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Symmetry Reductions and New Exact Non-Traveling Wave Solutions of b -Family Equations

M. H. M. Moussa and Zidan M. Abd Al-Halim[*]

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Hiliopolis, Cairo, Egypt.

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Abstract: In this work, b-family equations has been analyzed via symmetry method.We obtained similarity reductions for b-family equations. These similarity reductions are ordinary differential equations (ODEs) of the third order. Some of these ODEs has been solved via Exp-function method. Finally, we arrive at some new similarity solutions for the equation under consideration.

Keywords: b-family equations, symmetry method, similarity solutions

1 Introduction

we consider the following b-family equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \qquad (1)$$

where b is a dimensionless constant. Eq.(1) has been investigated in the literature [1]-[11].The quadratic terms in Eq.(1) represent the competition, or balance, in fluid convection between nonlinear transport and amplification due to b-dimensional stretching [12,13]. Recently, a study of soliton equations, it was found that for any $b \neq -1$ Eq.(1) was included in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations [14]. Degasperis and Procesi [2] showed that the family of equations (1) cannot be integrable unless $b = 2$ or $b = 3$ by using the method of asymptotic integrability. The preceding two values of b are corresponding to two important equations the Camassa–Holm (CH) equation and Degasperis–Procesi (DP) equation respectively. The CH and the DP equations are bi-Hamiltonian and have an associated isospectral problem, therefore they are both formally integrable [15,16,17,18]. Moreover, both equations admit peaked solitary wave solutions and present similarities although they are truly different [19, 20,21].

2 Symmetry method

We briefly outlined Steinberg's similarity method of finding explicit solutions of both linear and non-linear partial differential equations [22]. The method based on finding the symmetries of the differential equation as follows:

Assume that, the differential operator L can be written in the following form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u), \qquad (2)$$

where $u = u(x.t)$ and H may depend on x,t,u and any derivative of u as long the derivative of u does not contain more than $p-1,t$ derivatives. Consider the symmetry operator called infinitesimal symmetry, which being quasi-linear partial differential operator of first order has the form

$$S(u) = A(x,t,u)\frac{\partial u}{\partial t} + \sum_{i=1}^{n} B_i(x,t,u)\frac{\partial u}{\partial x_i} + C(x,t,u). \qquad (3)$$

Define the Frècht derivative of $L(u)$ by

$$F(L,u,v) = \frac{d}{d\varepsilon}L(u+\varepsilon v)\mid_{\varepsilon=0}. \qquad (4)$$

With these definitions in the mind we need to follow the following steps

[*] Corresponding author e-mail: zidanmhm@gmail.com

(i) Compute $F(L,u,v)$
(ii) Compute $F(L,u,S(u))$
(iii) Substituting $H(u)$ for $\left(\frac{\partial^p u}{\partial t^p}\right)$ in $F(L,u,S(u))$
(iv) Set this expression to zero and perform a polynomial expansion.
(v) Solve the resulting partial differential equations. Once this system of partial differential equations is solved for the coefficients of $S(u)$, equation under study can be used to obtain the functional form of the solutions.

3 *b*-family equations

b-family equations can be represented by the form

$$L(u) = u_{xxt} + u_t + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} = 0, \quad (5)$$

where $u = u(x,t)$, b is a dimensionless constant.

3.1 Determination of the symmetries

In order to find the symmetries of Eq.(5) we set

$$S(u) = A(x,t,u)\,u_t + B(x,t,u)\,u_x + C(x,t,u). \quad (6)$$

Calculating the Frèchet derivative $F(L,u,\Psi)$ of $L(u)$ in the direction of Ψ, given by Eq.(4), and replacing Ψ in F by $S(u)$, we get

$$\begin{array}{l} F(L,u,S(u)) = [S(u)]_{txx} - [S(u)]_t \\ -(b+1)u_x[S(u)] - (b+1)u[S(u)]_x + bu_{xx}[S(u)]_{\boldsymbol{x}} \\ +bu_{\boldsymbol{x}}[S(u)]_{\boldsymbol{xx}} + u_{xxxx}[S(u)] + u[S(u)]_{xxx} = 0. \end{array} \quad (7)$$

Substituting the values of different derivatives of u in F, we get a polynomial expansion in $u_t, u_x, u_x u_t, u_{xt}, \ldots$,etc. On making use of Eq.(5) in the polynomial expression for F, rearranging terms of various powers of derivatives of u and equating them to zero, we arrive at the following equations:

$$\begin{array}{l} A_x = A_u = B_u = C_{uu} = 0, \\ 2B_x + C_{xxu} = 0, \\ B_{xx} + 2C_{xu} = 0, \\ bB_x + bC_u - bA_t = 0, \\ B_t + uB_x + C - uA_t = 0, \\ C_{xxt} - C_t + uC_{xxx} - (b+1)uC_x = 0, \\ 2B_{xt} + bC_x + 3uC_{xu} + 3uB_{xx} + C_{tu} = 0, \\ u(b+1)(A_t + B_x) - B_t + B_{xxt} + 2C_{xtu} \\ +bC_{xx} - (b+1)C + u(3C_{xxu} + B_{xxx}) = 0. \end{array} \quad (8)$$

On solving system of Eqs.(8) we see that, the infinitesimals A, B and C satisfying these equations are

$$\begin{array}{l} A = c_1 t + c_2, \\ B = c_3, \\ C = c_1 u, \end{array} \quad (9)$$

where c_1, c_2 and c_3 are the integration constants. In order to study the group theoretic structure, the vector field operator V is written as

$$V = \chi_1(c_1) + \chi_2(c_2) + \chi_3(c_3), \quad (10)$$

where

$$\begin{array}{l} \chi_1(c_1) = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ \chi_2(c_2) = \frac{\partial}{\partial t}, \\ \chi_3(c_3) = \frac{\partial}{\partial x}. \end{array} \quad (11)$$

The commutator relations are given in table 1.
It is clear that the vector field V in Eq.(10) constitutes a finite dimensional Lie algebra.
Furthermore, from the symmetries given in Eq.(9) the following possibilities exist for the solution of Eq.(1).
1. $c_1 \neq 0, c_2 \neq 0, c_3 \neq 0$.
2. $c_1 \neq 0, c_2 = 0, c_3 \neq 0$.
3. $c_1 \neq 0, c_2 \neq 0, c_3 = 0$.
4. $c_1 = 0, c_2 \neq 0, c_3 \neq 0$.

Table 1. commutator relations.

$[\chi_i, \chi_j]$	χ_1	χ_2	χ_3
χ_1	0	$-\chi_2$	0
χ_2	χ_2	0	0
χ_3	0	0	0

Where $i = 1,2,3$ and $j = 1,2,3$. In order to obtain the invariant transformation, we write the characteristic equation in the form

$$\frac{dt}{A(x,t,u)} = \frac{dx}{B(x,t,u)} = -\frac{du}{C(x,t,u)}. \quad (12)$$

To get the similarity variables we integrate the first two terms, to get the similarity solution for the reduced ordinary differential equation, we integrate the first and the third ratios for the four cases as follows.

Case (1)

Corresponding to the choice $c_1 \neq 0, c_2 \neq 0$ and $c_3 \neq 0$ we obtain the following invariant ξ and the form of u :

$$\begin{array}{l} \xi(x,t) = \dfrac{\exp\left[\dfrac{c_1}{c_3}x\right]}{\left(t + \dfrac{c_2}{c_1}\right)}, \\ u(x,t) = \dfrac{\eta(\xi)}{\left(t + \dfrac{c_2}{c_1}\right)}. \end{array} \quad (13)$$

Substituting Eq.(13) into Eq.(1), we have the following ordinary differential equation of third order

$$\begin{aligned}&\left(\tfrac{c_1}{c_3}\right)^2\left(1-\left(\tfrac{c_1}{c_3}\right)\eta\right)\xi^3\eta'''-b\left(\tfrac{c_1}{c_3}\right)^3\xi^3\eta'\eta''\\&+4\left(\tfrac{c_1}{c_3}\right)^2\xi^2\eta''-3\left(\tfrac{c_1}{c_3}\right)^3\xi^2\eta\eta''-\left(\tfrac{c_1}{c_3}\right)^3\xi^2\eta'^2\\&+\tfrac{c_1}{c_3}(b+1)\xi\eta\eta'-\left(\tfrac{c_1}{c_3}\right)^3\xi\eta\eta'\\&+\left(2\left(\tfrac{c_1}{c_3}\right)^2-1\right)\xi\eta'-\eta=0.\end{aligned}\tag{14}$$

In order to solve Eq.(14), we set $\tau(\xi)=\ln(\xi)$, then Eq.(14) can be transformed to the ODE

$$\begin{aligned}&\left(\tfrac{c_1}{c_3}\right)^2\dddot{\eta}-\left(\tfrac{c_1}{c_3}\right)^3\eta\dddot{\eta}+\left(\tfrac{c_1}{c_3}\right)^2\ddot{\eta}-b\left(\tfrac{c_1}{c_3}\right)^3\dot{\eta}\ddot{\eta}\\&-\left(\tfrac{c_1}{c_3}\right)^3(b+1)\dot{\eta}^2+\tfrac{c_1}{c_3}(b+1)\eta\dot{\eta}-\dot{\eta}-\eta=0,\end{aligned}\tag{15}$$

where $(\cdot)$ denotes to the differentiation to τ.

Case (2)

Corresponding to the choice $c_1\neq 0, c_2=0$ and $c_3\neq 0$, Eq.(9) becomes

$$\begin{aligned}A&=c_1t,\\B&=c_3,\\C&=c_1u,\end{aligned}\tag{16}$$

and we obtain the following invariant ξ and the form of u

$$\begin{aligned}\xi(x,t)&=\frac{\exp\left[\dfrac{c_1}{c_3}x\right]}{t},\\u(x,t)&=\frac{\eta(\xi)}{t}.\end{aligned}\tag{17}$$

As doing in case (1), we arrive at the same ODE in the first case, which reads

$$\begin{aligned}&\left(\tfrac{c_1}{c_3}\right)^2\dddot{\eta}-\left(\tfrac{c_1}{c_3}\right)^3\eta\dddot{\eta}+\left(\tfrac{c_1}{c_3}\right)^2\ddot{\eta}-b\left(\tfrac{c_1}{c_3}\right)^3\dot{\eta}\ddot{\eta}\\&-\left(\tfrac{c_1}{c_3}\right)^3(b+1)\dot{\eta}^2+\tfrac{c_1}{c_3}(b+1)\eta\dot{\eta}-\dot{\eta}-\eta=0\end{aligned}\tag{18}$$

Case (3)

Corresponding to the choice $c_1\neq 0, c_2\neq 0$ and $c_3=0$, hence the symmetries takes the form

$$\begin{aligned}A&=c_1t+c_2,\\B&=0,\\C&=c_1u.\end{aligned}\tag{19}$$

Consequently, the invariant ξ and the form of u are as under

$$\begin{aligned}\xi(x,t)&=x,\\u(x,t)&=\frac{\eta(\xi)}{\left(t+\dfrac{c_2}{c_1}\right)}.\end{aligned}\tag{20}$$

From Eq.(20) in Eq.(1) yields

$$\eta\eta'''+b\eta'\eta''-\eta''-(b+1)\eta\eta'+\eta=0.\tag{21}$$

Now, we are going to use Exp-function method $[23,24]$ for solving Eq.(21). In view of the Exp-funcation method, we suppose that the solution of Eq. (21) can be written in the form

$$\eta(\xi)=\frac{\sum_{n=-c}^{d}a_n\exp(n\xi)}{\sum_{m=-p}^{q}b_m\exp(m\xi)},\tag{22}$$

where c,d,p and q are positive integers which are unknown to be determined later, a_n and b_m are unknown constants. Eq. (22) can be re-written in an alternative form as follows

$$\eta(\xi)=\frac{a_c\exp(c\xi)+...+a_{-d}\exp(-d\xi)}{b_p\exp(p\xi)+...+b_{-q}\exp(-q\xi)}.\tag{23}$$

In order to determine values of c and p, we balance the linear term of the highest order in Eq.(21) with the highest order nonlinear term η'' and $\eta\eta'''$ we have

$$\eta''=\frac{c_1\exp[(c+3p)\xi]+...}{c_2\exp[4p\xi]+...},\tag{24}$$

$$\eta\eta'''=\frac{c_3\exp[(2c+7p)\xi]+...}{c_4\exp[9p\xi]+...},\tag{25}$$

where c_i are coefficients of Exp-function for simplicity. By balancing highest order of Exp-function in Eqs. (24) and (25) provides $c+8p=2c+7p$, which leads to

$$p=c.\tag{26}$$

Proceeding the same manner as illustrated above, we can determine values of d and q . Balancing the linear term of lowest order in Eq.(21)

$$\eta''=\frac{...+d_1\exp[-(d+3q)\xi]}{...+d_2\exp[-4q\xi]},\tag{27}$$

$$\eta\eta'''=\frac{...+d_3\exp[-(2d+7q)\xi]}{...+d_4\exp[-9q\xi]},\tag{28}$$

where d_i are determined coefficients only for simplicity, we have $-(d+8q)=-(2d+7q)$, which leads to the result

$$q=d\,.\tag{29}$$

We can freely choose the values of c and d. For simplicity, we set $p=c=1$ and $d=q=1$, Eq.(23) can be expressed as

$$\eta(\xi)=\frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)}\,. \tag{30}$$

Substituting Eq.(30) into Eq.(21). Equating to zero the coefficients of all powers of $exp(n\xi)$ yields a set of algebraic equations for $a_o,a_1,a_{-1},b_0,b_1,b_{-1},k,l$ and c_0, solving this system of algebraic equations with the aid of Maple, we have the following four solutions

$$\begin{aligned}
&a_{-1}=b_1=0, b_{-1}=b_{-1}, a_o=a_o, b_0=b_0, b=b, a_1=\tfrac{a_o b_o}{b_{-1}}.\\
&a_o=b_{-1}=b_1=0, a_{-1}=a_{-1}, b_0=b_0, b=b, a_1=a_1.\\
&a_{-1}=a_1=b_{-1}=b_0=0, a_0=a_0, b=b, b_1=b_1.\\
&a_1=b_{-1}=0, a_{-1}=a_{-1}, b_0=b_0, b_1=b_1, b=b,\\
&a_0=\tfrac{a_{-1}b_1}{b_0}.
\end{aligned} \tag{31}$$

On making use of Eqs.(31) and (30) with Eq.(20) we obtain the next exact solutions for Eq.(1) (see figures (1-5))

$$u_1(x,t)=\frac{a_0b_0\exp[x]+a_0b_{-1}}{\left(b_{-1}b_0+b_{-1}^2\exp[-x]\right)\left(t+\frac{c_2}{c_1}\right)} \tag{32}$$

$$u_2(x,t)=\frac{a_1\exp[x]+a_{-1}\exp[-x]}{b_0\left(t+\frac{c_2}{c_1}\right)}, \tag{33}$$

we can obtain special two solutions from relation (33) as follows, if $a_{-1}=a_1$ yields

$$u_3(x,t) = \frac{2a_1\cosh(x)}{b_0\left(t+\frac{c_2}{c_1}\right)}, \tag{34}$$

and in Eq.(33) if $a_{-1}=-a_1$, the solution takes the form

$$u_4(x,t)=\frac{2a_1\sinh(x)}{b_0\left(t+\frac{c_2}{c_1}\right)}. \tag{35}$$

$$u_5(x,t)=\frac{a_0\exp[-x]}{b_1\left(t+\frac{c_2}{c_1}\right)}. \tag{36}$$

$$u_6(x,t)=\frac{a_{-1}b_0\exp[-x]+a_{-1}b_1}{\left(b_0b_1\exp[x]+b_0^2\right)\left(t+\frac{c_2}{c_1}\right)}. \tag{37}$$

More solutions for Eq.(1) can be obtained by choosing different values for c and d.

Case (4)

Corresponding to the choice $c_1=0, c_2\neq 0$ and $c_3\neq 0$, then the symmetries can be written as

$$\begin{aligned} A&=c_2,\\ B&=c_3,\\ C&=0. \end{aligned} \tag{38}$$

As listed above, we have the following form of the invariant ξ and u

$$\begin{aligned} \xi(x,t)&=t-\frac{c_2}{c_3}x,\\ u(x,t)&=\eta(\xi). \end{aligned} \tag{39}$$

Substituting Eq.(39) into Eq.(1) provides

$$\begin{aligned} &\left(\tfrac{c_2}{c_3}\right)^2\eta'''-\left(\tfrac{c_2}{c_3}\right)^3\eta\eta'''-b\left(\tfrac{c_2}{c_3}\right)^3\eta'\eta''\\ &\quad+\left(\tfrac{c_2}{c_3}\right)(b+1)\eta\eta'-\eta'=0. \end{aligned} \tag{40}$$

In Eq.(39), set $b=1$ and integrating that equation, therefore Eq.(40) becomes

$$\left(\frac{c_2}{c_3}\right)^2\eta''-\left(\frac{c_2}{c_3}\right)^3\eta\eta''+\left(\frac{c_2}{c_3}\right)\eta^2-\eta=0, \tag{41}$$

which has the following solution

$$\eta(\xi)=c_4\exp\left[\frac{c_3}{c_2}\xi\right]+c_5\exp\left[-\frac{c_3}{c_2}\xi\right], \tag{42}$$

where c_4 and c_5 are the integration constants. Combining Eq.(42) with Eq.(39), we obtain the following exact solutions of Eq.(1) (see figure (6))

$$u(x,t)=c_4\exp\left[\frac{c_3}{c_2}t-x\right]+c_5\exp\left[-\left(\frac{c_3}{c_2}t-x\right)\right], \tag{43}$$

setting $c_4=c_5$, leads to (see figure (7))

$$u(x,t)=2c_4\cosh\left(\frac{c_3}{c_2}t-x\right). \tag{44}$$

On the other hand, if we take $c_4=-c_5$ in Eq.(42), the solution can be given as (see figure (8))

$$u(x,t)=2c_4\sinh\left(\frac{c_3}{c_2}t-x\right). \tag{45}$$

By back substitutions we find that the above three solutions (Eq.(43), Eq.(44) and Eq.(45)) are true for any b.

4 Figures

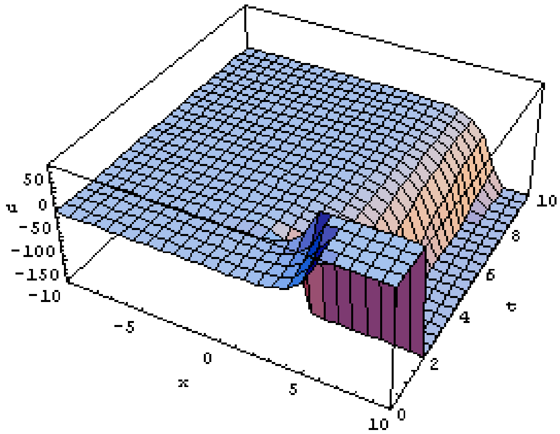

Fig. 1: solution (32), at $a_0 = -1, b_0 = 1, b_{-1} = 2, c_1 = -2, c_2 = 4$.

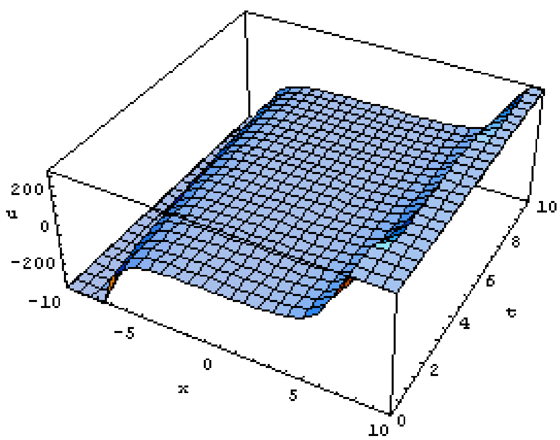

Fig. 2: solution (33), at $a_1 = 3, a_{-1} = -2, b_0 = 4, c_1 = 2, c_2 = 3$

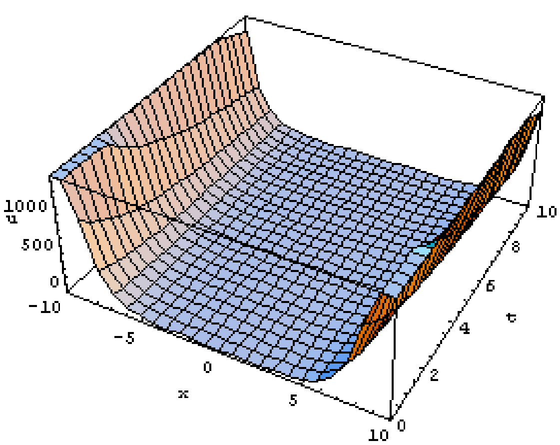

Fig. 3: solution (34), at $a_1 = 3, b_0 = 4, c_1 = 2, c_2 = 3$.

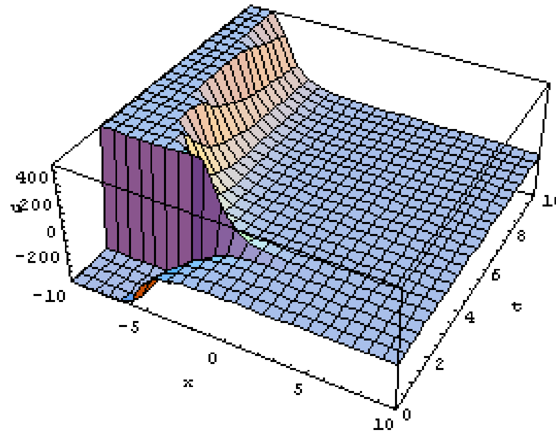

Fig. 4: solution (36), at $a_0 = -5, b_0 = -3, c_1 = -1, c_2 = 2$.

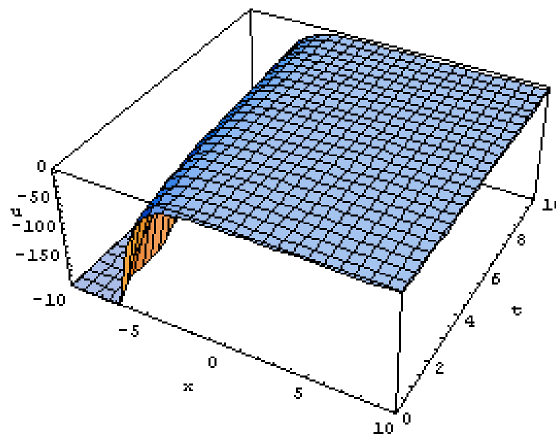

Fig. 5: solution (37), at $a_{-1} = -1, b_0 = 2, b_1 = -3, c_1 = -2, c_2 = -4$.

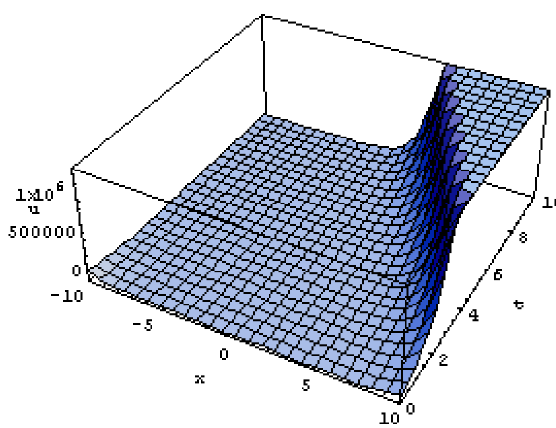

Fig. 6: solution (43), at $c_2 = 2, c_3 = -2, c_4 = 5, c_5 = 3$.

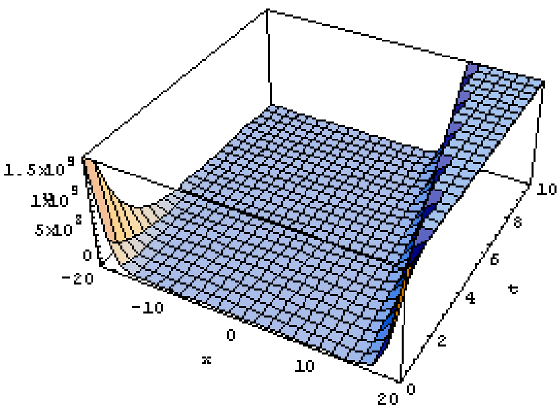

Fig. 7: solution (44), at $c_2 = 2, c_3 = -2, c_4 = 5$.

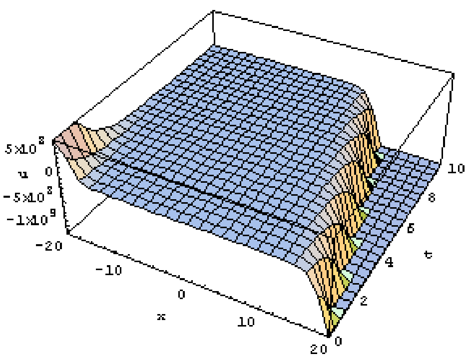

Fig. 8: solution (45), at $c_2 = 2, c_3 = -2, c_4 = 5$.

5 Conclusion

The application of the symmetry method to b-family equations has led three similarity reductions for the mentioned equations. On solving the second reduction via Exp-function method we have obtained new exact similarity solutions for the equation under consideration. On the other hand, we have arrived at three similarity solutions from the last reduction. While, the first reduction gives an ODE which is under consideration.

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Mohsen Hanafy Mohamad Moussa Assistant professor of applied math. Department of Math. Faculty of education, Ain Shams University, Egypt. His research interests are in the areas of applied mathematic and mathematical physics, nonlinear differential Equations, Dynamical systems, Computational Mathematics, Fluid mechanics. He has published many research articles in reputed international journals of mathematical and engineering sciences.

Zidan Mostafa Hendy Mostafa Abd Al-Halim received the PhD degree in pure mathematics from Faculty of education, Ain Shams University, Egypt. His research interests are in the areas of pure mathematic, nonlinear partial differential equations and mathematical physics. He has published research articles in reputed international journals of mathematical and engineering sciences. He is an expert teacher in the Egyptian Ministry of Education.