood derivatives with respect to model parameter exist and are continuous in an open subset of Ψ that contains the true parameters.
4.The expected information matrix is positive and finite.
5.The absolute value of the third order log-likelihood derivative with respect to the parameters are bounded by the expected finite function of x.

The parameters are calculated from maximum likelihood estimation technique. The parameters obtain by partial differentiate with respect to corresponding parameters and equating to zero, we have

Let us suppose, $\Psi_i = e^{\alpha x_i^{-\beta}} - 1,\ \ \phi_i = \left[1 - e^{-\alpha x_i^{-\beta}}\right]^{\lambda}$ and $\omega_i = \left[1 + \left\{1 - e^{-\alpha x_i^{-\beta}}\right\}^{\lambda}\right]$

$$\frac{\partial \ell n(\ell)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{1}{x_i^{\beta}}\right) + \sum_{i=1}^{n}\left(\frac{\lambda-1}{x_i^{\beta}\Psi_i}\right) - 2\lambda\sum_{i=1}^{n}\left(\frac{\phi_i}{x_i^{\beta}\Psi_i\omega_i}\right) = 0. \tag{31}$$

$$\begin{aligned}\frac{\partial \ell n(\ell)}{\partial \beta} &= \frac{n}{\beta} + \alpha\textstyle\sum_{i=1}^{n}\left(\ell n(x_i)x_i^{-\beta}\right) - \alpha(\lambda-1)\sum_{i=1}^{n}\left(\frac{\ell n(x_i)}{x_i^{\beta}\Psi_i}\right)\\ &- \textstyle\sum_{i=1}^{n}\ell n(x_i) + 2\alpha\lambda\sum_{i=1}^{n}\left(\frac{\ell n(x_i)\phi_i}{x_i^{\beta}\Psi_i\omega_i}\right) = 0.\end{aligned} \tag{32}$$

$$\frac{\partial \ell n(\ell)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\ell n\left(1 - e^{-\alpha x_i^{-\beta}}\right) - 2\sum_{i=1}^{n}\left\{\frac{\phi_i \ell n\left(1 - e^{-\alpha x_i^{-\beta}}\right)}{\omega_i}\right\} = 0. \tag{33}$$

To estimate unknown parameters α, β and $\lambda$by solving the non-linear equation (31), (32), and (33). It is difficult to solve analytically, therefore we have applied Newton-Raphson's iterative technique by using the *optim ()* function in R software [33,34].

4.1 Asymptotic Distribution

Let us denote the parameter vector $\underset{\sim}{\delta} = (\alpha,\beta,\lambda)^T$ and corresponding MLE of $\underset{\sim}{\delta}$ is $\hat{\underset{\sim}{\delta}} = (\hat{\alpha},\hat{\beta},\hat{\lambda})^T$, then asymptotic theory of normality result of parameter is$\left(\hat{\underset{\sim}{\delta}} - \underset{\sim}{\delta}\right) \to N_3\left[0,(I(\underset{\sim}{\delta}))^{-1}\right]$; where, $I(\underset{\sim}{\delta})$ is Fisher's information matrix. In practice, it is difficult to compute. Therefore, we calculate observed information matrix$O(\hat{\underset{\sim}{\delta}})$; as an estimate of Fisher's information matrix$I(\underset{\sim}{\delta})$; from Newton Raphson algorithm. The observed information matrix $O(\hat{\underset{\sim}{\delta}})$is

$$O(\hat{\underset{\sim}{\delta}}) = -\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\,\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\,\partial\lambda} \\ \frac{\partial^2\ell}{\partial\alpha\,\partial\beta} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\,\partial\lambda} \\ \frac{\partial^2\ell}{\partial\alpha\,\partial\lambda} & \frac{\partial^2\ell}{\partial\beta\,\partial\lambda} & \frac{\partial^2\ell}{\partial\lambda^2}\end{pmatrix} = -H(\underset{\sim}{\delta})_{|\underset{\sim}{\delta}=\hat{\underset{\sim}{\delta}}}. \tag{34}$$

From equation (34), we get Hessian matrix and inverse of Hessian matrix is variance-covariance matrix, which is

$$\left(-H(\underset{\sim}{\delta})_{|\underset{\sim}{\delta}=\hat{\underset{\sim}{\delta}}}\right)^{-1} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\alpha},\hat{\beta}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda}) \\ cov(\hat{\alpha},\hat{\lambda}) & cov(\hat{\beta},\hat{\lambda}) & var(\hat{\lambda})\end{pmatrix}. \tag{35}$$

Finally, from the asymptotic normality of MLE's, approximate $100(1-\gamma)$ % confidence interval for α, β and λ can be constructed as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{var(\hat{\alpha})}\ ,\ \hat{\beta} \pm z_{\gamma/2}\sqrt{var(\hat{\beta})} \text{ and } \hat{\lambda} \pm z_{\gamma/2}\sqrt{var(\hat{\lambda})}$$

where, $Z_{\gamma/2}$ is the upper percentile of standard normal variate.

5 Data Analysis

Data analysis is a technique to provide a valid conclusion based on some facts or information. We have used two different techniques in data analysis, (i) simulation study and (ii) real data analysis

5.1 Simulation Study

It is difficult to validation of theoretical performance of maximum likelihood estimators of new probability distribution. Therefore, Monte Carlo simulation study is used to compare the performance of estimation methods mainly with respect to their average bias and mean square error (MSEs) for difference sample sizes.

1.For the estimation purpose, 10000 random samples of size 50, 100, 200, and 500 are generated from the R software of the proposed model.

2.Compute the MLEs for 10000 samples, say $\left(\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i\right)$;for i=1, 2, ..., 10000.

3.Compute the average MLEs, average bias, and mean sum of square (MSE) of different sample sizes. The average bias and MSE of $\underset{\sim}{\delta} = (\alpha, \beta, \lambda)$ are

$$bias_{\underset{\sim}{\delta}}(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{\underset{\sim}{\delta}}_i - \underset{\sim}{\delta}\right)$$

$$MSE_{\underset{\sim}{\delta}}(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{\underset{\sim}{\delta}}_i - \underset{\sim}{\delta}\right)^2.$$

Setting the initial parameter values of proposed distribution;$(\alpha = 1.0, \beta = 1.5$ and $\lambda = 2.0)$ for sample sizes, n=50, 100, 200, 500. We observed that average bias and MSEs for individual parameters fall to zero when the sample size increases. Similarly, we repeated same sample sizes for different parameters values; $(\alpha = 1.0, \beta = 2.0,and\lambda = 1.5)$, $(\alpha = 1.5, \beta = 2.0$ and $\lambda = 2.5)$ &$(\alpha = 2.0, \beta = 1.0$ and$\lambda = 1.5)$. The value of MSEs and average bias decreases to zero as the sample size increases. Therefore, we conclude that these results agree with the asymptotic theory. This emphasizes the effectiveness of the method of maximum likelihood in estimating the parameter of the proposed distribution [Table 2].

Table 2:MLE, Average bias and MSEs of HLIW distribution with different parameter's value

Parameter values		$\alpha = 1.0, \beta = 1.5, \lambda = 2.0$			$\alpha = 1.5, \beta = 2.0, \lambda = 2.5$		
Parameters	n	MLE	AB*	MSEs	MLE	AB*	MSEs
α	50	0.9061	-0.0939	0.1234	1.4838	-0.0162	0.2545
	100	0.7863	-0.2137	0.1154	1.1864	-0.3135	0.2060
	200	0.7645	-0.2355	0.1107	0.9990	-0.5009	0.1859
	500	0.7145	-0.2855	0.0917	0.9859	-0.5141	0.1190
β	50	1.7735	0.2735	0.3708	2.6633	0.6633	1.3143
	100	1.7623	0.2623	0.3373	2.6285	0.6285	0.7929
	200	1.7355	0.2355	0.2919	2.6756	0.6156	0.6711
	500	1.6234	0.1234	0.2733	2.6107	0.6107	0.6662
λ	50	1.6000	-0.3999	0.9653	1.9898	-0.5102	1.2020
	100	1.2754	-0.7246	0.6569	1.5208	-0.9791	1.1339
	200	0.9982	-1.0017	0.5714	1.0714	-1.4285	1.1121
	500	0.6443	-1.3556	0.5421	1.0056	-1.4944	0.7645
Parameter values		$\alpha = 1.0, \beta = 2.0, \lambda = 1.5$			$\alpha = 2.0, \beta = 1.0, \lambda = 1.5$		
α	50	1.4422	0.4112	0.3781	2.1089	0.1088	0.7834
	100	1.2836	0.2836	0.2003	1.9885	-0.0114	0.6679
	200	0.9585	-0.0414	0.1532	1.4173	-0.5826	0.4154
	500	1.1398	-0.1398	0.1110	1.4120	-0.5879	0.3466
β	50	2.1587	0.1587	0.5475	2.3130	1.3130	2.7696
	100	2.0931	0.0931	0.2406	2.1459	1.1459	2.2067
	200	2.0636	0.0360	0.1624	2.0889	1.0889	1.3824
	500	1.9149	-0.0851	0.0365	2.0618	1.0618	1.1328
λ	50	1.8100	0.3100	1.4709	0.5705	-0.9294	0.9819
	100	1.4551	-0.0449	0.1053	0.5305	-0.9695	0.8946
	200	1.2112	-0.2888	0.0943	0.4907	-1.0092	0.4317
	500	1.5016	0.0016	0.0253	0.3486	-1.1514	0.0288

**Average Bias*

5.2 Real Data Analysis

In literature, the discrete Type-II half-logistics exponential (DTIIHLE) distribution is proposed to apply to COVID-19 new cases in Pakistan and Saudi Arabia [35]. Similarly, we proposed a HLIW distribution for the application of Polymerase Chain Reaction (PCR) test positive rate (in percent) in Nepal during the first wave of COVID-19 [36] and survival time (in days) of 72 guinea pigs infected with virulent tubercular bacilli [45].

5.2.1 Test Positive Rate of PCR in Nepal

COVID-19 is a worldwide pandemic of coronavirus disease in 2019 including Nepal. The first COVID case was confirmed on 23 January 2020 and the first death was on 14 May in Nepal. Due to the COVID-19 pandemic, the government has emphasized a nationwide lockdown from March 24, 2020, to July 21, 2020. Following that, the government concentrated its efforts on PCR tests and other health-related initiatives. Every day, the ministry of health and population has provided data regarding COVID-19 issues, such as test positive rate, the number of deaths, the number of infected, and many others. During the research period, researchers collected the data on a daily basis from 23 January 2019 to 24 December 2019 all over the country. Every day, the ministry of health and population of Nepal (MOHP) reported data [36]. Among these data, we selected test positive rate data to assess the validity of the proposed model. Data were collected in 332 days during the study period, and the summary finding of the test positive rate is shown in the following [Table 3].

Table 3: Descriptive statistic of test positive rate of COVID -19

Reporting Days	Minimum	Q_1	Mean	Median	Q_3	Maximum
332	0.00	0.67	5.83	5.42	8.52	33.33

In (n=332) days, the maximum positive test rate was 33.33% and the minimum was zero. Every day, an average of 5.83% of the test positive rate was reported. To validate the proposed model, we chose 116 (34.93 %) samples where the test positive rate was less than 5%, as reported by MOHP[36]. The sample data have been presented as follows: (4.17, 4.17, 2.94, 2.94, 2.86, 0.48, 0.47, 0.46, 0.46, 0.46, 0.45,0.45, 0.45, 0.41, 0.24, 0.23, 0.23, 0.23, 0.22, 0.22, 0.22, 0.22, 0.21, 0.21, 0.20, 0.20, 0.19, 0.18, 0.17, 0.33, 0.33, 0.44, 0.40, 0.50, 0.57, 0.55, 0.50, 0.47, 0.44, 0.51, 0.47, 0.59, 0.55,0.48, 0.42, 0.38, 0.31, 0.26, 0.20, 0.23, 0.25, 0.25, 0.23, 0.22, 0.40, 0.39, 0.38, 0.37, 0.48, 0.50 ,0.52, 0.52, 0.51, 0.52, 0.50, 0.50, 0.49, 0.47, 0.47, 0.45, 0.56, 0.55, 0.59, 0.70, 0.70, 0.66, 0.67, 0.65, 0.75, 1.14 ,1.14, 1.10, 1.12, 1.05, 1.05, 1.22, 1.22, 1.20,1.18, 1.21, 1.27, 1.24, 1.32, 1.41, 1.52, 1.71, 1.89 ,2.10 ,2.26, 2.52, 2.79, 2.87, 3.13, 3.30, 3.50, 3.58, 3.73, 3.84, 3.94, 3.98, 4.15, 4.19, 4.32, 4.48, 4.59, 4.79). We used exploratory data analysis on the collected sample and displayed a summary of our findings [Fig. 2].

Likewise, we have estimated the value of parameters with the standard error by using the method of maximum likelihood estimation which maximize the log-likelihood function (30) directly by using *optim ()* function in the "CG" method from R software [34]. The estimated parameter $\hat{\alpha}$ is statistically insignificant (p-value =0.4923) whereas, $\hat{\beta}$ and $\hat{\lambda}$ are statistically significant (p-value=0.0088) and (p-value =0.0409) respectively [Table 4].

Table 4: Estimated value with SE and p-value of parameters

Parameters	MLE	SE	t-value	p-value
$\hat{\alpha}$	0.06356	0.09257	0.687	0.4923
$\hat{\beta}$	2.11362	0.80775	2.617	0.0088
$\hat{\lambda}$	0.60120	0.29418	2.044	0.0409

Here, one of the assumptions of the MLE is that the expected information matrix is positive and finite. The expected information matrix is used to create the variance-covariance matrix. If there exists a variance-covariance matrix, it is clear that the above assumption of MLE holds. Finally, the asymptotic normality results of MLE is $\left(\hat{\underset{\sim}{\delta}} - \underset{\sim}{\delta}\right) \to N_3\left[0, (I(\underset{\sim}{\delta}))^{-1}\right]$ valid. Hence, the variance-covariance matrix is

$$\left(-H(\delta)_{|\underset{\sim}{\delta}=\hat{\underset{\sim}{\delta}}}\right)^{-1} = \begin{matrix} \\ \alpha \\ \beta \\ \lambda \end{matrix} \begin{matrix} \qquad\alpha \qquad\qquad \beta \qquad\qquad \lambda \\ \begin{pmatrix} 0.008570 & -0.074234 & 0.026883 \\ -0.074234 & 0.652465 & -0.233182 \\ 0.026883 & -0.233182 & 0.086541 \end{pmatrix} \end{matrix}.$$

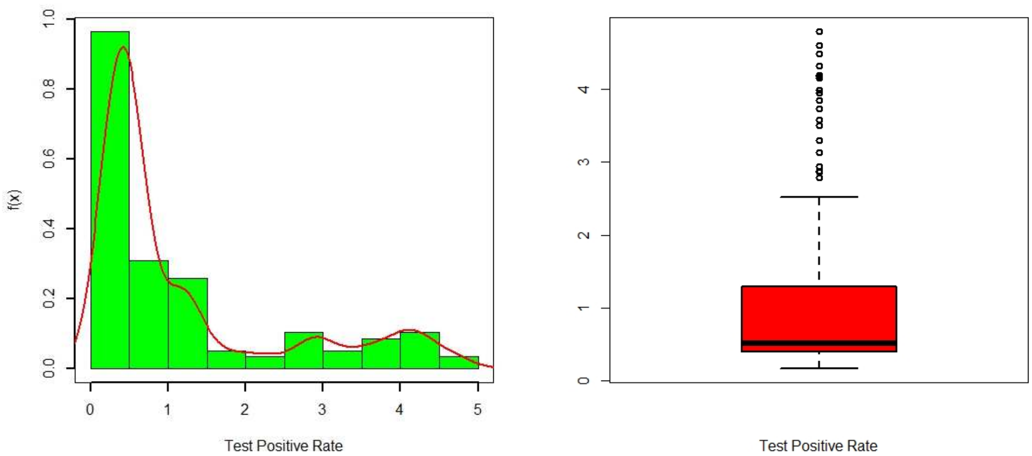

Fig. 2: Histogram and density plot (left panel) and boxplot (right panel) along with the data points.

Moreover, after estimate the parameter value, we have to check the model validation by Kolmogorov-Smirnov (KS) test, Anderson-Darling (A^2) and Cramér-Von Mises (W) statistics. The calculated value of each statistics are 0.08276 (p-value= 0.2041), 1.9029 (p-value= 0.104) and 0.29271 (p-value= 0.1417) respectively. The p-value of each statistics supported the null hypothesis, which indicating that the proposed model satisfied the goodness of fit and it is suitable for further data analysis.

Further, we have checked our model by inspecting the probability-probability (P-P) and quantile-quantile (Q-Q) plots. P-P plot uses cumulative distribution function of theoretical distribution verses empirical cumulative distribution function. P-P plot depicts the points;$\left(F(x_{(i)}), F(x_{(i)};\hat{\delta})\right); i = 1,2,...,n$; where, $\hat{\delta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and $x_{(i)}$ are order statistics. $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X \leq x)$ is an empirical distribution function and $I(.)$ is the indicator function. Similarly, Q-Q plots are the plots of sample order statistics versus the expected quantile of the proposed distribution. The Q-Q plot depicts the points: $\left(x_{(i)}, F^{-1}\left(\frac{i}{n+1};\hat{\delta}\right)\right); i = 1,2,....,n$. The Q-Q plot is more sensitive toward the tail while the P-P plot is more sensitive toward the center value. The P-P and Q-Q plots of the proposed model are well-fitting in the given data set [Fig.3].
Similarly, the proposed model is valid with the graphical representation of empirical cumulative distribution versus the theoretical cumulative distribution of the proposed model. This criterion is used to verify that the individual CDF is closely resembles the empirical distribution of the data set [Fig.4 (left panel)].

Likewise, for validation of the hazard rate function, we exhibited the *Total Time on Test* plot. TTT plot is an important graphical method to check whether or not our data set can be applied to a particular model. The empirical version of the TTT plot is$T\,(r/n) = \sum_{i=1}^{r} y_{i:n} + (n-r)y_{r:n} \big/ \sum_{i=1}^{r} y_{i:n}$; where,$(y_{r:n} = 1,2,...,n)$and $(y_{i:n} = 1,2,...,n)$ are the order statistics of the sample [37]. Hence, we observed that TTT plot has a bathtub shape, indicating that the hazard rate shape of the proposed distribution is unimodal [Fig.4, (right panel)]. Finally, we concluded that the data set and model both are well suited for further research into survival and reliability analysis.

Furthermore, we have considered seven alternative models, namely exponentiated half logistic exponential (EHLE), generalized inverted exponential (GIE), Kumaraswamy half logistic (Kw-HL)[38,39,40], exponentiated generalized inverted exponential (EGIE)[10], generalized inverted generalized exponential (GIGE) , odd Lomax exponential (OLE) and transmuted Weibull (TW) [41,42,43], compared with the proposed model. Firstly, we have estimated the parameters of the proposed model and competitive models. Each model's parameters are estimated by the maximum likelihood estimation technique by maximize the likelihood function or equivalent to log-likelihood function by using R [34,44].

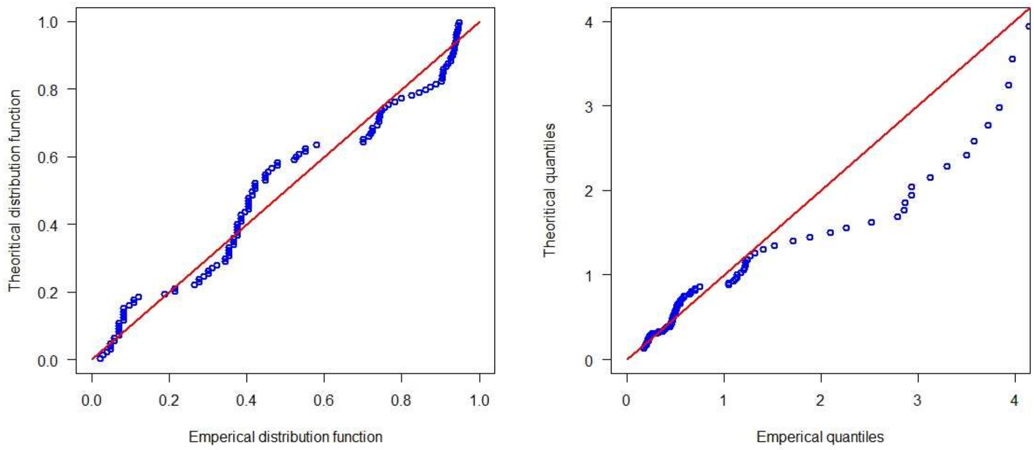

Fig. 3: P-P (left panel) and Q-Q (right panel) plots, using estimated value of MLE's from COVID -19 data set of test positive rate.

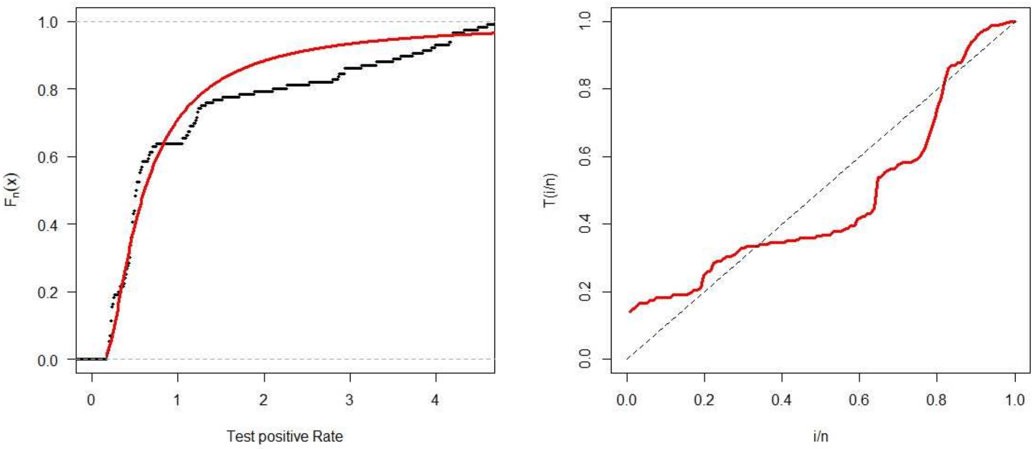

Fig. 4: Plot of EDF verses CDF (left panel) and TTT plot (right panel)

The estimated value of parameters are presented in the following table [Table 5].

Table 5: Estimated value of competitive models, along with proposed model

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	$\hat{\gamma}$	$\hat{\theta}$
HLIW	0.06356	2.11362	0.60120	-	-	-	-
EHLE	0.6086	-	1.69690	0.8707	-	-	-
GIE	1.49698	-	0.63610	-	-	-	-
Kw-HL	-	-	5.18358	1.4965	0.19098	-	-
EGIE	2.52013	0.13610	4.09676	-	-	-	-
GIGE	1.4970	-	0.36870	-	-	1.7250	-
OLE	1.2879	0.6150	-	-	-	-	0.4892
TW	-	-	0.45959	1.091^{η}	1.5334^{σ}	-	-

η ,σ are the parameters of transmuted Weibull distribution

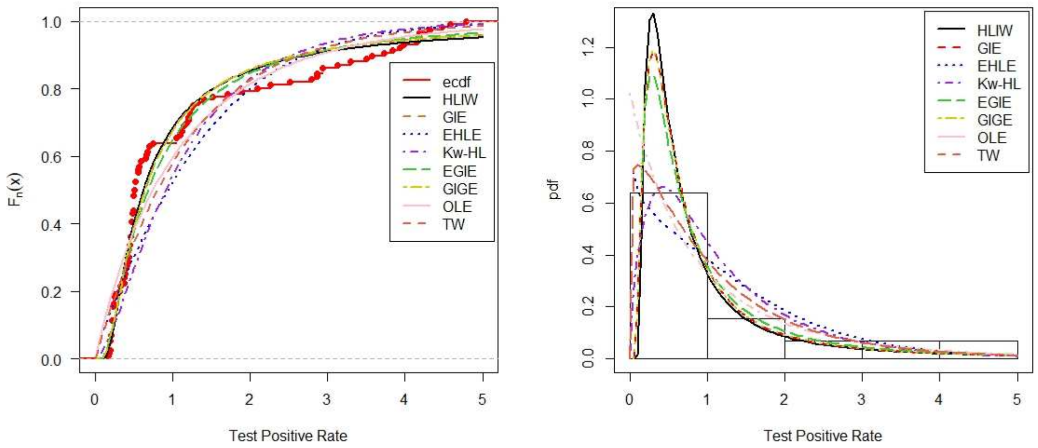

Fig. 5: Estimated fitted CDFs (left panel), Estimated fitted densities (right panel).

Now, we have compared the proposed model with all other competitive models by different goodness of fit criteria's like; (i) Value of log-likelihood, (ii) Akaike's information criterion, (iii) Bayesian information criterion, (iv) Corrected Akaike's information criterion, and (v) Hannan-Quinn information criterion. The different criteria are calculated by the following relations:AIC=$-2\ell(\hat{\delta})+2k$, BIC=$-2\ell(\hat{\delta})+k\ell n(n)$, CAIC=AIC$+\frac{2k(k+1)}{n-k-1}$, and HQIC =$-2\ell(\hat{\delta})+2k\ell n(\ell n(n))$, where k is the number of parameters, n is the total sample under consideration. Among all goodness of fit criteria, the least value supports the best model among all competitive models. Based on the finding, we revealed that, the least value of $\ell(\hat{\delta})$, AIC, BIC, CIAC, and HQIC in the proposed model as compared to all other competitive models followed by EGIE. Hence, the proposed model is superior than other competitive models, followed by EGIE. The model EHLE is inferior suitability among the competitive models [Table 6].

Table 6: Goodness of fit value of competitive models and proposed model

Models	$\ell(\hat{\delta})$	AIC	BIC	CIAC	HQIC
HLIW	-115.601	237.2029	245.463	237.4142	240.5554
EGIE	-117.445	240.8894	249.1502	241.1036	244.2428
GIE	-117.541	239.083	244.5892	239.1892	244.4354
GIGE	-117.5415	241.083	249.3436	241.2973	244.4362
Kw-HL	-134.130	274.8367	282.5207	275.2653	277.6134
TW	-134.714	275.4279	283.6887	275.6422	278.7814
OLE	-135.464	276.928	285.1987	277.1423	280.2814
EHLE	-142.968	291.9378	300.1967	292.1521	295.2894

Likewise, the theoretical CDF of EHLE, GIE, Kw-HL, EGIE, GIGE, OLE, and TW compared with the theoretical CDF of the proposed model. Also, the theoretical PDF of the intended model is compared with the theoretical PDFs of all other seven competitive models. The finding suggests that the proposed model adequately fits in given data set than all other competitive models [Fig 5].

5.2.2 **Survival Times Data Analysis**

Further validation of the proposed model, survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli data set has been used [45]. The data set has been used in the analysis of lognormal survival distribution, logistic-X family of distribution, exponential-gamma distribution, Lindley exponential distribution, and new extremely flexible version of the Weibull exponential distribution[46,47,48,49,50]. (10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116,120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176,183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327,342, 347, 361, 402, 432, 458, 555)

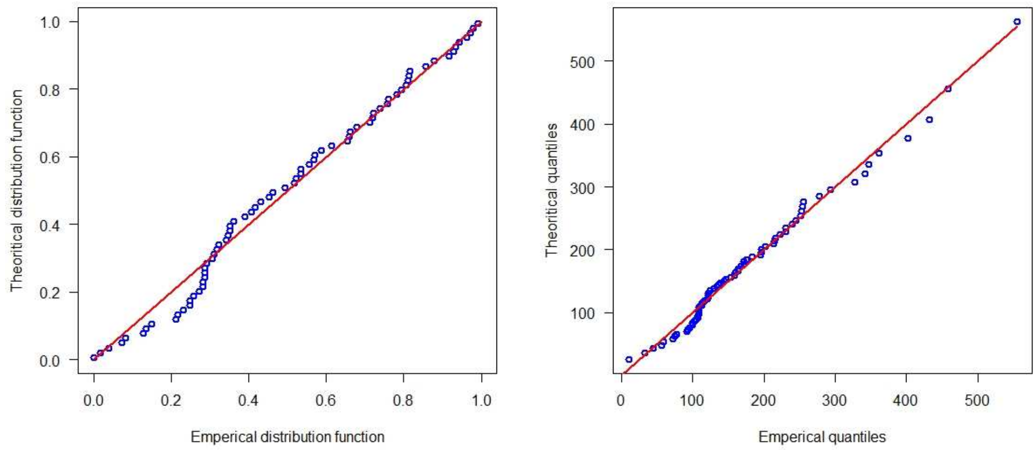

Fig. 6: P-P (left panel) and Q-Q (right panel) plots, using the estimated value of MLEs of survival data.

We estimated the value of parameters by using maximum likelihood estimation technique, which maximize the log-likelihood function (30) directly by using *maxLik ()* function from "BFGS" method[34,44]. The estimated value of parameters are $\hat{\alpha}$=31.5947, $\hat{\beta}$= 0.4201 and $\hat{\lambda}$=48.5975.

Likewise, we have checked the model validation by Kolmogorov-Smirnov test, Anderson-Darling (A^2), and Cramér-Von Mises (W) statistics. The calculated value of each statistics are 0.10157 (p-value= 0.2264), 0.58561 (p-value= 0.6611) and 0.087242 (p-value= 0.6525). The p-value of each statistic supports the null hypothesis, which indicates that the proposed model satisfied the goodness of fit and it is suitable for further data analysis. Similarly, we have checked the graphical presentation of P-P plot and Q-Q plot. The finding of the P-P and Q-Q plots of the proposed model, both graphs are well-fitting in the given data set [Fig.6].

Furthermore, we compared five alternative models, namely expononented inverted Weibull (EIW), generalized inverted exponential, compound Rayleigh (CR) [7,39,51], exponentiated generalized inverted exponential and generalized inverted generalized exponential[10,41] with proposed model. Here, we estimated the parameters of the proposed model as well as competitive models [Table 7].

Table 7: Estimated value of competitive models, along with proposed model

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$
HLIW	31.5947	0.4201	48.5975	-	-
EIW	-	1.148	-	-	210.751
GIE	2.8705	-	209.4839	-	-
CR	0.2268	316.6371	-	-	-
EGIE	3.0325	0.7716	245.7861	-	-
GIGE	2.888	-	20.453	10.291	-

Likewise, after the estimation of parameters, we have to compare the proposed model with competitive models. The finding revealed that, the least value of $\ell(\hat{\delta})$, AIC, BIC, CIAC, and HQIC in the proposed model as compared to all other competitive models followed by EGIE. Hence, the proposed model is superior than the other competitive models, followed by EGIE [Table 8].

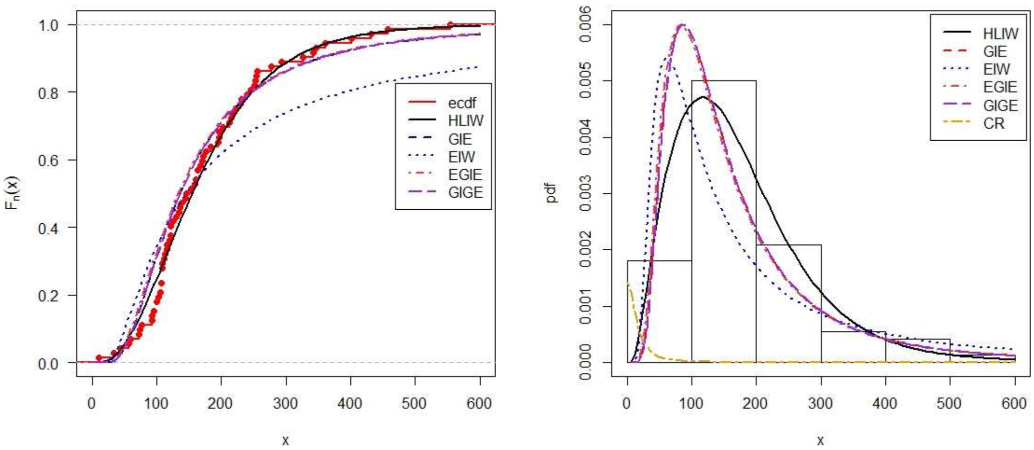

Fig. 7: Estimated CDFs with empirical distribution (left panel) and estimated fitted densities (right panel) for survival data.

Table 8: Goodness of fit values of competitive models and proposed model

Models	$\ell(\hat{\delta})$	AIC	BIC	CIAC	HQIC
HLIW	-426.6802	859.3603	866.1904	859.7132	862.0794
EGIE	-438.6044	883.2088	890.0387	883.5617	885.9278
GIGE	-439.5945	885.189	892.0189	885.5419	887.9080
GIE	-439.5951	883.1903	892.0201	883.3642	885.0029
EIW	-449.7964	903.5928	912.4227	903.7667	905.4055
CR	-850.7812	1705.562	1710.1157	1705.7359	1707.3751

Similarly, we can select the suitable model from hypotheses testing. Among competitive models, EIW, GIG, and CR have two parameters, while the proposed model has three parameters. Both competitive and proposed models have satisfied the properties of the asymptotic theory of normality. Therefore, these models are referred to as nested models. In nested models, we can use hypothesis testing by the likelihood ratio test. The value of LR test H_0 :ELW versus H_α : HLIW is 25.8298 (p-value =0.00000037), H_0 :GIE versus H_α : HLIW is 46.2324 (p-value <0.001), H_0 :WIE versus H_α : HLIW is 848.202 (p-value <0.001). Hence proposed model supports the alternative hypothesis among the nested models. It indicates that the proposed model is superior to the nested models.

Finally, we compared the empirical distribution and theoretical cumulative distribution of the proposed model, indicating that both curves are closer in the illustrative data set. Likewise, the theoretical CDF of EGIE, GIGE, GIE, and IEW compared to the theoretical CDF of the proposed model. Also, the theoretical PDF of the intended model is compared with all other competitive models. The finding suggests that the proposed model adequately fits in given data set than all other competitive models [Fig.7].

6 Conclusion

This study is based on proposed a new distribution having three parameters called *Half Logistic Inverted Weibull* distribution by compounding the half logistic-G family with inverted Weibull distribution. It is positively skewed and unimodal distribution. We have derived some important properties like; quintile and median, asymptotic behavior with mode, skewness and kurtosis, moments, moment generating function, residual life function; Rényi and q entropy, probability weighted moments, and order statistic. Simulation studies has been done to verify the properties of MLE. Two real data sets, the positive test rate of PCR during the period of COVID-19 first wave in Nepal and survival data, have been used to verify the efficiency of proposed model. We analyzed the two data sets and we concluded that the proposed model provides a reasonably better fit as compared to other well-known models. Therefore, the HLIW distribution can be used an alternative model for lifetime data (survival data) or reliability data.

Acknowledgement

We would like to grateful to the referees for their valuable comments and suggestions which are helpful in improving the original manuscript.

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