

An Inverse Power Lifetime Model with Decreasing and Unimodal Hazard Rates: Applications to Flood and Engineering Data

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Abstract: Decreasing and inverted bathtub-shaped hazard rates arise in lifetime data and require models that can describe both patterns. This paper introduces the inverse power Garhy distribution (IPGD), a two-parameter model obtained through an inverse power transformation of the Garhy distribution. We derive its moments, incomplete moments, survival function, and Mills ratio. The parameters are estimated using fifteen classical methods, and a simulation study examines their performance across different sample sizes. Applications to flood and engineering datasets show that the IPGD provides a better fit than the Garhy distribution and several competing models considered in the study. These results support its use in modeling data from environmental and engineering studies.

Keywords: Inverse power transformation, lifetime distribution, hazard rate, classical parameter estimation, flood data, engineering applications.

1 Introduction

Probability distributions are useful for describing real-life phenomena. They are used in engineering to model machine and component failure times. They are applied in environmental studies to describe rainfall, flood levels, and extreme events. They are used in medicine to study survival times and treatment outcomes. These examples show that probability models are needed to study uncertainty and to make decisions under risk.

Although many probability distributions have been proposed in the literature, there is still a need to develop

new ones because real-world data often exhibit characteristics that cannot be adequately described by existing models. As a result, many flexible classes of probability distributions have been developed to model complex data.

Mixture distributions are an important class of probability distributions. They are obtained by combining two or more component distributions to model complex data. They have been widely used in engineering, finance, environmental studies, and many other fields. Examples include the xgamma distribution [1], Shukla distribution [2], the new exponential gamma distribution [3],

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Chris-Jerry distribution [4], Komal distribution [5], EX-Lindley distribution [6], Lindley modified Lindley distribution [7], and the Obulezi distribution [8].

Among these models, the Garhy distribution proposed by [9] has attracted research interest because of its simple form and useful statistical properties. Ref. [9] applied the distribution to two real-life datasets, where it outperformed several existing mixture distributions. The Garhy distribution has also been studied in recent works. For example, [10] proposed the Half-Logistic Garhy distribution, while [11] studied the stress-strength reliability parameter of the Garhy distribution under different record sampling schemes using both classical and Bayesian estimation methods. These studies show the continued interest in the Garhy distribution and its potential for further development.

The classical Garhy distribution has a limitation. Its hazard rate is strictly increasing, which limits its application to data with other hazard rate patterns. This limitation motivates the development of new extensions that can accommodate additional hazard rate shapes.

One useful approach for extending existing distributions is the inverse power transformation. This method introduces an additional parameter and increases the flexibility of the baseline distribution. It has been used to develop several flexible lifetime distributions. Examples include the inverse power Akshaya distribution [12], the inverse-power logistic-exponential distribution [13], the inverse power half-logistic distribution [14], the inverse power Perk distribution [15], the inverse power novel two-parameter quadratic exponential distribution [16], and the inverse power induced Bilal distribution [17]. These studies show that the inverse power transformation is an effective method for developing flexible probability distributions.

Motivated by these studies, this paper proposes a new extension of the Garhy distribution using the inverse power transformation. The resulting model is called the Inverse Power Garhy distribution (IPGD). The proposed distribution introduces one additional parameter, which increases its flexibility. It can accommodate decreasing and non-monotonic hazard rate shapes that are not supported by the classical Garhy distribution. Several statistical properties of the proposed distribution are derived, its parameters are estimated using fifteen classical estimation methods, and its performance is evaluated through simulation studies and applications to environmental and engineering datasets.

Motivation and Contributions

The main motivations and contributions of this study are summarized as follows:

- To develop a new extension of the Garhy distribution using the inverse power transformation.
- To introduce a more flexible distribution that can accommodate a wider range of hazard rate shapes than the classical Garhy distribution.

-To derive important statistical properties of the proposed distribution, including the moments, survival function, incomplete moments, and harmonic mean.

-To estimate the model parameters using fifteen classical estimation methods and evaluate their performance through a simulation study.

-To demonstrate the usefulness of the proposed distribution using real-life environmental and engineering datasets.

-To compare the performance of the proposed distribution with the Garhy distribution and other competing models.

The organization of the remaining sections of this study is illustrated in Figure 1.

2 Formulation of the Inverse Power Garhy Distribution

In this section, we construct the Inverse Power Garhy Distribution (IPGD) as a transformation of the Garhy distribution [9] and derive its reliability and survival characteristics. Let Y be a random variable following the Garhy distribution with cumulative distribution function (CDF)

$$F(y; \kappa) = 1 - \left(\frac{3\kappa^2 y^2}{\kappa^3 + 6} + (\kappa y + 1) \right) e^{-\kappa y}, \quad (1)$$

$$y > 0, \kappa > 0.$$

Using the transformation $X = Y^{-1/\phi}$, we obtain the inverse power Garhy distribution (IPGD). The CDF of X is obtained as

$$F(x; \kappa, \phi) = e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right], \quad (2)$$

$$x > 0, \kappa > 0, \phi > 0.$$

Differentiating the CDF (2) with respect to x , the probability density function is obtained as

$$f(x; \kappa, \phi) = \frac{\kappa^3 \phi x^{-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3)}{\kappa^3 + 6}, \quad x > 0. \quad (3)$$

The survival (reliability) function is defined as

$$\bar{F}(x; \kappa, \phi) = 1 - e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right].$$

The hazard rate function is defined by

$$h(x; \kappa, \phi) = \frac{f(x; \kappa, \phi)}{\bar{F}(x; \kappa, \phi)}$$

$$= \frac{\frac{\kappa^3 \phi x^{-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3)}{\kappa^3 + 6}}{1 - e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right]}.$$

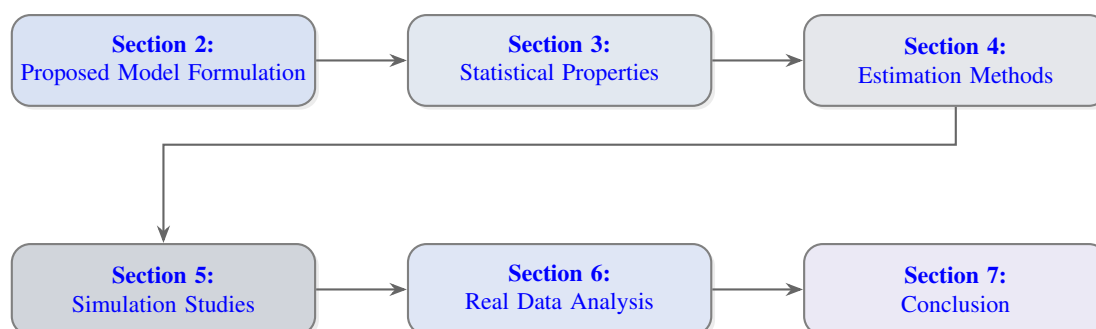


Fig. 1: Flowchart of the Remaining Sections of the Study.

The reversed hazard rate function is defined as

$$\begin{aligned}\tau(x; \kappa, \phi) &= \frac{f(x; \kappa, \phi)}{F(x; \kappa, \phi)} \\ &= \frac{\kappa^3 \phi x^{-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3)}{\kappa^3 + 6} \\ &= \frac{e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right]}{e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right]}.\end{aligned}$$

The cumulative hazard function is given by

$$\begin{aligned}H(x; \kappa, \phi) &= -\ln [\bar{F}(x; \kappa, \phi)] \\ &= -\ln \left[1 - e^{-\kappa x^{-\phi}} \left(\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right) \right].\end{aligned}$$

The odds function and the Mills ratio are defined as follows

$$\begin{aligned}O_X(x; \kappa, \phi) &= \frac{F(x; \kappa, \phi)}{\bar{F}(x; \kappa, \phi)} \\ &= \frac{e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right]}{1 - e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right]},\end{aligned}$$

and

$$\begin{aligned}M(x; \kappa, \phi) &= \frac{\bar{F}(x; \kappa, \phi)}{f(x; \kappa, \phi)} \\ &= \frac{1 - e^{-\kappa x^{-\phi}} \left[\kappa x^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x^\phi \right) + 1 \right]}{\kappa^3 \phi x^{-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3)}.\end{aligned}$$

Figure 2 presents the two-dimensional and three-dimensional plots of the probability density function of the IPGD for selected parameter values. The density function is positively skewed and unimodal, depending on the parameter values. These shapes show that the proposed distribution is flexible and suitable for modelling positively skewed lifetime data.

Figure 3 shows the hazard rate function of the IPGD for selected parameter values. The hazard rate can take

either a decreasing or an inverted bathtub shape, depending on the parameter values. This shows that the proposed distribution is flexible and can model different lifetime behaviours.

3 Statistical Properties

In this section, some important statistical properties of the IPGD are presented.

3.1 Mode

In this subsection, the following theorem provides the mode of the IPGD.

Theorem 1. Let X be a random variable following the IPGD with PDF (3), then, the mode of the IPGD is given by

$$x_m = \left[\frac{\kappa^3 \phi - 3(3\phi + 1) + \sqrt{[\kappa^3 \phi - 3(3\phi + 1)]^2 + 12\kappa^3 \phi (2\phi + 1)}}{2\kappa^2 (2\phi + 1)} \right]^{\frac{1}{\phi}}. \quad (4)$$

Proof. The mode is obtained by maximizing the PDF. Taking the natural logarithm yields

$$\begin{aligned}\ln(f(x)) &= \ln \left(\frac{\kappa^3 \phi}{\kappa^3 + 6} \right) - (3\phi + 1) \ln(x) \\ &\quad - \kappa x^{-\phi} + \ln(\kappa^2 x^\phi + 3).\end{aligned}$$

Differentiating with respect to x , we obtain

$$\frac{d}{dx} \ln(f(x)) = -\frac{3\phi + 1}{x} + \kappa \phi x^{-\phi-1} + \frac{\kappa^2 \phi x^{\phi-1}}{\kappa^2 x^\phi + 3}.$$

The critical point satisfies $\frac{d}{dx} \ln(f(x)) = 0$, which implies

$$-\frac{3\phi + 1}{x} + \kappa \phi x^{-\phi-1} + \frac{\kappa^2 \phi x^{\phi-1}}{\kappa^2 x^\phi + 3} = 0.$$

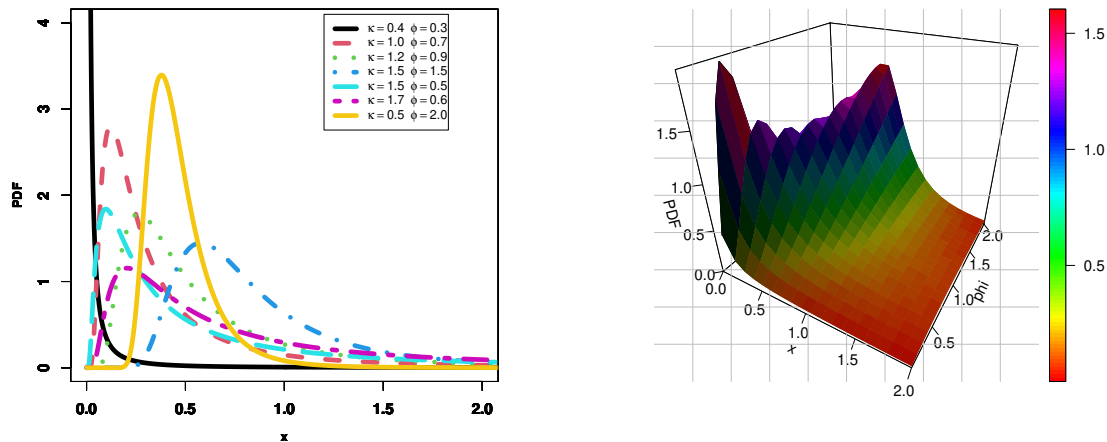


Fig. 2: 2D Plots of PDF and 3D Plots of PDF at $\kappa = 1.5$ for the IPGD.

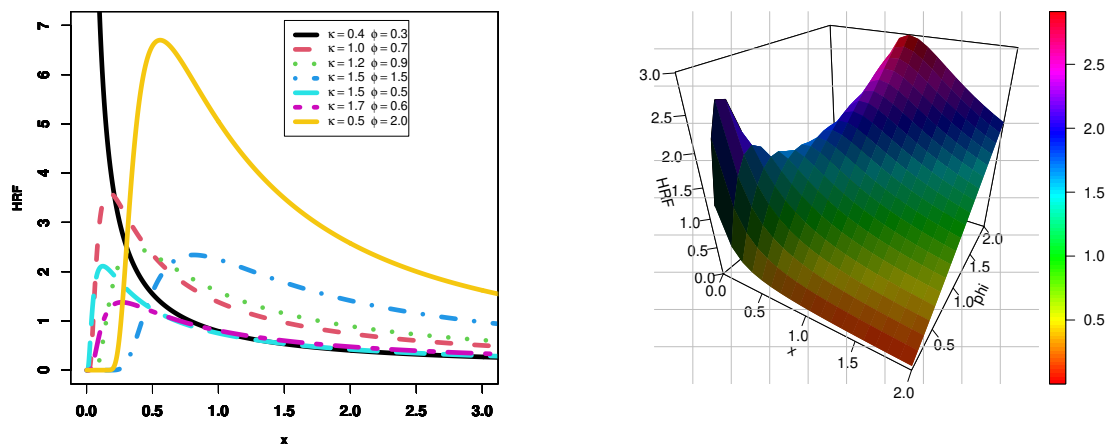


Fig. 3: 2D Plots of HRF and 3D Plots of HRF at $\kappa = 1.5$ for the IPGD.

Multiplying both sides by x gives

$$-(3\phi + 1) + \kappa\phi x^{-\phi} + \frac{\kappa^2\phi x^\phi}{\kappa^2 x^\phi + 3} = 0.$$

Let $y = x^\phi$. Then the above equation becomes

$$-(3\phi + 1) + \frac{\kappa\phi}{y} + \frac{\kappa^2\phi y}{\kappa^2 y + 3} = 0.$$

Multiplying by $y(\kappa^2 y + 3)$, we get

$$-(3\phi + 1)y(\kappa^2 y + 3) + \kappa\phi(\kappa^2 y + 3) + \kappa^2\phi y^2 = 0.$$

After expansion and simplification, the above expression reduces to

$$\kappa^2(2\phi + 1)y^2 + [3(3\phi + 1) - \kappa^3\phi]y - 3\kappa\phi = 0.$$

By using the quadratic formula, we have

$$y = \frac{\kappa^3\phi - 3(3\phi + 1) \pm \sqrt{[\kappa^3\phi - 3(3\phi + 1)]^2 + 12\kappa^3\phi(2\phi + 1)}}{2\kappa^2(2\phi + 1)}.$$

Finally, recalling that $y = x^\phi$, the mode of the IPGD is given by

$$x_m = (y_m)^{1/\phi} = \left[\frac{\kappa^3 \phi - 3(3\phi + 1) + \sqrt{[\kappa^3 \phi - 3(3\phi + 1)]^2 + 12\kappa^3 \phi(2\phi + 1)}}{2\kappa^2(2\phi + 1)} \right]^{1/\phi}.$$

This completes the proof.

3.2 Ordinary Moments

In this subsection, we derive an explicit expression for the r th ordinary moment of the IPGD.

Theorem 2. Let X be a random variable following the IPGD with PDF (3), then, the r th ordinary moment of X is given by

$$\mu'_r = E(X^r) = \frac{\kappa^{r/\phi} ((\kappa^3 + 6)\phi - 3r) \Gamma\left(2 - \frac{r}{\phi}\right)}{(\kappa^3 + 6)\phi}, \quad r < 2\phi. \quad (5)$$

Proof. By definition, the r th ordinary moment is

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x) dx.$$

Substituting the PDF (3) of the IPGD, we obtain

$$\begin{aligned} \mu'_r &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \int_0^\infty x^{r-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3) dx \\ &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \int_0^\infty x^{r-2\phi-1} e^{-\kappa x^{-\phi}} dx + 3 \int_0^\infty x^{r-3\phi-1} e^{-\kappa x^{-\phi}} dx \right]. \end{aligned}$$

To evaluate these integrals, consider the transformation $t = \kappa x^{-\phi}$, then

$$\mu'_r = \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \left(\frac{1}{\phi} \kappa^{\frac{r}{\phi}-2} \int_0^\infty t^{1-\frac{r}{\phi}} e^{-t} dt \right) + 3 \left(\frac{1}{\phi} \kappa^{(r-3\phi)/\phi} \int_0^\infty t^{2-r/\phi} e^{-t} dt \right) \right].$$

Then,

$$\begin{aligned} \mu'_r &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \left(\frac{1}{\phi} \kappa^{r/\phi-2} \Gamma\left(2 - \frac{r}{\phi}\right) \right) + 3 \left(\frac{1}{\phi} \kappa^{r/\phi-3} \Gamma\left(3 - \frac{r}{\phi}\right) \right) \right]. \end{aligned}$$

After simplification, we obtain

$$\mu'_r = \frac{\kappa^{r/\phi} ((\kappa^3 + 6)\phi - 3r) \Gamma\left(2 - \frac{r}{\phi}\right)}{(\kappa^3 + 6)\phi}, \quad r < 2\phi.$$

This completes the proof.

Corollary 1. The mean of the IPGD is obtained by setting $r = 1$ in the previous theorem, yielding

$$\mu = E(X) = \frac{\kappa^{1/\phi} ((\kappa^3 + 6)\phi - 3) \Gamma\left(2 - \frac{1}{\phi}\right)}{(\kappa^3 + 6)\phi}.$$

Similarly, the second ordinary moment is

$$E(X^2) = \frac{\kappa^{2/\phi} ((\kappa^3 + 6)\phi - 6) \Gamma\left(2 - \frac{2}{\phi}\right)}{(\kappa^3 + 6)\phi}.$$

Therefore, the variance of X is given by

$$\begin{aligned} \text{Var}(X) &= \frac{\kappa^{2/\phi} ((\kappa^3 + 6)\phi - 6) \Gamma\left(2 - \frac{2}{\phi}\right)}{(\kappa^3 + 6)\phi} \\ &\quad - \left[\frac{\kappa^{1/\phi} ((\kappa^3 + 6)\phi - 3) \Gamma\left(2 - \frac{1}{\phi}\right)}{(\kappa^3 + 6)\phi} \right]^2. \end{aligned}$$

Figure 4 shows the three-dimensional plots of the mean, variance, skewness, and kurtosis of the IPGD. The plots illustrate the effect of the model parameters on these measures.

3.3 Incomplete Moments

Incomplete moments play an important role in reliability analysis, income inequality measures, mean residual life functions, and actuarial applications. In this subsection, explicit expressions for the lower and upper incomplete moments of the IPGD are derived.

Theorem 3. Let X be a random variable following the IPGD with PDF (3), then, the s th lower incomplete moment of X , is given by

$$\begin{aligned} m_s(t) &= \frac{\kappa^{\frac{s}{\phi}}}{\kappa^3 + 6} \left[\kappa^3 \Gamma\left(2 - \frac{s}{\phi}, \kappa t^{-\phi}\right) + 3 \Gamma\left(3 - \frac{s}{\phi}, \kappa t^{-\phi}\right) \right], \end{aligned} \quad (6)$$

where $\Gamma(s, z)$ denotes the upper incomplete gamma function.

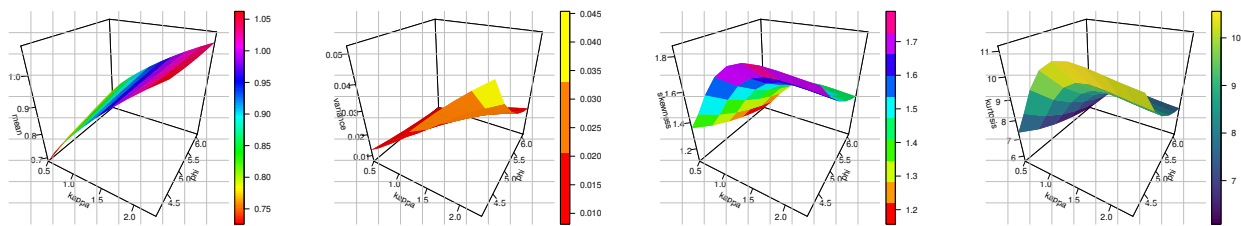


Fig. 4: 3D plots of the mean, the variance, the skewness, and the kurtosis for the IPGD

Proof. By definition,

$$m_s(t) = \int_0^t x^s f(x) dx.$$

Substituting the PDF (3) of the IPGD yields

$$\begin{aligned} m_s(t) &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \int_0^t x^{s-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3) dx \\ &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \int_0^t x^{s-2\phi-1} e^{-\kappa x^{-\phi}} dx \right. \\ &\quad \left. + 3 \int_0^t x^{s-3\phi-1} e^{-\kappa x^{-\phi}} dx \right]. \end{aligned}$$

To evaluate these integrals, consider the transformation $u = \kappa x^{-\phi}$, then

$$\begin{aligned} m_s(t) &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \frac{1}{\phi} \kappa^{\frac{s}{\phi}-2} \int_{\kappa t^{-\phi}}^{\infty} u^{1-\frac{s}{\phi}} e^{-u} du \right. \\ &\quad \left. + 3 \frac{1}{\phi} \kappa^{\frac{s}{\phi}-3} \int_{\kappa t^{-\phi}}^{\infty} u^{2-\frac{s}{\phi}} e^{-u} du \right]. \end{aligned}$$

After some simplifications, we obtain

$$\begin{aligned} m_s(t) &= \frac{\kappa^{\frac{s}{\phi}}}{\kappa^3 + 6} \left[\kappa^3 \Gamma \left(2 - \frac{s}{\phi}, \kappa t^{-\phi} \right) \right. \\ &\quad \left. + 3 \Gamma \left(3 - \frac{s}{\phi}, \kappa t^{-\phi} \right) \right]. \end{aligned}$$

This completes the proof.

Theorem 4. The s th upper incomplete moment of the IPGD, is given by

$$\begin{aligned} M_s(t) &= \frac{\kappa^{s/\phi}}{\kappa^3 + 6} \left[\kappa^3 \gamma \left(2 - \frac{s}{\phi}, \kappa t^{-\phi} \right) \right. \\ &\quad \left. + 3 \gamma \left(3 - \frac{s}{\phi}, \kappa t^{-\phi} \right) \right], \end{aligned} \quad (7)$$

where $\gamma(s, z)$ denotes the lower incomplete gamma function.

Proof. By definition,

$$M_s(t) = \int_t^\infty x^s f(x) dx.$$

Substituting the density function yields

$$\begin{aligned} M_s(t) &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \int_t^\infty x^{s-2\phi-1} e^{-\kappa x^{-\phi}} dx \right. \\ &\quad \left. + 3 \int_t^\infty x^{s-3\phi-1} e^{-\kappa x^{-\phi}} dx \right]. \end{aligned}$$

Following the derivation of the lower incomplete moments, we get

$$\begin{aligned} M_s(t) &= \frac{\kappa^{\frac{s}{\phi}}}{\kappa^3 + 6} \left[\kappa^3 \gamma \left(2 - \frac{s}{\phi}, \kappa t^{-\phi} \right) \right. \\ &\quad \left. + 3 \gamma \left(3 - \frac{s}{\phi}, \kappa t^{-\phi} \right) \right]. \end{aligned}$$

Hence, the desired result follows.

3.4 Inverse Moments and Harmonic Mean

Inverse moments are useful in reliability theory, actuarial science, and survival analysis. In this subsection, explicit expressions for the inverse moments and the harmonic mean of the IPGD are derived.

Theorem 5. Let X be a random variable following the IPGD with PDF (3), then, the s th inverse moment of X is given by

$$E(X^{-s}) = \frac{\kappa^{-s/\phi}}{\kappa^3 + 6} \left[\kappa^3 \Gamma \left(\frac{s}{\phi} + 2 \right) + 3 \Gamma \left(\frac{s}{\phi} + 3 \right) \right], \quad (8)$$

$s > 0.$

Proof. By definition, the s th inverse moment is

$$E(X^{-s}) = \int_0^\infty x^{-s} f(x) dx.$$

Substituting the PDF of the IPGD, we obtain

$$\begin{aligned} E(X^{-s}) &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \int_0^\infty x^{-s-3\phi-1} e^{-\kappa x^{-\phi}} (\kappa^2 x^\phi + 3) dx \\ &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \int_0^\infty x^{-s-2\phi-1} e^{-\kappa x^{-\phi}} dx \right. \\ &\quad \left. + 3 \int_0^\infty x^{-s-3\phi-1} e^{-\kappa x^{-\phi}} dx \right]. \end{aligned}$$

Consider the transformation $u = \kappa x^{-\phi}$, then

$$\begin{aligned} E(X^{-s}) &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \frac{1}{\phi} \kappa^{-s/\phi-2} \int_0^\infty u^{s/\phi+1} e^{-u} du \right. \\ &\quad \left. + 3 \frac{1}{\phi} \kappa^{-s/\phi-3} \int_0^\infty u^{s/\phi+2} e^{-u} du \right]. \end{aligned}$$

Then

$$\begin{aligned} E(X^{-s}) &= \frac{\kappa^3 \phi}{\kappa^3 + 6} \left[\kappa^2 \left(\frac{1}{\phi} \kappa^{-s/\phi-2} \Gamma\left(\frac{s}{\phi} + 2\right) \right) \right. \\ &\quad \left. + 3 \left(\frac{1}{\phi} \kappa^{-s/\phi-3} \Gamma\left(\frac{s}{\phi} + 3\right) \right) \right]. \end{aligned}$$

After simplification, we obtain

$$E(X^{-s}) = \frac{\kappa^{-s/\phi}}{\kappa^3 + 6} \left[\kappa^3 \Gamma\left(\frac{s}{\phi} + 2\right) + 3 \Gamma\left(\frac{s}{\phi} + 3\right) \right].$$

This completes the proof.

Corollary 2. The first inverse moment of the IPGD is obtained by setting $s = 1$ in Equation (8) of the previous theorem. Thus,

$$E(X^{-1}) = \frac{\kappa^{-1/\phi}}{\kappa^3 + 6} \left[\kappa^3 \Gamma\left(2 + \frac{1}{\phi}\right) + 3 \Gamma\left(3 + \frac{1}{\phi}\right) \right].$$

Corollary 3. The harmonic mean of a positive random variable X is defined by

$$\begin{aligned} H &= \frac{1}{E(X^{-1})} \\ &= \left[\frac{\kappa^{-1/\phi}}{\kappa^3 + 6} \left(\kappa^3 \Gamma\left(2 + \frac{1}{\phi}\right) + 3 \Gamma\left(3 + \frac{1}{\phi}\right) \right) \right]^{-1}. \end{aligned}$$

Therefore, the harmonic mean of the IPGD is

$$H = \frac{(\kappa^3 + 6) \kappa^{1/\phi}}{\kappa^3 \Gamma\left(2 + \frac{1}{\phi}\right) + 3 \Gamma\left(3 + \frac{1}{\phi}\right)}.$$

4 Estimation methods

The fifteen estimation methods used to estimate the parameters of the proposed distribution are presented in this section.

4.1 Method of maximum likelihood

Maximum likelihood estimation has also been used in related studies [18, 19, 20].

The log-likelihood function of the IPGD is given by:

$$\begin{aligned} \ell(\kappa, \phi) &= 3n \log \kappa + n \log \phi - (3\phi + 1) \sum_{i=1}^n \log x_i \\ &\quad - \kappa \sum_{i=1}^n x_i^{-\phi} + \sum_{i=1}^n \log(\kappa^2 x_i^\phi + 3) \\ &\quad - n \log(\kappa^3 + 6). \end{aligned}$$

The maximum likelihood estimates are obtained by numerically maximizing the log-likelihood function.

4.2 Method of Anderson-Darling

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$\begin{aligned} AD &= -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\log \left(e^{-\kappa x_{i:n}^{-\phi}} \right. \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{i:n}^\phi \right) + 1 \right) \right) \\ &\quad + \log \left(1 - e^{-\kappa x_{n-i+1:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{n-i+1:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{n-i+1:n}^\phi \right) + 1 \right) \right) \right]. \end{aligned}$$

4.3 Method of Cramer von Mises

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$\begin{aligned} CVM &= \frac{1}{12n} + \sum_{i=1}^n \left[e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{i:n}^\phi \right) + 1 \right) - \frac{2i-1}{2n} \right]^2. \end{aligned}$$

4.4 Method of maximum product of spacings

The parameter estimates of the proposed distribution using this method are obtained by maximizing the following function:

$$MPS = \frac{1}{n+1} \sum_{i=1}^{n+1} \log I_i(x_i),$$

where,

$$\begin{aligned} I_i(x_i) &= F(x_{i:n}) - F(x_{i-1:n}) \\ &= e^{-\kappa x_{i:n}^{-\phi}} \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) \\ &\quad - e^{-\kappa x_{i-1:n}^{-\phi}} \left(\kappa x_{i-1:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i-1:n}^{\phi} \right) + 1 \right), \end{aligned}$$

$$F(x_{0:n}) = 0 \text{ and } F(x_{n+1:n}) = 1.$$

4.5 Method of least squares

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$\begin{aligned} LS &= \sum_{i=1}^n \left[e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) - \frac{i}{n+1} \right]^2. \end{aligned}$$

4.6 Method of right tail Anderson Darling

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$\begin{aligned} RTAD &= \frac{n}{2} - 2 \sum_{i=1}^n \left(e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) \right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \log \left(1 - e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) \right). \end{aligned}$$

4.7 Method of weighted least squares

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$\begin{aligned} WLS &= \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) - \frac{i}{n+1} \right]^2. \end{aligned}$$

4.8 Method of left tail Anderson Darling

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$\begin{aligned} LTAD &= -\frac{3}{2}n + 2 \sum_{i=1}^n \left(e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) \right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \log \left(e^{-\kappa x_{i:n}^{-\phi}} \right. \\ &\quad \times \left. \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3+6} + x_{i:n}^{\phi} \right) + 1 \right) \right). \end{aligned}$$

4.9 Method of minimum spacing absolute distance

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$MSAD = \sum_{i=1}^{n+1} \left| I_i - \frac{1}{n+1} \right|.$$

4.10 Method of minimum spacing absolute-log distance

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$MSALD = \sum_{i=1}^{n+1} \left| \log I_i - \log \frac{1}{n+1} \right|.$$

4.11 Method of Anderson Darling left tail second order

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$ADLTSO = 2 \sum_{i=1}^n \log \left(e^{-\kappa x_{i:n}^{-\phi}} \times \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{i:n}^{\phi} \right) + 1 \right) \right) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{e^{-\kappa x_{i:n}^{-\phi}} \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{i:n}^{\phi} \right) + 1 \right)}.$$

4.12 Kolmogorov Method

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$KM = \max_{1 \leq i \leq n} \left[\frac{i}{n} - e^{-\kappa x_{i:n}^{-\phi}} \times \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{i:n}^{\phi} \right) + 1 \right), e^{-\kappa x_{i:n}^{-\phi}} \left(\kappa x_{i:n}^{-2\phi} \left(\frac{3\kappa}{\kappa^3 + 6} + x_{i:n}^{\phi} \right) + 1 \right) - \frac{i-1}{n} \right].$$

4.13 Method of minimum spacing square distance

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$MSSD = \sum_{i=1}^{n+1} \left(I_i - \frac{1}{n+1} \right)^2.$$

4.14 Method of minimum spacing square-log distance

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$MSSLD = \sum_{i=1}^{n+1} \left(\log I_i - \log \frac{1}{n+1} \right)^2.$$

4.15 Method of minimum spacing Linex distance

The parameter estimates of the proposed distribution using this method are obtained by minimizing the following function:

$$MSLD = \sum_{i=1}^{n+1} \left[e^{I_i - \frac{1}{n+1}} - \left(I_i - \frac{1}{n+1} \right) - 1 \right].$$

5 Numerical simulation

We performed a comprehensive Monte Carlo simulation analysis to evaluate the estimators of the IPGD, its flexibility, and its usefulness across various sample sizes. All calculations were performed according to the settings provided by the R software environment. In the simulation we performed, we used the following exact metrics, outlined as follows:

–Average bias (BIAS):

$$|Bias(\hat{\mathbf{J}})| = \frac{1}{L} \sum_{i=1}^L |\hat{\mathbf{J}} - \mathbf{J}|.$$

–Mean squared errors (MSE):

$$MSE = \frac{1}{L} \sum_{i=1}^L (\hat{\mathbf{J}} - \mathbf{J})^2.$$

–Mean relative errors (MRE):

$$MRE = \frac{1}{L} \sum_{i=1}^L |\hat{\mathbf{J}} - \mathbf{J}| / \mathbf{J}.$$

–Average absolute difference (D_{abs}):

$$D_{abs} = \frac{1}{nL} \sum_{i=1}^L \sum_{j=1}^n |F(x_{ij}; \mathbf{J}) - F(x_{ij}; \hat{\mathbf{J}})|.$$

–Maximum absolute difference (D_{max}):

$$D_{max} = \frac{1}{L} \sum_{i=1}^L \max_{j=1, \dots, n} |F(x_{ij}; \mathbf{J}) - F(x_{ij}; \hat{\mathbf{J}})|.$$

–Average scaled absolute error (ASAE):

$$ASAE = \frac{1}{n} \sum_{i=1}^n \frac{|x_i - \hat{x}_i|}{x_n - x_1},$$

where $\mathbf{J} = (\phi, \kappa)$, L is the number of simulation replications, and n is the sample size.

Tables 1 to 5 present the Monte Carlo simulation results for the IPGD across different sample sizes, with $L = 1,000$ simulation replications for each setting. The results generally show lower values of the evaluation measures as the sample size increases, suggesting improved estimation performance. The simulation rankings in Table 6 indicate that MLE achieves the smallest overall rank, followed by MPSE.

Table 1: Results of simulation measures when $\phi = 0.2$, $\kappa = 0.3$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSSE	LTADSE	MSADSE	MSALDE	ADSOE	KE	MSDSE	MSLDE	MSLNDE
20	BIAS(ϕ)	0.20806(4)	0.21588(3)	0.27147(10)	0.19377(2)	0.26401(8)	0.27535(12)	0.20367(3)	0.24090(6)	0.27312(11)	0.26308(7)	0.28201(13)	0.26842(9)	0.35580(15)	0.19312(11)	0.30163(14)
20	BIAS(κ)	0.21121(6)	0.20303(4)	0.24088(13)	0.18418(2)	0.17741(1)	0.23592(11)	0.21878(7)	0.20643(5)	0.22006(9)	0.23334(10)	0.23779(12)	0.24392(14)	0.21972(8)	0.19880(5)	0.25993(15)
20	MSE(ϕ)	0.07746(4)	0.08246(5)	0.13808(13)	0.05914(2)	0.09662(6)	0.13306(11)	0.06815(3)	0.09867(7)	0.10824(8.5)	0.10824(8.5)	0.14048(14)	0.12645(10)	0.17362(15)	0.05895(1)	0.13679(12)
20	MSE(κ)	0.07120(5)	0.06299(3)	0.09145(13)	0.04929(2)	0.04873(1)	0.09353(14)	0.08582(6)	0.07761(7)	0.07980(6)	0.08326(9)	0.09017(11)	0.09080(12)	0.07669(6)	0.06861(4)	0.10147(15)
20	MRE(ϕ)	0.13871(4)	0.14392(5)	0.18098(10)	0.13291(2)	0.17601(8)	0.18357(12)	0.13578(7)	0.16060(6)	0.18208(11)	0.17539(9)	0.18801(13)	0.17890(15)	0.23720(15)	0.12875(1)	0.20109(14)
20	MRE(κ)	0.10560(6)	0.10151(4)	0.12044(13)	0.09209(2)	0.08886(1)	0.11796(11)	0.10397(7)	0.10321(5)	0.11003(9)	0.11667(10)	0.11890(12)	0.12196(14)	0.10986(8)	0.09940(3)	0.12546(15)
20	D_{obs}	0.05668(1)	0.05820(3)	0.06547(9)	0.05830(4)	0.05792(2)	0.06371(11)	0.06108(7)	0.05916(6)	0.06596(10)	0.07415(13)	0.06648(12)	0.06448(8)	0.07490(14)	0.05862(5)	0.08127(15)
20	D_{max}	0.09115(1)	0.09201(4)	0.10747(10)	0.09167(3)	0.09362(5)	0.10781(12)	0.09680(7)	0.09483(6)	0.10619(8)	0.11516(13)	0.10773(11)	0.10675(9)	0.12156(14)	0.09141(2)	0.12645(15)
20	ASAE	0.04696(4)	0.04655(3)	0.04863(6)	0.04957(7)	0.04764(5)	0.04487(1)	0.04542(2)	0.05426(10)	0.05565(12)	0.05565(12)	0.07120(15)	0.05201(8)	0.06539(14)	0.05295(9)	0.06257(13)
20	$\Sigma Ranks$	35(3)	36(4)	97(12)	26(1)	37(5)	95(11)	49(6)	58(7)	85.5(8)	89.5(8)	113(4)	93(10)	109(13)	29(2)	128(15)
70	BIAS(ϕ)	0.12578(7)	0.14044(4)	0.17060(10)	0.14936(6)	0.15197(7)	0.18320(12)	0.13364(3)	0.17001(9)	0.17751(11)	0.14214(5)	0.18757(14)	0.15946(8)	0.18748(13)	0.12801(2)	0.22233(15)
70	BIAS(κ)	0.13979(5)	0.14102(6)	0.15448(9)	0.13001(1)	0.14117(7)	0.15298(8)	0.12598(3)	0.16688(4)	0.15518(10)	0.13954(4)	0.16715(15)	0.13438(2)	0.15941(13)	0.15905(12)	0.15864(14)
70	MSE(ϕ)	0.02511(1)	0.03192(4)	0.04793(9)	0.03229(5)	0.03515(1)	0.05115(11)	0.02763(3)	0.04919(9)	0.05539(13)	0.03288(6)	0.06499(14)	0.04134(8)	0.05282(12)	0.02549(2)	0.08592(15)
70	MSE(κ)	0.02883(4)	0.03255(7)	0.04124(11)	0.02529(1)	0.02842(2)	0.03776(8)	0.02920(5)	0.04313(13)	0.03817(9)	0.02977(6)	0.04689(15)	0.02844(3)	0.04476(14)	0.04046(6)	0.04187(12)
70	MRE(ϕ)	0.08385(1)	0.09363(4)	0.11373(10)	0.09957(6)	0.10131(7)	0.12213(12)	0.08099(3)	0.11334(9)	0.11834(11)	0.09676(5)	0.12505(14)	0.10630(6)	0.12499(13)	0.08534(1)	0.14882(15)
70	MRE(κ)	0.06990(5)	0.07051(6)	0.07724(9)	0.06501(1)	0.07039(5)	0.07648(11)	0.06742(3)	0.08344(14)	0.07759(10)	0.06977(4)	0.08358(15)	0.06719(2)	0.07957(13)	0.07953(12)	0.07932(11)
70	D_{obs}	0.03857(2)	0.04089(5)	0.04368(9)	0.04100(6)	0.04058(4)	0.04445(10)	0.03817(7)	0.04559(11)	0.04652(12)	0.04188(7)	0.04748(13)	0.03907(3)	0.04875(14)	0.04346(8)	0.05121(15)
70	D_{max}	0.06118(1)	0.06525(4)	0.07134(9)	0.06527(5)	0.06612(6)	0.07296(10)	0.06132(2)	0.07342(11)	0.07509(12)	0.06622(7)	0.07637(13)	0.06453(3)	0.07812(14)	0.06876(8)	0.08344(15)
70	ASAE	0.02986(10)	0.02780(4)	0.02982(9)	0.02712(2)	0.02895(7)	0.02401(1)	0.02401(1)	0.03102(11)	0.03373(12)	0.02971(8)	0.04092(15)	0.02882(5)	0.03764(14)	0.02887(6)	0.03542(13)
70	$\Sigma Ranks$	30(2)	44(3)	85(10)	33(3)	54(7)	80(9)	26(1)	102(12)	100(11)	52(6)	128(13)	42(4)	120(13)	62(8)	122(14)
130	BIAS(ϕ)	0.09353(2)	0.09616(3)	0.11079(7)	0.09759(4)	0.10095(5)	0.13071(12)	0.08891(1)	0.12167(10)	0.11847(9)	0.1112(8)	0.13705(14)	0.12576(11)	0.15040(15)	0.10895(6)	0.13128(13)
130	BIAS(κ)	0.08803(2)	0.09131(4)	0.10003(10)	0.09035(3)	0.09808(7)	0.08701(1)	0.09229(5)	0.09889(8)	0.12115(14)	0.10030(11)	0.10108(12)	0.09253(6)	0.11814(13)	0.09939(9)	0.12467(15)
130	MSE(ϕ)	0.01439(2)	0.01418(1)	0.02084(8)	0.01527(4)	0.01647(5)	0.02958(14)	0.01462(3)	0.02338(11)	0.02313(9)	0.01864(7)	0.02768(13)	0.02322(10)	0.03981(15)	0.01768(6)	0.022715(12)
130	MSE(κ)	0.01232(1)	0.01249(2)	0.01499(8)	0.01261(3.5)	0.01435(7)	0.01261(3.5)	0.01322(5)	0.01588(10)	0.01555(9)	0.01601(11)	0.01601(11)	0.01367(6)	0.02231(13)	0.01757(12)	0.02256(15)
130	MRE(ϕ)	0.06235(2)	0.06411(3)	0.07386(7)	0.06506(4)	0.06730(5)	0.08714(12)	0.05928(1)	0.08111(10)	0.07898(9)	0.04708(8)	0.09137(14)	0.08384(11)	0.10027(15)	0.07263(6)	0.08752(13)
130	MRE(κ)	0.0402(2)	0.04566(4)	0.05001(10)	0.04517(3)	0.04904(7)	0.04351(1)	0.04614(5)	0.04944(8)	0.06057(14)	0.05054(12)	0.05054(12)	0.04627(6)	0.05907(13)	0.04969(9)	0.06233(15)
130	D_{obs}	0.02463(1)	0.02622(3)	0.03226(7)	0.02697(4)	0.02848(8)	0.02778(5)	0.02608(2)	0.02869(9)	0.03392(13)	0.03035(11)	0.03123(12)	0.02792(6)	0.03611(14)	0.02888(10)	0.03636(15)
130	D_{max}	0.04037(1)	0.04340(4)	0.04621(6)	0.04323(3)	0.04623(5)	0.04610(5)	0.04186(2)	0.04726(10)	0.05508(13)	0.04834(11)	0.05092(12)	0.04681(9)	0.05946(15)	0.04669(8)	0.05748(14)
130	ASAE	0.01616(4)	0.01673(6)	0.01673(6)	0.01614(3)	0.01623(5)	0.01518(1)	0.01683(7)	0.01829(9)	0.01872(12)	0.01882(12)	0.02194(14)	0.01850(10)	0.02374(15)	0.01799(8)	0.02136(13)
130	$\Sigma Ranks$	17(1)	26(2)	69(7)	31.5(4)	56(6)	54.5(5)	31(3)	85(10)	106(12)	88(11)	114(13)	75(9)	128(15)	74(8)	125(14)
200	BIAS(ϕ)	0.06806(1)	0.08920(8)	0.10426(11)	0.07660(2)	0.08502(7)	0.09517(9)	0.08236(3)	0.08304(4)	0.11481(12)	0.08437(5)	0.12488(15)	0.09747(10)	0.11707(13)	0.08492(6)	0.11760(14)
200	BIAS(κ)	0.08005(5)	0.07246(3)	0.06791(6)	0.07916(6)	0.08095(8)	0.06869(2)	0.08440(10)	0.07581(5)	0.10629(14)	0.08119(9)	0.08942(12)	0.07283(4)	0.11270(15)	0.08683(11)	0.09618(13)
200	MSE(ϕ)	0.00729(1)	0.01502(10)	0.01192(7)	0.01184(6)	0.01184(6)	0.01029(3)	0.01029(3)	0.01181(12)	0.01181(12)	0.01041(4)	0.02647(15)	0.01555(11)	0.02142(13)	0.01054(5)	0.02142(13)
200	MSE(κ)	0.01059(9.5)	0.00806(3)	0.00934(6)	0.00963(4)	0.01029(7)	0.00750(2)	0.01040(8)	0.00891(8)	0.01676(14)	0.01059(9.5)	0.02647(15)	0.00847(4)	0.02024(15)	0.01143(11)	0.01604(13)
200	MRE(ϕ)	0.04538(1)	0.05947(8)	0.06951(11)	0.05107(2)	0.05668(7)	0.06344(9)	0.05491(3)	0.05536(4)	0.07654(12)	0.05625(5)	0.08325(15)	0.06498(10)	0.07804(13)	0.05661(6)	0.07840(14)
200	MRE(κ)	0.04003(3)	0.03623(3)	0.0396(1)	0.03958(6)	0.04048(8)	0.04343(2)	0.04220(10)	0.03790(5)	0.05315(14)	0.04059(9)	0.04471(12)	0.03643(4)	0.05635(15)	0.04341(11)	0.04809(13)
200	D_{obs}	0.02253(6)	0.02136(2)	0.02125(1)	0.02316(7)	0.02344(8)	0.02150(3)	0.02376(10)	0.02185(4)	0.03065(14)	0.02354(9)	0.02584(12)	0.02202(5)	0.03258(15)	0.02459(11)	0.02936(13)
200	D_{max}	0.03559(3)	0.03504(1)	0.03580(4)	0.03653(6)	0.03766(9)	0.03632(5)	0.03781(10)	0.03526(2)	0.04992(14)	0.03753(8)	0.04312(12)	0.03670(7)	0.05289(15)	0.03924(11)	0.04750(13)
200	ASAE	0.01163(2)	0.01175(3)	0.01268(7)	0.01208(5)	0.01154(1)	0.01217(6)	0.01206(4)	0.01425(11)	0.01507(12)	0.01376(9)	0.02036(15)	0.01375(8)	0.01617(14)	0.01420(10)	0.01611(13)
200	$\Sigma Ranks$	37.5(1)	38(2)	47(4.5)	42(3)	61(7.5)	47(4.5)	61(7.5)	48(6)	118(12)	67.5(10)	120(14)	63(9)	129(15)	82(11)	119(13)
240	BIAS(ϕ)	0.06242(3)	0.05745(1)	0.07821(10)	0.06674(5)	0.08385(11)	0.07160(7)	0.06026(2)	0.06454(4)	0.07582(9)	0.07377(8)	0.10483(15)	0.08361(12)	0.08598(13)	0.06787(6)	0.09398(14)
240	BIAS(κ)	0.07313(10)	0.05701(1)	0.06520(5)	0.05934(3)	0.07334(11)	0.06049(4)	0.05784(2)	0.06562(6)	0.07829(13)	0.07035(7)	0.08110(15)	0.07047(8)	0.07804(12)	0.07057(9)	0.07939(14)
240	MSE(ϕ)	0.00609(3)	0.00549(1)	0.01029(10)	0.00690(5)	0.01043(11)	0.00858(8)	0.00566(2)	0.00693(6)	0.00913(9)	0.00777(7)	0.01703(15)	0.01159(12)	0.01232(13)	0.00680(4)	0.01329(14)
240	MSE(κ)	0.00854(11)	0.00526(1)	0.00685(5)	0.00531(2)	0.00806(10)	0.00574(4)	0.00555(3)	0.00693(3)	0.00951(13)	0.00747(7)	0.01082(15)	0.00768(8)	0.00929(12)	0.00802(9)	0.00953(14)
240	MRE(ϕ)	0.04161(3)	0.03830(1)	0.05214(10)	0.04450(5)	0.05590(11)	0.04774(7)	0.04018(2)	0.04303(4)	0.05054(9)	0.04918(8)	0.06989(15)	0.05708(12)	0.05732(13)	0.04524(6)	0.06265(14)
240	MRE(κ)	0.03656(10)	0.02851(1)	0.03260(5)	0.02967(3)	0.03667(11)	0.03242(4)	0.02892(2)	0.03281(6)	0.03915(13)	0.03517(7)	0.04055(15)	0.03524(8)	0.03902(12)	0.03528(9)	0.03969(14)
240	D_{obs}	0.01992(7)	0.01650(1)	0.01939(6)	0.01707(3)	0.02125(11)	0.01850(4)	0.01673(2)	0.01875(5)	0.02265(12)	0.02068(10)	0.02318(13)	0.02051(9)	0.02343(15)	0.02018(8)	0.02335(14)
240	D_{max}	0.03196(7)	0.02639(1)	0.03172(6)	0.02766(3)	0.03477(11)	0.03056(5)	0.02706(2)	0.02969(4)	0.03603(12)	0.03354(9)	0.03798(13)	0.03372(10)	0.03800(14)	0.03200(8)	0.03853(15)
240	ASAE	0.01033(8)	0.00911(3)	0.00988(6)	0.00959(5)	0.00947(4)	0.00852(1)	0.00874(2)	0.01038(10)	0.01139(12)	0.01035(9)	0.01434(15)	0.01026(7)	0.01170(13)	0.01065(11)	0.01295(14)
240	$\Sigma Ranks$	62(6)	11(1)	63(7)	34(3)	91(11)	44(4)	19(2)	51(5)	102(12)	72(9)	131(15)	86(10)	117(13)	70(8)	127(14)
300	BIAS(ϕ)	0.05826(4)	0.05310(2)	0.06271(7)	0.04761(1)	0.06418(8)	0.06587(12)	0.05968(5)	0.05774(3)	0.06564(11)	0.06469(9)	0.10066(15)	0.06755(10)	0.06698(13)	0.06054(6)	0.07671(14)
30																

Table 2: Results of simulation measures when $\phi = 2.0$, $\kappa = 3.5$.

n	Est.	MILE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\phi}$)	0.21703(9)	0.20437(6)	0.22330(8)	0.17575(1)	0.18598(2)	0.19818(5)	0.19732(4)	0.19387(2)	0.25253(14)	0.23805(12)	0.25381(15)	0.20640(7)	0.20872(8)	0.23923(13)	0.22747(11)
20	BIAS($\hat{\kappa}$)	0.47808(2)	0.48699(3)	0.55918(4)	0.50267(4)	0.56427(4)	0.54641(4)	0.60582(4)	0.61615(12)	0.51668(6)	0.61150(11)	0.80859(15)	0.5101(5)	0.72042(14)	0.43618(11)	0.71138(13)
20	MSE($\hat{\phi}$)	0.07646(6)	0.06726(7)	0.08141(9)	0.05466(1)	0.05967(2)	0.06347(4)	0.06546(6)	0.06347(4)	0.10049(13)	0.09888(12)	0.11569(15)	0.06194(5)	0.08569(11)	0.08282(10)	0.11379(14)
20	MRE($\hat{\phi}$)	0.41441(3)	0.42524(4)	0.51538(7)	0.35618(2)	0.49341(5)	0.49341(5)	0.68980(12)	0.65906(11)	0.52381(12)	0.52381(12)	0.10490(15)	0.57604(10)	0.83003(14)	0.27942(11)	0.82257(13)
20	MSE($\hat{\kappa}$)	0.10851(9)	0.11165(10)	0.11665(10)	0.08788(1)	0.09299(2)	0.09909(5)	0.09866(1)	0.09694(3)	0.12626(14)	0.11903(12)	0.12690(15)	0.10320(7)	0.10436(8)	0.11962(13)	0.11373(11)
20	MRE($\hat{\kappa}$)	0.13659(2)	0.13914(3)	0.15977(8)	0.14362(4)	0.16135(9)	0.15612(7)	0.17309(1)	0.17604(1)	0.14762(6)	0.17472(11)	0.23103(15)	0.14600(5)	0.20583(14)	0.12462(11)	0.20325(13)
20	D_{bias}	0.05939(5)	0.06073(8)	0.06282(9)	0.05745(2)	0.05905(5)	0.05829(3)	0.06072(7)	0.05742(1)	0.07456(15)	0.07255(12)	0.07083(11)	0.06037(6)	0.07277(13)	0.06835(10)	0.07334(14)
20	D_{mse}	0.09585(6)	0.09572(5)	0.10254(7)	0.09492(1)	0.09353(2)	0.09693(3)	0.09978(8)	0.09530(3)	0.11593(13)	0.11273(13)	0.11603(12)	0.09571(4)	0.11603(12)	0.10381(10)	0.11923(14)
20	ASAE	0.04392(6)	0.03987(1)	0.04153(4)	0.04401(7)	0.03992(2)	0.04406(8)	0.04086(3)	0.04239(5)	0.06114(15)	0.05704(11)	0.05800(12)	0.04640(9)	0.05848(13)	0.04975(10)	0.06080(14)
20	$\Sigma Ranks$	50(4)	43(3)	74(10)	23(1)	38(2)	52(6)	64(8)	54(6)	103(12)	102(11)	128(15)	56(7)	107(13)	69(9)	117(14)
70	BIAS($\hat{\phi}$)	0.11279(7)	0.10671(5)	0.12730(13)	0.09153(1)	0.11088(6)	0.09969(3)	0.09593(2)	0.11373(8)	0.12858(14)	0.11803(9)	0.12561(12)	0.10452(4)	0.14030(15)	0.12519(11)	0.11873(10)
70	BIAS($\hat{\kappa}$)	0.25990(6)	0.23383(2)	0.33567(11)	0.23593(3)	0.29552(10)	0.26338(7)	0.23977(4)	0.33653(13)	0.27783(9)	0.27751(8)	0.47554(15)	0.23339(1)	0.34823(12)	0.24807(5)	0.40451(14)
70	MSE($\hat{\phi}$)	0.01887(7)	0.01669(4)	0.02459(13)	0.01289(1)	0.01845(6)	0.01453(2)	0.02003(8)	0.02565(14)	0.02565(14)	0.02127(9)	0.02244(11)	0.01842(5)	0.02274(15)	0.02214(10)	0.04445(12)
70	MSE($\hat{\kappa}$)	0.11269(9)	0.09349(2)	0.18920(12)	0.08610(1)	0.14115(10)	0.12566(7)	0.10368(4)	0.20547(13)	0.12907(14)	0.13171(9)	0.35027(15)	0.10794(5)	0.17360(11)	0.09883(3)	0.23126(14)
70	MRE($\hat{\phi}$)	0.05639(7)	0.05336(5)	0.06365(13)	0.04577(1)	0.05544(10)	0.04984(4)	0.04797(2)	0.05686(8)	0.06429(14)	0.05902(9)	0.06280(12)	0.05226(4)	0.07015(15)	0.06259(11)	0.05937(10)
70	MRE($\hat{\kappa}$)	0.07426(6)	0.06738(2)	0.09591(11)	0.06741(3)	0.08444(10)	0.07531(7)	0.06850(4)	0.10187(13)	0.07938(9)	0.07929(8)	0.13387(15)	0.06668(1)	0.09949(12)	0.07088(5)	0.11558(14)
70	D_{bias}	0.03232(6)	0.03094(5)	0.03785(11)	0.02829(1)	0.03364(7)	0.02982(3)	0.02908(2)	0.03545(8)	0.03824(12)	0.03557(9)	0.04198(14)	0.03073(4)	0.04283(15)	0.03621(10)	0.03975(13)
70	D_{mse}	0.05153(6)	0.04981(5)	0.06103(11)	0.04521(1)	0.05435(7)	0.04824(3)	0.04728(2)	0.05862(10)	0.06143(12)	0.05668(8)	0.07364(15)	0.04945(4)	0.06885(14)	0.05671(9)	0.06568(13)
70	ASAE	0.02132(7)	0.01937(2)	0.02078(4)	0.02090(5)	0.02031(3)	0.02095(6)	0.01905(1)	0.02165(9)	0.02853(12)	0.02570(11)	0.03125(14)	0.02153(8)	0.02877(13)	0.02438(10)	0.03132(15)
70	$\Sigma Ranks$	58(6)	32(1)	99(1)	17(1)	65(7)	42(3)	23(2)	90(10)	104(12)	80(9)	123(15)	36(4)	122(14)	74(9)	115(13)
130	BIAS($\hat{\phi}$)	0.06420(1)	0.06793(2)	0.08070(9)	0.06894(3)	0.08913(14)	0.07323(4)	0.08290(11)	0.07810(8)	0.07373(5)	0.07640(7)	0.08690(13)	0.07638(6)	0.09031(15)	0.08204(12)	0.08633(12)
130	BIAS($\hat{\kappa}$)	0.17803(3)	0.16773(1)	0.21071(10)	0.19347(6)	0.20720(9)	0.17621(2)	0.22386(13)	0.21878(11)	0.19580(8)	0.19468(7)	0.40744(15)	0.18302(5)	0.24032(14)	0.18079(4)	0.21939(12)
130	MSE($\hat{\phi}$)	0.00641(1)	0.00685(2)	0.00956(9)	0.00774(3)	0.01248(14)	0.00869(4)	0.01054(10)	0.00871(5)	0.00910(7)	0.00922(8)	0.01086(11)	0.00889(6)	0.01356(15)	0.01157(12)	0.01232(13)
130	MSE($\hat{\kappa}$)	0.04956(3)	0.04722(1)	0.07236(10)	0.05578(5)	0.07171(9)	0.04939(2)	0.07834(13)	0.07772(12)	0.06790(8)	0.05985(7)	0.24695(15)	0.05842(6)	0.08397(14)	0.05385(4)	0.07542(11)
130	MRE($\hat{\phi}$)	0.03210(1)	0.03397(2)	0.04035(6)	0.03447(1)	0.04547(14)	0.03661(4)	0.04451(11)	0.03905(8)	0.03687(5)	0.03820(7)	0.04345(13)	0.03819(6)	0.04516(15)	0.04102(10)	0.04316(12)
130	MRE($\hat{\kappa}$)	0.05087(3)	0.04792(1)	0.06002(10)	0.05528(6)	0.07520(9)	0.05035(2)	0.06396(11)	0.06251(11)	0.05594(8)	0.05562(7)	0.11641(15)	0.05229(5)	0.08666(14)	0.05161(5)	0.06268(12)
130	D_{bias}	0.01947(1)	0.02046(2)	0.02434(10)	0.02148(3)	0.02632(12)	0.02164(4)	0.02465(11)	0.02378(8)	0.02221(6)	0.02305(7)	0.03089(15)	0.02220(5)	0.02816(14)	0.02402(9)	0.02718(13)
130	D_{mse}	0.03180(1)	0.03331(2)	0.03943(10)	0.03491(4)	0.04255(12)	0.03465(3)	0.04030(6)	0.03935(9)	0.03637(6)	0.03760(7)	0.05606(15)	0.03615(5)	0.04517(14)	0.03847(8)	0.04323(13)
130	ASAE	0.01434(6)	0.01338(3)	0.01325(1)	0.01437(7)	0.01254(2)	0.01465(8)	0.01385(4)	0.01402(5)	0.01864(12)	0.01711(11)	0.02347(15)	0.01491(4)	0.02194(14)	0.01693(13)	0.02067(13)
130	$\Sigma Ranks$	20(2)	16(1)	78(10)	40(4)	95(11)	33(3)	97(12)	77(9)	65(6)	68(7)	127(14)	53(5)	121(13)	71(8)	111(13)
200	BIAS($\hat{\phi}$)	0.06308(5)	0.05620(1)	0.06396(8)	0.06332(7)	0.05960(3)	0.06314(6)	0.06053(4)	0.05700(2)	0.07197(13)	0.07218(14)	0.07062(11)	0.06847(10)	0.07191(12)	0.06692(9)	0.07541(15)
200	BIAS($\hat{\kappa}$)	0.14767(1)	0.15752(6)	0.18249(1)	0.16891(8)	0.17961(10)	0.15371(5)	0.14883(2)	0.19916(14)	0.15294(3)	0.18331(12)	0.32354(15)	0.15306(4)	0.17076(9)	0.16002(7)	0.19180(13)
200	MSE($\hat{\phi}$)	0.00631(6)	0.00515(2)	0.00594(5)	0.00632(7)	0.00568(3.5)	0.00661(8)	0.00568(3.5)	0.00510(1)	0.00857(14)	0.00749(11)	0.00823(13)	0.00732(9.5)	0.00795(12)	0.00702(9.5)	0.00891(15)
200	MSE($\hat{\kappa}$)	0.03720(6)	0.03838(6)	0.05346(12)	0.03802(4)	0.04802(9)	0.03692(1)	0.03737(3)	0.06059(14)	0.03851(7)	0.05012(11)	0.16199(15)	0.03834(5)	0.04899(10)	0.03865(8)	0.05540(13)
200	MRE($\hat{\phi}$)	0.03154(5)	0.02810(1)	0.03198(8)	0.03166(7)	0.02980(3)	0.03157(6)	0.03027(4)	0.02850(2)	0.03599(13)	0.03609(14)	0.03531(11)	0.03424(10)	0.03595(12)	0.03346(9)	0.03770(15)
200	MRE($\hat{\kappa}$)	0.04219(1)	0.04501(6)	0.05214(11)	0.04826(8)	0.05132(10)	0.04392(5)	0.04252(2)	0.05690(14)	0.04370(13)	0.05238(12)	0.09244(15)	0.04373(4)	0.04879(9)	0.04572(7)	0.05480(13)
200	D_{bias}	0.01854(4)	0.01691(1)	0.01947(8)	0.01944(7)	0.01857(5)	0.01896(6)	0.01793(2)	0.01844(3)	0.02143(11)	0.02165(13)	0.02547(15)	0.01978(10)	0.02157(12)	0.01998(9)	0.02332(14)
200	D_{mse}	0.02964(3.5)	0.02784(1)	0.03226(10)	0.03148(7)	0.03051(5)	0.03002(4)	0.02895(2)	0.03061(6)	0.03419(11)	0.03489(12)	0.04588(15)	0.03202(9)	0.03520(13)	0.03176(8)	0.03772(14)
200	ASAE	0.01096(3.5)	0.01103(6)	0.01084(2)	0.01148(8)	0.01096(3.5)	0.01097(5)	0.00991(1)	0.01109(7)	0.01523(12)	0.01417(11)	0.01835(15)	0.01173(9)	0.01608(13)	0.01407(10)	0.01679(14)
200	$\Sigma Ranks$	31.5(3)	30(2)	75(9)	63(6.5)	52(5)	47(4)	21.5(1)	63(6.5)	87(11)	110(13)	125(14)	70.5(8)	102(12)	76.5(10)	126(15)
240	BIAS($\hat{\phi}$)	0.05397(5)	0.04887(1)	0.06118(10)	0.05105(2)	0.05789(6)	0.05800(7.5)	0.05307(4)	0.05235(3)	0.06628(12)	0.05800(7.5)	0.06862(13)	0.05817(9)	0.07064(14)	0.06395(11)	0.07073(15)
240	BIAS($\hat{\kappa}$)	0.11247(1)	0.13686(4)	0.16365(11)	0.14249(6)	0.14418(7)	0.15068(10)	0.12910(2)	0.17276(13)	0.13388(8)	0.14205(5)	0.26892(15)	0.14704(8)	0.16762(12)	0.14941(9)	0.19162(14)
240	MSE($\hat{\phi}$)	0.00476(4)	0.00362(1)	0.00581(10)	0.00393(2)	0.00535(8)	0.00500(5)	0.00504(6)	0.00696(12)	0.00504(7)	0.00547(9)	0.00731(13)	0.00357(7)	0.00794(15)	0.00639(11)	0.00759(14)
240	MSE($\hat{\kappa}$)	0.01965(1)	0.02711(3)	0.04103(11)	0.03249(6)	0.03378(8)	0.03343(7)	0.02569(2)	0.04771(13)	0.02936(6)	0.03001(5)	0.11010(15)	0.03640(10)	0.04545(12)	0.03418(9)	0.05215(14)
240	MRE($\hat{\phi}$)	0.02699(5)	0.02444(1)	0.03059(9)	0.02553(2)	0.02894(6)	0.02900(7.5)	0.02654(4)	0.02618(3)	0.03314(12)	0.02900(7.5)	0.03431(13)	0.02908(9)	0.03532(14)	0.03198(11)	0.03537(15)
240	MRE($\hat{\kappa}$)	0.03214(1)	0.03910(4)	0.04676(11)	0.04071(6)	0.04419(7)	0.04305(10)	0.03689(2)	0.04936(13)	0.03825(5)	0.04059(5)	0.07683(15)	0.04201(8)	0.04789(13)	0.04269(9)	0.05475(14)
240	D_{bias}	0.01585(4)	0.01857(9)	0.01857(9)	0.01737(7)	0.01737(7)	0.01737(7)	0.01737(7)	0.01737(7)	0.01915(11)	0.02272(15)	0.02272(15)	0.02272(15)	0.02111(13)	0.02193(12)	0.02159(14)
240	D_{mse}	0.02500(2)	0.02438(1)	0.03047(10)	0.02607(4)	0.02814(7)	0.02789(5)	0.02553(3)	0.02827(8)	0.03054(11)	0.02811(6)	0.04138(15)	0.02872(9)	0.03406(13)	0.03073(12)	0.03530(14)
240	ASAE	0.01029(7)	0.00980(5)	0.00980(5)	0.00999(6)	0.00972(4)	0.01030(8)	0.00924(1)	0.00953(1)	0.01306(12)	0.01284(11)	0.01685(15)	0.01057(9)	0.01403(14)	0.01103(10)	0.01445(13)
240	$\Sigma Ranks$	30(3)	18(1)	88(11)	36.5(4)	60(5)	65(7)	23.5(2)	68(8)	80(10)	64(6)	129(15)	78(9)	119(13)	94(12)	127(14)
300	BIAS($\hat{\phi}$)	0.05017(8)	0.04597(5)	0.06310(11)	0.04476(2)	0.04942(6)	0.04960(7)	0.04472(5)	0.04322(1)	0.05918(12)						

Table 3: Results of simulation measures when $\phi = 3.0$, $\kappa = 0.8$.

n	Est.	MLE	ADL	CVME	MPSE	OLSE	RTADE	WLSE	LJADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSIDE	MSLNDE
20	BIAS(ϕ)	0.3632(6)	0.3248(4)	0.4286(9)	0.3192(2)	0.3646(7)	0.3975(11)	0.3688(8)	0.4399(14)	0.3702(9)	0.3529(4)	0.6246(15)	0.3170(4)	0.3930(10)	0.3610(5)	0.4169(12)
20	BIAS(ϕ)	0.1240(5)	0.1184(5)	0.1564(13)	0.1193(3)	0.1354(3)	0.1403(4)	0.1281(5)	0.1372(6)	0.1468(11)	0.1320(6)	0.1713(15)	0.1405(10)	0.1450(10)	0.1262(4)	0.1559(12)
20	MSE(ϕ)	0.2547(5)	0.2076(4)	0.3504(6)	0.1539(5)	0.2680(8)	0.3128(12)	0.2683(9)	0.3705(14)	0.2688(10)	0.2360(5)	0.7210(15)	0.1952(1)	0.2308(4)	0.2034(3)	0.3108(14)
20	MSE(ϕ)	0.0258(5)	0.0207(3)	0.0437(6)	0.0206(7)	0.0307(8)	0.0367(11)	0.0264(6)	0.0293(7)	0.0345(8)	0.0258(4)	0.0450(15)	0.0363(9)	0.0236(5)	0.0399(12)	0.0399(12)
20	MRE(ϕ)	0.1212(6)	0.1082(8)	0.1429(13)	0.1064(1)	0.1221(7)	0.1342(11)	0.1229(6)	0.1466(14)	0.1234(9)	0.1176(4)	0.2082(15)	0.1056(8)	0.1310(10)	0.1203(5)	0.1389(12)
20	MRE(ϕ)	0.1550(6)	0.1486(1)	0.1955(10)	0.1491(6)	0.1692(9)	0.1754(9)	0.1601(8)	0.1709(8)	0.1831(10)	0.1640(6)	0.2154(15)	0.1806(10)	0.2052(14)	0.1577(8)	0.1949(12)
20	D_{bias}	0.0535(1)	0.0585(2)	0.0613(7)	0.0586(5)	0.0586(6)	0.0612(5)	0.0586(6)	0.0606(7)	0.0682(10)	0.0628(8)	0.0658(12)	0.0586(4)	0.0732(15)	0.0615(10)	0.0712(14)
20	D_{max}	0.0883(8)	0.0892(2)	0.1024(8)	0.0912(5)	0.0950(1)	0.0985(9)	0.0938(4)	0.0951(8)	0.1084(12)	0.1002(8)	0.1130(13)	0.0950(9)	0.1175(15)	0.0982(7)	0.1156(14)
20	ASAE	0.0448(7)	0.0433(4)	0.0461(8)	0.0415(6)	0.0445(5)	0.0396(1)	0.0425(3)	0.0553(12)	0.0470(10)	0.0481(11)	0.0836(15)	0.0465(9)	0.0627(13)	0.0447(6)	0.0629(14)
20	$\Sigma Ranks$	37(3)	22(2)	107(12)	20(1)	60(7)	81(9)	51(5)	92(10)	95(11)	61(8)	130(15)	52(6)	109(13)	49(4)	114(14)
70	BIAS(ϕ)	0.1909(6)	0.1762(1)	0.2102(11)	0.1863(4)	0.2105(12)	0.1797(2)	0.2002(8)	0.2166(13)	0.1891(5)	0.2056(7)	0.3198(15)	0.1843(4)	0.2238(14)	0.1952(7)	0.2084(10)
70	BIAS(ϕ)	0.0648(5)	0.0583(1)	0.0794(12)	0.0606(2)	0.0690(3)	0.0739(11)	0.0633(3)	0.0763(8)	0.0779(11)	0.0709(8)	0.0937(14)	0.0597(14)	0.0907(15)	0.0656(6)	0.0850(13)
70	MSE(ϕ)	0.0660(4)	0.0503(1)	0.0706(11)	0.0562(4)	0.0707(12)	0.0545(6)	0.0660(5)	0.0741(6)	0.0624(5)	0.0634(6)	0.1945(15)	0.0570(3)	0.0791(14)	0.0605(5)	0.0633(7)
70	MSE(ϕ)	0.0067(6)	0.0052(1)	0.0102(12)	0.0055(8)	0.0071(4)	0.0065(5)	0.0064(3)	0.0098(11)	0.0097(10)	0.0073(4)	0.0137(15)	0.0095(7)	0.0113(14)	0.0065(4)	0.0105(4)
70	MRE(ϕ)	0.0633(6)	0.0586(1)	0.0707(11)	0.0622(1)	0.0702(12)	0.0599(2)	0.0666(7)	0.0720(10)	0.0633(5)	0.0685(6)	0.1060(15)	0.0614(5)	0.0746(14)	0.0650(7)	0.0694(10)
70	MRE(ϕ)	0.0810(6)	0.0758(2)	0.0993(12)	0.0758(5)	0.0862(9)	0.0795(3)	0.0803(7)	0.0967(4)	0.0967(4)	0.0887(8)	0.1213(15)	0.0936(6)	0.1213(15)	0.0820(6)	0.1062(14)
70	D_{bias}	0.0304(2)	0.0291(1)	0.0339(8)	0.0314(2)	0.0342(10)	0.0317(2)	0.0327(6)	0.0342(7)	0.0364(12)	0.0345(11)	0.0376(15)	0.0312(9)	0.0421(15)	0.0342(8)	0.0379(14)
70	D_{max}	0.0497(9)	0.0467(3)	0.0556(9)	0.0503(5)	0.0550(8)	0.0510(5)	0.0527(6)	0.0560(12)	0.0581(10)	0.0562(11)	0.0645(14)	0.0514(7)	0.0694(15)	0.0544(7)	0.0610(13)
70	ASAE	0.0184(5)	0.0193(6)	0.0200(7)	0.0182(4)	0.0197(3)	0.0175(1)	0.0183(9)	0.0241(12)	0.0233(10)	0.0221(10)	0.0352(15)	0.0215(4)	0.0279(14)	0.0269(8)	0.0277(13)
70	$\Sigma Ranks$	45(4)	13(1)	92(11)	27(2)	82(8)	27(2)	52(5)	100(12)	83(10)	82(8)	130(15)	53(6)	118(13)	58(7)	106(13)
130	BIAS(ϕ)	0.1205(1)	0.1353(6)	0.1482(9)	0.1194(3)	0.1521(10)	0.1295(4)	0.1298(5)	0.1669(12)	0.1528(11)	0.1627(7)	0.2572(15)	0.1255(3)	0.1768(14)	0.1368(8)	0.1756(13)
130	BIAS(ϕ)	0.0418(3)	0.0447(7)	0.0560(11)	0.0406(4)	0.0539(8)	0.0483(6)	0.0474(3)	0.0545(8)	0.0578(8)	0.0527(8)	0.0719(15)	0.0545(9)	0.0577(12)	0.0404(3)	0.0622(14)
130	MSE(ϕ)	0.0263(1)	0.0290(1)	0.0368(9)	0.0213(9)	0.0334(9)	0.0284(8)	0.0263(6)	0.0428(12)	0.0394(11)	0.0279(5)	0.1169(15)	0.0272(4)	0.0475(14)	0.0305(9)	0.0474(13)
130	MSE(ϕ)	0.0029(4)	0.0035(4)	0.0051(12)	0.0025(7)	0.0044(8)	0.0030(6)	0.0036(5)	0.0048(10)	0.0050(5)	0.0042(1)	0.0081(15)	0.0044(9)	0.0051(13)	0.0028(2)	0.0062(4)
130	MRE(ϕ)	0.0401(7)	0.0451(2)	0.0494(3)	0.0398(1)	0.0507(10)	0.0431(8)	0.0432(9)	0.0505(12)	0.0505(11)	0.0421(7)	0.0857(15)	0.0418(5)	0.0586(14)	0.0456(8)	0.0585(13)
130	MRE(ϕ)	0.0522(8)	0.0596(4)	0.0701(11)	0.0508(2)	0.0674(8)	0.0604(5)	0.0592(5)	0.0682(10)	0.0723(13)	0.0659(7)	0.0897(15)	0.0681(4)	0.0721(12)	0.0505(1)	0.0778(14)
130	D_{bias}	0.0201(4)	0.0228(5)	0.0244(9)	0.0213(3)	0.0239(8)	0.0223(4)	0.0217(8)	0.0241(9)	0.0283(12)	0.0250(6)	0.0298(15)	0.0228(2)	0.0286(13)	0.0227(5)	0.0291(10)
130	D_{max}	0.0326(8)	0.0367(6)	0.0404(11)	0.0338(5)	0.0389(8)	0.0362(7)	0.0354(4)	0.0399(9)	0.0455(12)	0.0400(6)	0.0515(15)	0.0374(3)	0.0465(13)	0.0363(4)	0.0472(14)
130	ASAE	0.0117(8)	0.0129(5)	0.0130(7)	0.0118(5)	0.0120(4)	0.0115(1)	0.0122(4)	0.0150(12)	0.0152(10)	0.0152(9)	0.0245(15)	0.0144(8)	0.0174(13)	0.0133(8)	0.0183(14)
130	$\Sigma Ranks$	19(2)	49(6)	90(10)	15(1)	75(9)	41(4)	38(3)	96(11)	105(12)	71(8)	135(15)	59(7)	118(13)	46(5)	123(14)
200	BIAS(ϕ)	0.1105(6)	0.1056(4)	0.1106(7)	0.1007(4)	0.1113(8)	0.1053(3)	0.1125(9)	0.1450(14)	0.1217(10)	0.1238(11)	0.2077(15)	0.1091(2)	0.1448(13)	0.1050(2)	0.1458(12)
200	BIAS(ϕ)	0.0335(1)	0.0347(2)	0.0415(9)	0.0360(5)	0.0435(8)	0.0386(6)	0.0428(8)	0.0488(12)	0.0398(1)	0.0398(1)	0.0572(15)	0.0410(2)	0.0478(13)	0.0400(4)	0.0536(14)
200	MSE(ϕ)	0.0192(9)	0.0166(2)	0.0211(8)	0.0163(2)	0.0213(9)	0.0180(7)	0.0188(5)	0.0337(14)	0.0246(11)	0.0234(6)	0.0652(15)	0.0204(5)	0.0325(12)	0.0169(3)	0.0327(13)
200	MSE(ϕ)	0.0018(7)	0.0019(3)	0.0029(9)	0.0019(3)	0.0031(4)	0.0024(4)	0.0023(8)	0.0303(18)	0.0300(10)	0.0025(1)	0.0052(15)	0.0025(8)	0.0037(4)	0.0023(4)	0.0042(6)
200	MRE(ϕ)	0.0368(4)	0.0351(2)	0.0368(7)	0.0335(8)	0.0371(8)	0.0351(3)	0.0375(1)	0.0485(14)	0.0405(7)	0.0413(11)	0.0691(15)	0.0363(7)	0.0482(13)	0.0350(2)	0.0479(12)
200	MRE(ϕ)	0.0419(4)	0.0434(1)	0.0451(2)	0.0450(6)	0.0544(12)	0.0467(7)	0.0482(5)	0.0534(8)	0.0534(8)	0.0497(6)	0.0717(15)	0.0512(7)	0.0597(6)	0.0500(5)	0.0670(14)
200	D_{bias}	0.0176(5)	0.0176(1)	0.0183(5)	0.0180(3)	0.0190(8)	0.0180(3)	0.0189(3)	0.0201(9)	0.0206(12)	0.0202(11)	0.0246(15)	0.0185(6)	0.0236(13)	0.0186(5)	0.0241(14)
200	D_{max}	0.0284(7)	0.0282(1)	0.0303(2)	0.0289(7)	0.0317(9)	0.0292(4)	0.0302(6)	0.0333(10)	0.0337(12)	0.0328(10)	0.0424(15)	0.0303(8)	0.0381(13)	0.0302(5)	0.0393(14)
200	ASAE	0.0092(4)	0.0094(4)	0.0103(9)	0.0089(4)	0.0101(6)	0.0084(5)	0.0095(5)	0.0113(11)	0.0113(12)	0.0112(10)	0.0194(15)	0.0083(9)	0.0137(13)	0.0104(8)	0.0142(14)
200	$\Sigma Ranks$	28(3)	23(2)	67(8)	19(1)	84(10)	31(4)	58(5)	108(12)	97(11)	82(9)	135(15)	64(7)	116(13)	46(5)	121(14)
240	BIAS(ϕ)	0.0915(1)	0.0957(4)	0.1028(7)	0.0927(2)	0.1086(11)	0.0957(4)	0.0956(5)	0.1187(9)	0.1068(9)	0.1073(10)	0.1836(15)	0.0952(3)	0.1285(13)	0.1049(8)	0.1239(13)
240	BIAS(ϕ)	0.0304(1)	0.0328(3)	0.0358(8)	0.0320(2)	0.0372(9)	0.0345(4)	0.0357(3)	0.0421(9)	0.0398(5)	0.0345(4)	0.0521(15)	0.0397(10)	0.0454(13)	0.0345(4)	0.0457(14)
240	MSE(ϕ)	0.0137(9)	0.0149(5)	0.0182(9)	0.0136(8)	0.0192(11)	0.0141(3)	0.0149(4)	0.0240(12)	0.0183(10)	0.0175(8)	0.0543(15)	0.0153(6)	0.0292(14)	0.0167(7)	0.0260(13)
240	MSE(ϕ)	0.0013(4)	0.0016(2)	0.0021(5)	0.0016(8)	0.0021(3)	0.0024(7)	0.0019(4)	0.0269(12)	0.0240(10)	0.0019(4)	0.0042(15)	0.0024(5)	0.0032(14)	0.0019(2)	0.0031(13)
240	MRE(ϕ)	0.0305(1)	0.0319(1)	0.0342(9)	0.0309(1)	0.0362(11)	0.0318(4)	0.0318(5)	0.0396(12)	0.0356(9)	0.0357(10)	0.0612(15)	0.0317(4)	0.0428(14)	0.0349(8)	0.0413(13)
240	MRE(ϕ)	0.0380(1)	0.0410(4)	0.0448(8)	0.0400(3)	0.0465(9)	0.0418(6)	0.0446(6)	0.0524(18)	0.0498(11)	0.0431(13)	0.0562(15)	0.0496(3)	0.0568(13)	0.0431(4)	0.0571(14)
240	D_{bias}	0.0147(1)	0.0156(4)	0.0167(7)	0.0167(6)	0.0174(8)	0.0159(2)	0.0162(3)	0.0182(12)	0.0181(11)	0.0181(11)	0.0216(15)	0.0159(3)	0.0223(10)	0.0181(2)	0.0213(16)
240	D_{max}	0.0240(5)	0.0255(3)	0.0272(9)	0.0267(2)	0.0286(8)	0.0254(2)	0.0263(4)	0.0294(11)	0.0302(12)	0.0291(10)	0.0377(15)	0.0266(7)	0.0358(14)	0.0289(9)	0.0347(13)
240	ASAE	0.0089(4)	0.0079(1)	0.0087(6)	0.0080(3)	0.0088(9)	0.0078(1)	0.0081(4)	0.0105(11)	0.0108(7)	0.0094(6)	0.0165(15)	0.0095(9)	0.0121(13)	0.0095(9)	0.0124(14)
240	$\Sigma Ranks$	13(1)	33(3)	68(8)	27(2)	82(10)	35(4)	48(5)	103(12)	96(11)	70(9)	134(15)	61(6)	124(14)	66(7)	120(13)
300	BIAS(ϕ)	0.0811(6)	0.0848(3)	0.0912(12)	0.0866(4)	0.0880(6)	0.0828(2)	0.0885(7)	0.0975(9)	0.0975(11)	0.0972(10)	0.1527(15)	0.0872(8)	0.1106(13)	0.0891(8)	0.1175(14)
300	BIAS(ϕ)	0.0270(7)	0.0288(4)	0.0341(9)	0.0288(2)	0.0354(11)	0.0318(3)	0.0346(8)	0.0436(18)	0.0359(6)	0.0316(7)	0.0473(15)	0.0352(10)	0.0383(13)	0.0303(5)	0.0391(14)
300	MSE(ϕ)	0.0110(6)	0.0116(2)	0.0165(4)	0.0116(4)	0.0120(5)	0.0100(1)	0.0128(8)	0.0149(10)	0.0153(9)	0.0141(8)	0.0369(15)	0.0122(4)	0.0200(13)	0.0127(7)	0.0229(14)
300	MSE(ϕ)	0.0011(7)	0.0014(5)	0.00186(5)	0.00132(2)	0.00202(6)	0.00167(7)	0.00133(3)	0.00186(8)	0.0021(10)	0.0015(6)	0.0032(15)	0.0020(6)	0.0022(13)	0.0014(6)	0.0024(14)
300	MRE(ϕ)	0.0270(5)	0.0282(3)	0.0337(12)	0.0289(4)	0.0337(12)										

Table 4: Results of simulation measures when $\phi = 0.9$, $\kappa = 1.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSE	MSLSE	MSLNDE
20	BIAS(ϕ)	0.11830(2)	0.12396(3)	0.14936(10)	0.12895(4)	0.14357(7)	0.15202(12)	0.14227(6)	0.14849(9)	0.14034(5)	0.16164(15)	0.15074(11)	0.11513(1)	0.15302(13)	0.14645(8)	0.15959(14)
20	BIAS(κ)	0.13932(1)	0.16523(5)	0.21213(13)	0.15540(4)	0.18159(7)	0.19152(11)	0.18335(8)	0.19017(9)	0.16768(6)	0.19028(10)	0.26790(12)	0.14072(2)	0.21852(14)	0.15129(3)	0.21204(12)
20	MSE(ϕ)	0.02200(2)	0.02402(3)	0.03591(12)	0.02525(4)	0.03496(2)	0.03496(2)	0.03496(2)	0.03496(2)	0.03496(2)	0.03496(2)	0.03496(2)	0.02155(1)	0.03939(14)	0.03436(10)	0.04224(15)
20	MRE(ϕ)	0.03296(1)	0.04359(5)	0.08234(14)	0.03496(2)	0.05277(7)	0.06214(11)	0.05816(10)	0.05691(9)	0.04625(6)	0.05377(8)	0.12428(11)	0.03569(3)	0.07570(13)	0.03878(4)	0.06792(12)
20	MRE(κ)	0.13145(2)	0.13773(3)	0.16596(10)	0.14328(4)	0.15953(7)	0.16892(12)	0.15808(6)	0.16993(9)	0.15593(5)	0.17960(15)	0.16749(11)	0.12792(3)	0.17103(13)	0.16272(8)	0.17732(14)
20	MRE(ϕ)	0.09288(1)	0.11015(5)	0.14142(13)	0.10227(4)	0.12106(7)	0.12768(11)	0.12224(8)	0.12678(9)	0.11179(6)	0.12685(10)	0.17860(15)	0.09381(2)	0.14568(14)	0.10086(3)	0.14136(12)
20	D_{max}	0.05568(1)	0.07773(2)	0.06047(5)	0.05817(3)	0.06102(6)	0.06028(4)	0.06210(6)	0.06210(6)	0.06413(10)	0.06853(12)	0.06977(14)	0.06114(7)	0.06896(14)	0.06484(11)	0.07241(15)
20	D_{max}	0.03030(3)	0.03030(3)	0.01098(9)	0.09062(2)	0.09062(2)	0.09062(2)	0.09062(2)	0.09062(2)	0.10250(10)	0.11020(12)	0.12299(15)	0.09734(4)	0.11014(15)	0.10145(8)	0.11671(14)
20	ASAE	0.08657(1)	0.16646(5)	0.43459(15)	0.28742(2)	0.22859(10)	0.3631(14)	0.29312(12)	0.29312(12)	0.11659(3)	0.22208(9)	0.26037(11)	0.2580(4)	0.33750(13)	0.21347(7)	0.20613(6)
20	$\Sigma Ranks$	12(1)	34(4)	101(11)	29(3)	65(7)	89(10)	72(8)	79(9)	56(5)	104(12)	118(14)	25(2)	120(15)	114(13)	124(15)
70	BIAS(ϕ)	0.06457(1)	0.06472(2)	0.08150(11)	0.07217(6)	0.07277(7)	0.07653(9)	0.07181(5)	0.06551(3)	0.09185(14)	0.08241(12)	0.08093(10)	0.07090(4)	0.08738(13)	0.07535(8)	0.09267(15)
70	BIAS(κ)	0.07120(1)	0.08217(2)	0.09761(11)	0.08334(5)	0.08754(7)	0.09152(9)	0.09128(8)	0.08620(9)	0.10556(12)	0.09740(10)	0.14020(15)	0.08316(4)	0.11261(13)	0.08254(3)	0.11956(14)
70	MSE(ϕ)	0.00657(1)	0.00643(1)	0.01074(12)	0.00820(6)	0.00799(5)	0.00905(8)	0.00838(7)	0.00722(3)	0.01302(14)	0.01016(10)	0.01500(11)	0.00788(4)	0.01121(13)	0.00915(9)	0.01307(15)
70	MSE(κ)	0.00867(1)	0.01099(4)	0.01514(10)	0.01081(3)	0.01198(6)	0.01297(7)	0.01359(9)	0.01353(8)	0.01721(12)	0.01521(11)	0.03255(15)	0.01163(5)	0.01880(13)	0.01065(2)	0.02165(14)
70	MRE(ϕ)	0.07174(1)	0.07191(2)	0.09056(11)	0.08019(6)	0.08086(7)	0.08503(9)	0.07979(5)	0.07279(3)	0.10205(14)	0.09157(12)	0.08992(10)	0.07878(4)	0.09709(13)	0.08372(8)	0.10297(15)
70	MRE(κ)	0.04747(1)	0.05478(2)	0.06507(11)	0.05556(5)	0.05836(7)	0.06102(9)	0.06083(8)	0.05747(6)	0.07037(12)	0.06493(10)	0.09347(15)	0.05544(4)	0.07447(13)	0.05503(3)	0.07971(14)
70	D_{max}	0.03030(3)	0.02840(1)	0.03483(10)	0.03148(6)	0.03286(8)	0.03092(4)	0.02963(2)	0.03168(7)	0.03893(12)	0.03502(11)	0.04087(15)	0.03125(5)	0.04017(14)	0.03413(9)	0.03976(13)
70	D_{max}	0.04872(2)	0.04655(1)	0.05757(11)	0.05087(5)	0.05303(8)	0.05145(6)	0.04919(3)	0.05190(7)	0.06384(12)	0.05940(10)	0.07039(15)	0.05080(4)	0.06515(13)	0.05465(9)	0.06551(14)
70	ASAE	0.02876(4)	0.02474(2)	0.04691(14)	0.02346(1)	0.02565(3)	0.04444(12)	0.04076(9)	0.04027(8)	0.04559(13)	0.03718(7)	0.05245(15)	0.03014(5)	0.04324(11)	0.03485(6)	0.04149(10)
70	$\Sigma Ranks$	16(1)	17(2)	101(11)	43(4)	58(9)	73(9)	56(6)	51(5)	115(12)	93(10)	121(14)	39(3)	116(13)	57(7)	124(15)
130	BIAS(ϕ)	0.04984(2)	0.05039(6)	0.05032(3)	0.04542(1)	0.05189(5)	0.06162(11)	0.05546(8)	0.05417(7)	0.06912(14)	0.05636(9)	0.06369(12)	0.05110(4)	0.06536(13)	0.06023(10)	0.07148(15)
130	BIAS(κ)	0.05570(3)	0.05971(4)	0.06037(5)	0.05293(1)	0.06568(8)	0.07196(12)	0.06109(6)	0.07276(12)	0.07265(11)	0.06159(11)	0.11782(15)	0.05427(2)	0.08469(14)	0.07239(10)	0.08569(13)
130	MSE(ϕ)	0.00379(2)	0.00438(6)	0.00397(3)	0.00329(1)	0.00420(5)	0.00562(11)	0.00484(8)	0.00468(7)	0.00776(14)	0.00515(9)	0.00640(12)	0.00418(4)	0.00664(13)	0.00561(10)	0.00816(15)
130	MSE(κ)	0.00479(2)	0.00579(5)	0.00577(4)	0.00463(1)	0.00643(8)	0.00783(9)	0.00615(7)	0.00873(11)	0.00886(12)	0.00599(6)	0.02263(15)	0.00534(3)	0.01107(14)	0.00803(10)	0.01067(13)
130	MRE(ϕ)	0.05538(2)	0.05899(6)	0.0591(3)	0.05047(1)	0.05766(5)	0.06847(11)	0.06162(8)	0.06019(7)	0.07679(14)	0.06262(9)	0.07076(12)	0.05678(4)	0.07262(13)	0.06692(10)	0.07942(15)
130	MRE(κ)	0.03713(3)	0.03981(4)	0.04024(5)	0.03529(1)	0.04379(8)	0.04797(9)	0.04851(12)	0.04843(11)	0.04067(12)	0.04106(7)	0.07855(15)	0.03618(2)	0.05646(14)	0.04826(10)	0.05580(13)
130	D_{max}	0.02252(3)	0.02254(4)	0.02208(2)	0.02181(1)	0.02298(5)	0.02427(7)	0.02382(6)	0.02515(9)	0.02978(13)	0.02517(10)	0.03129(15)	0.02472(8)	0.02946(12)	0.02588(11)	0.03080(14)
130	D_{max}	0.03606(3)	0.03670(4)	0.03609(2)	0.03502(1)	0.03804(5)	0.04002(8)	0.03853(6)	0.04198(10)	0.04787(12)	0.04046(9)	0.05506(15)	0.03966(7)	0.04852(11)	0.04237(11)	0.05067(14)
130	ASAE	0.01338(1)	0.01676(5)	0.01377(3)	0.01426(5)	0.01544(4)	0.01697(7)	0.01715(9)	0.01995(11)	0.02768(12)	0.01913(10)	0.02323(15)	0.01709(8)	0.02338(12)	0.02433(13)	0.02433(13)
130	$\Sigma Ranks$	21(2)	44(5)	29(3)	11(1)	53(6)	84(9)	62(7)	86(10)	115(12)	76(8)	126(15)	40(4)	118(13)	90(11)	125(14)
200	BIAS(ϕ)	0.03973(2)	0.04088(4)	0.04480(7)	0.04324(6)	0.03927(1)	0.04729(9)	0.04839(11)	0.04194(5)	0.05011(13)	0.04664(8)	0.04965(12)	0.04029(3)	0.05420(14)	0.04757(10)	0.05588(15)
200	BIAS(κ)	0.04385(1)	0.05216(5)	0.05575(8)	0.04750(3)	0.05380(6)	0.05769(12)	0.05669(9)	0.05675(10)	0.05702(11)	0.05554(7)	0.09476(15)	0.04636(2)	0.06324(13)	0.05030(4)	0.06846(14)
200	MSE(ϕ)	0.00244(1)	0.00281(3)	0.00312(2)	0.00282(6)	0.00250(2)	0.00353(12)	0.00341(10)	0.00276(4)	0.00384(12)	0.00319(8)	0.00388(13)	0.00259(3)	0.00453(14)	0.00327(9)	0.00493(15)
200	MSE(κ)	0.00314(1)	0.00433(5)	0.00476(8)	0.00347(2)	0.00438(6)	0.00502(12)	0.00407(7)	0.00492(11)	0.00498(11)	0.00468(7)	0.01388(15)	0.00285(3)	0.00643(13)	0.00413(4)	0.00702(14)
200	MRE(ϕ)	0.04414(2)	0.04543(4)	0.04978(7)	0.04803(6)	0.04363(1)	0.05254(9)	0.05377(11)	0.04660(5)	0.05568(13)	0.05182(8)	0.05516(12)	0.04477(3)	0.06022(14)	0.05285(10)	0.06209(15)
200	MRE(κ)	0.02923(2)	0.03478(5)	0.03716(8)	0.03167(3)	0.03593(6)	0.03846(12)	0.03779(9)	0.03783(10)	0.03801(11)	0.03703(7)	0.06317(15)	0.03091(2)	0.04216(13)	0.03353(4)	0.04564(14)
200	D_{max}	0.01832(4)	0.01773(1)	0.01914(8)	0.01907(7)	0.01820(2)	0.01888(6)	0.01873(5)	0.01920(9)	0.02240(12)	0.02017(10)	0.02513(15)	0.01830(3)	0.02401(13)	0.02044(11)	0.02424(14)
200	D_{max}	0.02954(2)	0.02953(1)	0.03181(6)	0.03064(5)	0.03035(4)	0.03141(7)	0.03128(6)	0.03199(9)	0.03637(12)	0.03301(11)	0.04424(15)	0.02975(5)	0.03884(13)	0.03286(10)	0.04000(14)
200	ASAE	0.01080(2)	0.01082(3)	0.01265(10)	0.01095(4)	0.01101(5)	0.01126(6)	0.01026(1)	0.01211(8)	0.01530(12)	0.01342(11)	0.02020(15)	0.01156(7)	0.01674(13)	0.01238(9)	0.01691(14)
200	$\Sigma Ranks$	16(1)	33(5)	71(8)	42(5)	33(3)	84(11)	71(8)	70(6)	107(12)	77(10)	127(14)	29(2)	120(13)	71(8)	129(15)
240	BIAS(ϕ)	0.03389(1)	0.03861(9)	0.03807(8)	0.03762(7)	0.03708(5)	0.03713(6)	0.03628(4)	0.03524(2)	0.04738(13)	0.04473(11)	0.04733(15)	0.03581(3)	0.05381(15)	0.04049(10)	0.05231(14)
240	BIAS(κ)	0.04025(1)	0.04545(6)	0.04706(8)	0.04364(4)	0.04519(5)	0.04670(7)	0.04206(2)	0.05137(11)	0.05132(12)	0.04925(10)	0.08303(15)	0.04452(5)	0.06154(14)	0.04781(9)	0.05934(13)
240	MSE(ϕ)	0.00185(1)	0.00243(9)	0.00218(6)	0.00213(4)	0.00212(3)	0.00222(8)	0.00214(5)	0.00200(2)	0.00345(12)	0.00299(11)	0.00299(11)	0.00219(7)	0.00431(14)	0.00248(10)	0.00434(15)
240	MSE(κ)	0.00278(1)	0.00347(7)	0.00338(6)	0.00279(2)	0.00315(4)	0.00355(8)	0.00295(3)	0.00431(11)	0.00445(12)	0.00365(10)	0.01067(15)	0.00322(5)	0.00576(14)	0.00355(9)	0.00549(15)
240	MRE(ϕ)	0.03765(1)	0.04290(9)	0.04230(8)	0.04180(7)	0.04120(5)	0.04125(6)	0.04032(4)	0.03915(2)	0.05265(13)	0.04970(11)	0.04970(11)	0.03979(3)	0.05979(15)	0.04499(10)	0.05812(14)
240	MRE(κ)	0.02683(1)	0.03030(6)	0.03137(8)	0.02909(4)	0.03013(5)	0.03113(7)	0.02804(2)	0.03425(11)	0.03541(12)	0.03284(10)	0.05353(15)	0.02834(5)	0.04103(14)	0.03187(9)	0.03956(13)
240	D_{max}	0.01546(1)	0.01663(6)	0.01616(3)	0.01718(8)	0.01777(9)	0.01633(4)	0.01611(2)	0.01661(2)	0.02136(12)	0.01944(11)	0.02239(15)	0.01699(7)	0.02239(15)	0.01780(10)	0.02217(13)
240	D_{max}	0.02525(1)	0.02716(5)	0.02688(5)	0.02774(7)	0.02895(9)	0.02710(4)	0.02649(3)	0.02800(8)	0.03444(12)	0.03147(11)	0.04098(15)	0.0272(6)	0.03637(14)	0.02897(10)	0.03611(13)
240	ASAE	0.00983(6)	0.00979(5)	0.00945(2)	0.00988(7)	0.0101(8)	0.00997(3)	0.00931(1)	0.00964(4)	0.01298(12)	0.01222(11)	0.01665(15)	0.01149(9)	0.01674(13)	0.01151(10)	0.01470(13)
240	$\Sigma Ranks$	14(1)	62(9)	52(5)	50(4)	33(6)	53(6)	25(2)	56(8)	111(12)	97(11)	124(14)	46(3)	128(15)	87(10)	121(13)
300	BIAS(ϕ)	0.02913(1)	0.03239(4)	0.03423(6)	0.03447(7)	0.03639(9)	0.03630(8)	0.03308(3)	0.03374(5)	0.04302(14)	0.04097(11)	0.04100(12)	0.03188(2)	0.04543(15)	0.03716(10)	0.04204(13)
300																

Table 5: Results of simulation measures when $\phi = 4.0$, $\kappa = 6.0$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSDSE	MISSLE	MSLNDE
20	BIAS(ϕ)	0.59672(4)	0.62509(7)	0.71388(13)	0.61030(5)	0.69311(11)	0.61272(6)	0.64440(9)	0.80265(14)	0.56559(2)	0.57527(3)	0.98713(15)	0.63820(8)	0.71085(12)	0.51428(1)	0.67619(10)
20	BIAS(ϕ)	0.97167(6)	0.89811(3)	0.98110(7)	0.96611(5)	1.02997(12)	1.01475(10)	0.99006(9)	1.03562(13)	0.88376(2)	0.98345(8)	1.02450(11)	0.71688(1)	0.82766(15)	0.93628(4)	1.13419(4)
20	MSE(ϕ)	0.69566(5)	0.79793(8)	1.06128(13)	0.67159(4)	0.77892(7)	0.75077(6)	0.77892(7)	1.22631(14)	0.51329(2)	0.54578(3)	1.62212(15)	0.43362(10)	0.83536(10)	0.83962(1)	0.80632(9)
20	MSE(ϕ)	1.55327(7)	1.38951(4)	1.68093(9)	1.42520(6)	1.63534(9)	1.89269(13)	1.62312(8)	1.72940(11)	1.25380(1)	1.39203(5)	1.74382(12)	0.58325(11)	2.08836(11)	1.36629(3)	2.05956(4)
20	MRE(ϕ)	0.14918(4)	0.15627(7)	0.17847(13)	0.15257(5)	0.17328(9)	0.15318(8)	0.16101(9)	0.20666(14)	0.14140(2)	0.14382(3)	0.24678(15)	0.15955(8)	0.17771(12)	0.12857(1)	0.16905(10)
20	MRE(ϕ)	0.16194(6)	0.14968(3)	0.16352(7)	0.16102(5)	0.17166(12)	0.16912(10)	0.16501(9)	0.21760(13)	0.14729(2)	0.16391(8)	0.17075(11)	0.11948(1)	0.20479(15)	0.15605(4)	0.18903(14)
20	D_{bias}	0.06013(7)	0.05657(1)	0.05982(6)	0.06097(9)	0.09674(7)	0.09479(4)	0.05832(4)	0.06329(10)	0.06459(12)	0.06368(11)	0.06506(13)	0.05736(2)	0.07277(14)	0.06061(8)	0.07277(15)
20	D_{max}	0.09678(8)	0.09145(2)	0.09767(9)	0.09546(6)	0.09674(7)	0.09479(4)	0.09436(3)	0.10231(12)	0.10102(11)	0.10036(10)	0.10683(14)	0.08828(1)	0.11686(15)	0.09522(5)	0.11486(4)
20	ASAE	0.04686(8)	0.04259(3)	0.04631(7)	0.04402(4)	0.04200(2)	0.04716(9)	0.03982(1)	0.04585(1)	0.05611(13)	0.05321(11)	0.05344(12)	0.04615(6)	0.06229(14)	0.05072(10)	0.06278(15)
20	$\Sigma Ranks$	55(6)	38(2)	85(11)	49(5)	81(10)	67(9)	59(7)	106(12)	45(4)	62(8)	117(14)	39(3)	122(15)	37(1)	115(13)
70	BIAS(ϕ)	0.32300(3)	0.34423(8)	0.40608(12)	0.28950(1)	0.33885(6)	0.29400(2)	0.33385(4)	0.18339(13)	0.33593(10)	0.33696(5)	0.59323(15)	0.35444(9)	0.42177(14)	0.34144(7)	0.39361(11)
70	BIAS(ϕ)	0.45783(2)	0.52192(10)	0.55934(12)	0.46629(3)	0.50792(7)	0.52004(9)	0.50729(6)	0.55597(11)	0.50020(4)	0.51334(8)	0.61823(14)	0.50551(5)	0.61823(14)	0.50551(5)	0.63370(14)
70	MSE(ϕ)	0.16946(3)	0.20930(8)	0.30497(13)	0.13039(1)	0.17141(4)	0.14743(2)	0.18770(6)	0.32082(14)	0.22406(10)	0.17180(5)	0.63002(15)	0.22101(9)	0.26499(12)	0.19295(7)	0.22698(11)
70	MSE(ϕ)	0.34776(6)	0.40865(9)	0.50163(11)	0.34301(1)	0.38641(5)	0.44510(10)	0.38606(4)	0.53339(12)	0.40358(7)	0.37681(4)	0.83092(15)	0.34368(2)	0.55765(13)	0.40505(8)	0.65055(14)
70	MRE(ϕ)	0.08075(5)	0.08606(8)	0.10152(12)	0.07237(1)	0.08472(6)	0.07350(2)	0.08346(4)	0.10460(13)	0.08883(10)	0.08824(5)	0.14831(15)	0.08861(9)	0.10544(14)	0.08536(7)	0.09890(11)
70	MRE(ϕ)	0.07631(2)	0.08699(1)	0.09322(12)	0.07771(3)	0.08465(6)	0.08667(9)	0.08455(6)	0.09266(11)	0.08337(8)	0.08556(8)	0.12356(15)	0.07205(1)	0.10304(13)	0.08425(5)	0.10562(14)
70	D_{bias}	0.03127(5)	0.03311(7)	0.03388(9)	0.03039(3)	0.03005(1)	0.03040(4)	0.03164(6)	0.03339(10)	0.03680(11)	0.03728(12)	0.03810(14)	0.03025(2)	0.04293(13)	0.03397(9)	0.03792(13)
70	D_{max}	0.05031(5)	0.05371(7)	0.05530(9)	0.04843(2)	0.04898(3)	0.04955(5)	0.05129(6)	0.05622(10)	0.05861(12)	0.05828(11)	0.06647(14)	0.04840(1)	0.06860(15)	0.05409(8)	0.06198(13)
70	ASAE	0.02233(8)	0.02023(1)	0.02056(3)	0.02131(6)	0.02093(5)	0.02265(9)	0.02093(5)	0.02069(4)	0.02770(12)	0.02635(11)	0.02934(13)	0.02217(7)	0.02970(14)	0.02489(10)	0.03067(15)
70	$\Sigma Ranks$	34(2)	68(8)	92(11)	21(1)	41(5)	51(6)	49(5)	98(12)	80(10)	69(9)	131(15)	41(5)	123(14)	66(7)	116(13)
130	BIAS(ϕ)	0.23597(5)	0.23521(4)	0.28729(13)	0.21205(1)	0.23333(2)	0.23371(3)	0.24616(6)	0.28724(12)	0.25146(8)	0.26145(9)	0.55203(15)	0.27172(10)	0.30267(14)	0.24932(7)	0.28283(14)
130	BIAS(ϕ)	0.31915(2)	0.32998(3)	0.38234(11)	0.30640(1)	0.34815(6)	0.36812(12)	0.34756(5)	0.38762(12)	0.37007(8)	0.38156(10)	0.57465(15)	0.33796(4)	0.44570(13)	0.37348(9)	0.48493(14)
130	MSE(ϕ)	0.09057(3)	0.08543(3)	0.13975(13)	0.06905(1)	0.09050(5)	0.09493(4)	0.09890(7)	0.14998(14)	0.10015(8)	0.10121(9)	0.55153(15)	0.12537(11)	0.13618(12)	0.09707(6)	0.12294(10)
130	MSE(ϕ)	0.16321(2)	0.17675(5)	0.24454(12)	0.15079(1)	0.19043(3)	0.24019(11)	0.19785(5)	0.23926(13)	0.22843(8)	0.22903(9)	0.53717(15)	0.21817(7)	0.29373(13)	0.21180(6)	0.34681(14)
130	MRE(ϕ)	0.05899(5)	0.05880(4)	0.07182(13)	0.05301(1)	0.05833(2)	0.06287(8)	0.06154(6)	0.07181(12)	0.06287(8)	0.06356(9)	0.13801(15)	0.06793(11)	0.07567(14)	0.06233(7)	0.07071(11)
130	MRE(ϕ)	0.05319(2)	0.05500(3)	0.06372(11)	0.05107(1)	0.05802(6)	0.06135(7)	0.05793(5)	0.06460(12)	0.06168(8)	0.06359(10)	0.09578(15)	0.05633(4)	0.07428(13)	0.06225(9)	0.08082(14)
130	D_{bias}	0.02397(7)	0.02350(6)	0.02225(5)	0.02103(1)	0.02157(2)	0.02302(5)	0.02262(4)	0.02446(8)	0.02842(12)	0.02495(10)	0.03104(15)	0.02456(9)	0.02964(13)	0.02595(11)	0.03018(14)
130	D_{max}	0.03812(7)	0.03765(6)	0.03717(4)	0.03367(1)	0.03512(4)	0.03719(15)	0.03666(3)	0.04034(9)	0.04475(12)	0.04040(10)	0.05499(15)	0.03903(8)	0.04779(13)	0.04151(11)	0.04889(14)
130	ASAE	0.01528(8)	0.01467(3)	0.01498(5)	0.01510(7)	0.01463(3)	0.01494(4)	0.01433(1)	0.01450(2)	0.01974(12)	0.01869(11)	0.02162(15)	0.01589(9)	0.02101(14)	0.01801(10)	0.02062(13)
130	$\Sigma Ranks$	41(4)	34(2)	85(10)	15(1)	34(5)	49(6)	42(5)	91(12)	84(9)	87(11)	135(15)	72(7)	119(14)	76(8)	115(13)
200	BIAS(ϕ)	0.16769(1)	0.18774(5)	0.21279(13)	0.17009(2)	0.21885(11)	0.19315(6)	0.18085(3)	0.23423(12)	0.21368(10)	0.21161(8)	0.42518(15)	0.20667(7)	0.25012(14)	0.18262(4)	0.23650(13)
200	BIAS(ϕ)	0.27683(3)	0.29038(6)	0.31597(10)	0.27236(2)	0.31664(11)	0.27688(4)	0.28603(5)	0.32140(12)	0.31143(9)	0.30728(8)	0.45193(15)	0.26478(1)	0.38686(14)	0.29983(7)	0.35581(13)
200	MSE(ϕ)	0.04991(2)	0.06128(6)	0.07877(10)	0.04332(1)	0.07863(11)	0.05892(5)	0.05542(4)	0.09127(13)	0.06558(7)	0.06961(8)	0.31979(15)	0.07106(9)	0.10141(14)	0.05536(3)	0.08857(12)
200	MSE(ϕ)	0.11760(2)	0.13751(7)	0.15610(11)	0.11271(1)	0.15372(10)	0.12461(3)	0.13678(6)	0.16251(12)	0.15293(9)	0.14500(8)	0.43716(15)	0.13153(4)	0.23765(14)	0.13509(5)	0.20175(13)
200	MRE(ϕ)	0.04192(1)	0.04695(5)	0.05320(9)	0.04252(2)	0.05471(11)	0.04829(6)	0.04521(3)	0.05856(12)	0.05342(10)	0.05296(8)	0.10630(15)	0.05167(7)	0.06253(14)	0.04565(4)	0.05913(13)
200	MRE(ϕ)	0.04614(3)	0.04840(6)	0.05266(10)	0.04539(2)	0.05277(11)	0.04615(4)	0.04777(5)	0.05357(12)	0.05190(9)	0.05121(8)	0.08665(15)	0.04413(1)	0.06448(14)	0.04997(7)	0.05930(13)
200	D_{bias}	0.01780(2)	0.01844(5)	0.01917(9)	0.01914(8)	0.01899(7)	0.01793(3)	0.01747(1)	0.01879(6)	0.02129(12)	0.02119(11)	0.02549(15)	0.01830(4)	0.02416(14)	0.02022(10)	0.02296(13)
200	D_{max}	0.02897(7)	0.02980(5)	0.03160(9)	0.03024(6)	0.03153(8)	0.02914(3)	0.02841(1)	0.03139(7)	0.03426(12)	0.03345(11)	0.04563(15)	0.02947(4)	0.03914(14)	0.03230(10)	0.03711(13)
200	ASAE	0.01277(9)	0.01182(3)	0.01205(6)	0.01176(2)	0.01203(5)	0.01222(7)	0.01166(1)	0.01197(4)	0.01572(12)	0.01519(11)	0.01844(15)	0.01276(8)	0.01624(13)	0.01458(10)	0.01698(14)
200	$\Sigma Ranks$	25(1)	48(6)	83(9)	26(2)	85(10)	41(4)	29(3)	90(11)	90(11)	81(8)	135(15)	45(5)	125(14)	60(7)	117(13)
240	BIAS(ϕ)	0.16533(4)	0.17082(5)	0.19418(11)	0.16318(2)	0.19077(8)	0.16019(1)	0.16421(3)	0.21663(13)	0.19171(9)	0.18334(7)	0.38843(15)	0.19329(10)	0.21213(12)	0.17153(6)	0.22275(14)
240	BIAS(ϕ)	0.20965(5)	0.26459(5)	0.31379(13)	0.24840(3)	0.27259(12)	0.26787(7)	0.27187(7)	0.29880(12)	0.29142(9)	0.29243(8)	0.45716(15)	0.23293(2)	0.29284(11)	0.26300(4)	0.33159(14)
240	MSE(ϕ)	0.04228(3)	0.04547(5)	0.05424(8)	0.04150(2)	0.06238(11)	0.04034(1)	0.04326(4)	0.07334(13)	0.06212(10)	0.05118(7)	0.26095(15)	0.06015(9)	0.06930(12)	0.04841(6)	0.08426(14)
240	MSE(ϕ)	0.07166(1)	0.11351(6)	0.14294(13)	0.09532(3)	0.12655(8)	0.10979(5)	0.11612(7)	0.13469(11)	0.13388(10)	0.12679(9)	0.34033(15)	0.09396(2)	0.13550(12)	0.10775(4)	0.16573(14)
240	MRE(ϕ)	0.04133(4)	0.04271(5)	0.04855(11)	0.04079(2)	0.04769(8)	0.04005(1)	0.04105(3)	0.05416(13)	0.04793(9)	0.04583(7)	0.09711(15)	0.04832(10)	0.05303(12)	0.04288(6)	0.05569(14)
240	MRE(ϕ)	0.03494(1)	0.04410(5)	0.05230(13)	0.04140(3)	0.04543(8)	0.04446(6)	0.04531(7)	0.04980(12)	0.04857(9)	0.04874(10)	0.07619(15)	0.03882(2)	0.04881(11)	0.04383(4)	0.05527(14)
240	D_{bias}	0.01541(2)	0.01695(5)	0.01680(4)	0.01701(6)	0.01752(7)	0.01665(3)	0.01475(1)	0.01860(10)	0.01972(12)	0.01960(11)	0.02319(15)	0.01780(8)	0.02094(14)	0.01811(9)	0.02246(14)
240	D_{max}	0.02476(2)	0.02745(5)	0.02796(6)	0.02723(4)	0.02850(8)	0.02690(3)	0.02424(1)	0.03072(10)	0.03169(12)	0.03122(11)	0.04147(15)	0.02839(7)	0.03328(13)	0.02914(9)	0.03615(14)
240	ASAE	0.01148(7)	0.01078(1)	0.01131(6)	0.01152(8)	0.01091(2)	0.01127(5)	0.01106(3)	0.01109(4)	0.01403(12)	0.01381(11)	0.01669(15)	0.01172(9)	0.01492(13)	0.01246(10)	0.01506(14)
240	$\Sigma Ranks$	25(1)	42(5)	85(10)	33(3)	68(8)	31(2)	36(4)	98(12)	92(11)	83(9)	135(15)	59(7)	109(13)	58(6)	126(14)
300	BIAS(ϕ)	0.13649(1)	0.15000(4)	0.17728(11)	0.14412(2)	0.16233(7)	0.15424(3)	0.14885(3)	0.20807(14)	0.16380(9)	0.16331(8)	0.34640(15)	0.19721(13)	0.21753(6)	0.15475(6)	0.19099(12)
300	BIAS(ϕ)	0.20465(2)	0.24273(7)	0.2411												

Table 6: Ranks for estimation methods simulation tables.

Parameter	<i>n</i>	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$\phi = 0.2, \kappa = 0.3$	20	3.0	4.0	12.0	1.0	5.0	11.0	6.0	7.0	8.0	9.0	14.0	10.0	13.0	2.0	15.0
	70	2.0	5.0	10.0	3.0	7.0	9.0	1.0	12.0	11.0	6.0	15.0	4.0	13.0	8.0	14.0
	130	1.0	2.0	7.0	4.0	6.0	5.0	3.0	10.0	12.0	11.0	13.0	9.0	15.0	8.0	14.0
	200	1.0	2.0	4.5	3.0	7.5	4.5	7.5	6.0	12.0	10.0	14.0	9.0	15.0	11.0	13.0
	240	6.0	1.0	7.0	3.0	11.0	4.0	2.0	5.0	12.0	9.0	15.0	10.0	13.0	8.0	14.0
	300	1.5	1.5	6.0	3.0	4.0	7.0	5.0	9.0	12.0	11.0	14.0	10.0	13.0	8.0	15.0
$\phi = 2.0, \kappa = 3.5$	20	4.0	3.0	10.0	1.0	2.0	5.0	8.0	6.0	12.0	11.0	15.0	7.0	13.0	9.0	14.0
	70	6.0	3.0	11.0	1.0	7.0	5.0	2.0	10.0	12.0	9.0	15.0	4.0	14.0	8.0	13.0
	130	2.0	1.0	10.0	4.0	11.0	3.0	12.0	9.0	6.0	7.0	14.0	5.0	15.0	8.0	13.0
	200	3.0	2.0	9.0	6.5	5.0	4.0	1.0	6.5	11.0	13.0	14.0	8.0	12.0	10.0	15.0
	240	3.0	1.0	11.0	4.0	5.0	7.0	2.0	8.0	10.0	6.0	15.0	9.0	13.0	12.0	14.0
	300	5.0	2.0	10.0	1.0	7.0	6.0	3.0	8.0	11.5	11.5	15.0	9.0	13.0	4.0	14.0
$\phi = 3.0, \kappa = 0.8$	20	3.0	2.0	12.0	1.0	7.0	9.0	5.0	10.0	11.0	8.0	15.0	6.0	13.0	4.0	14.0
	70	4.0	1.0	11.0	2.5	8.5	2.5	5.0	12.0	10.0	8.5	14.5	6.0	14.5	7.0	13.0
	130	2.0	6.0	10.0	1.0	9.0	4.0	3.0	11.0	12.0	8.0	15.0	7.0	13.0	5.0	14.0
	200	3.0	2.0	8.0	1.0	10.0	4.0	6.0	12.0	11.0	9.0	15.0	7.0	13.0	5.0	14.0
	240	1.0	3.0	8.0	2.0	10.0	4.0	5.0	12.0	11.0	9.0	15.0	6.0	14.0	7.0	13.0
	300	1.0	3.0	11.0	2.0	8.0	4.0	5.0	10.0	12.0	9.0	15.0	7.0	13.0	6.0	14.0
$\phi = 0.9, \kappa = 1.5$	20	1.0	4.0	11.0	3.0	7.0	10.0	8.0	9.0	5.0	12.0	14.0	2.0	15.0	6.0	13.0
	70	1.0	2.0	11.0	4.0	8.0	9.0	6.0	5.0	12.0	10.0	14.0	3.0	13.0	7.0	15.0
	130	2.0	5.0	3.0	1.0	6.0	9.0	7.0	10.0	12.0	8.0	15.0	4.0	13.0	11.0	14.0
	200	1.0	3.5	8.0	5.0	3.5	11.0	8.0	6.0	12.0	10.0	14.0	2.0	13.0	8.0	15.0
	240	1.0	9.0	5.0	4.0	6.5	6.5	2.0	8.0	12.0	11.0	14.0	3.0	15.0	10.0	13.0
	300	1.0	2.0	8.0	5.0	9.0	6.0	3.0	7.0	12.0	11.0	14.0	4.0	15.0	10.0	13.0
$\phi = 4.0, \kappa = 6.0$	20	6.0	2.0	11.0	5.0	10.0	9.0	7.0	12.0	4.0	8.0	14.0	3.0	15.0	1.0	13.0
	70	2.0	8.0	11.0	1.0	3.5	6.0	5.0	12.0	10.0	9.0	15.0	3.5	14.0	7.0	13.0
	130	4.0	2.0	10.0	1.0	3.0	6.0	5.0	12.0	9.0	11.0	15.0	7.0	14.0	8.0	13.0
	200	1.0	6.0	9.0	2.0	10.0	4.0	3.0	11.5	11.5	8.0	15.0	5.0	14.0	7.0	13.0
	240	1.0	5.0	10.0	3.0	8.0	2.0	4.0	12.0	11.0	9.0	15.0	7.0	13.0	6.0	14.0
	300	1.0	4.0	7.0	2.0	9.0	6.0	3.0	12.0	11.0	10.0	15.0	5.0	14.0	8.0	13.0
Σ Ranks		73.5	97.0	271.5	80.0	213.5	182.5	142.5	280.0	318.0	282.0	436.5	181.5	410.5	219.0	412.0
Overall Rank		1	3	9	2	7	6	4	10	12	11	15	5	13	8	14

6 Real data analysis

To show the usefulness of the proposed inverse power Garhy distribution (IPGD), we applied it to two real datasets.

The first dataset, obtained from [21], contains the failure times of 24 mechanical components. The data are: 30.94, 18.51, 16.62, 51.56, 22.85, 22.38, 19.08, 49.56, 17.12, 10.67, 25.43, 10.24, 27.47, 14.70, 14.10, 29.93, 27.98, 36.02, 19.40, 14.97, 22.57, 12.26, 18.14, 18.84

The second dataset, obtained from [22], gives the maximum annual flood discharges of the North Saskatchewan River at Edmonton over 47 years. The data are given below:

19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 185.560

For comparison purposes, the IPGD was fitted together with some well-known lifetime and continuous distributions, namely the inverse Gompertz distribution (IGD), gamma distribution (GAD), Weibull distribution (WD), Garhy distribution (GD), Rayleigh distribution (RD), Chris Jerry distribution (CJD), Lindley distribution (LD), and XLindley distribution (XLD).

The CDFs of the competing models are:

–Inverse Gompertz distribution (IGD) [23]:

$$F(x; \alpha, \beta) = e^{-\frac{\alpha}{\beta} \left(e^{\frac{\beta}{x}} - 1 \right)}.$$

–Gamma distribution (GAD):

$$F(x; \alpha, \beta) = 1 - \frac{\Gamma(\alpha, \beta x)}{\Gamma(\alpha)}$$

–Weibull distribution (WD) [24]:

$$F(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^{\kappa}}.$$

–Garhy distribution (GD) [9]:

$$F(x) = 1 - \left(\frac{3\kappa^2 x^2}{\kappa^3 + 6} + (\kappa x + 1) \right) e^{-\kappa x}.$$

–Rayleigh distribution (RD) [25]:

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\xi^2}\right).$$

–Chris Jerry Distribution (CJD) [4]:

$$F(x) = 1 - \left[1 + \frac{\theta x (\theta x + 2)}{\theta + 2} \right] e^{-\theta x}$$

–Lindley distribution (LD) [26]:

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}$$

–XLindley distribution (XLD) [27]:

$$F(x) = 1 - \left(1 + \frac{\theta x}{(1 + \theta)^2}\right) e^{-\theta x}.$$

Table 7 presents the descriptive statistics of the two datasets, while Figure 5 shows the nonparametric plots, including the histograms, kernel density estimates, and boxplots.

From Table 7, both datasets have positive skewness and kurtosis values greater than 3. Dataset II has higher skewness and kurtosis than Dataset I, indicating greater asymmetry and more pronounced extreme observations.

From Figure 5, the histograms and density plots confirm that both datasets are positively skewed. The boxplots also suggest unusually large observations, especially for Dataset II. These shapes agree well with the behaviour that the proposed IPGD can capture.

Table 7: Descriptive statistics for the two datasets.

Measure	Dataset I	Dataset II
size	24	47
Mean	22.97	50.00
Variance	115.67	960.78
Skewness	1.35	2.30
Kurtosis	4.36	9.52
Minimum	10.24	19.89
Maximum	51.56	185.56
Range	41.32	165.68

Table 8 presents the maximum likelihood estimates, standard errors, negative log-likelihood (NLL) values, Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICc), Bayesian Information Criterion (BIC), and Hannan–Quinn Information Criterion (HQIC) for all fitted models.

The rule used here is simple: the model with the smallest negative log-likelihood, AIC, AICc, BIC, and HQIC values is considered the best model among the competitors.

For Dataset I, Table 8 shows that the IPGD gives the smallest values of negative log-likelihood (85.7254), AIC (175.4508), AICc (176.0221), BIC (177.8069), and HQIC (176.0758). This clearly shows that the IPGD provides the best fit among all fitted models for the mechanical failure time data.

For Dataset II, the same pattern is observed. The IPGD again has the smallest negative log-likelihood (208.2477), AIC (420.4955), AICc (420.7682), BIC (424.1958), and HQIC (421.8879). This indicates that the proposed model also gives the best fit for the flood discharge data.

To further assess the fitting performance, Table 9 presents the Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CVM) statistics with their corresponding p -values, root mean

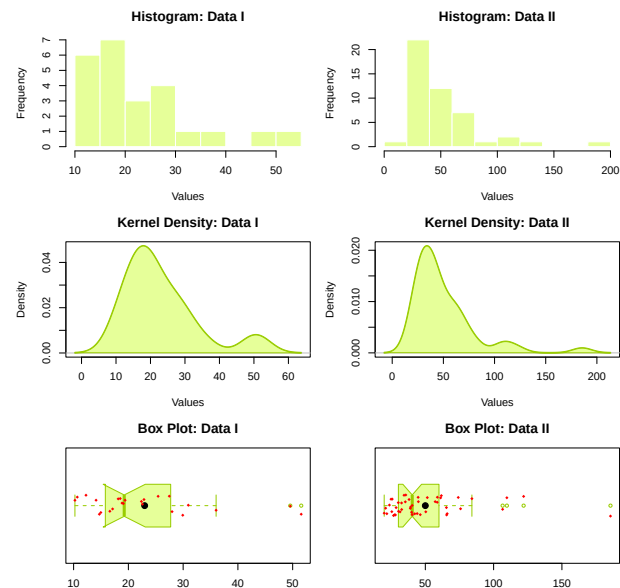


Fig. 5: Nonparametric plots of the two datasets, including histograms, kernel density estimates, and box plots.

square error (RMSE), coefficient of determination (COD), and weighted extropy (WEXT). Following Ref. [28], weighted extropy is also considered in the model comparison.

The results are interpreted as follows:

- Smaller KS, AD, CVM, and RMSE values indicate a better fit.
- Larger p -values indicate less evidence against the fitted model under the corresponding goodness-of-fit test.
- Larger COD values indicate a better fit.
- Lower WEXT values are preferred under the comparison approach in [28].

For Dataset I, Table 9 shows that the IPGD has the smallest KS (0.0710), AD (0.1627), CVM (0.0210), and RMSE (0.0286), together with the largest COD value (0.9900). It also has the highest p -values and the lowest WEXT value (−0.3473). These results favor the IPGD among the competing models for the first dataset.

For Dataset II, the IPGD again has the smallest KS (0.0669), AD (0.1564), CVM (0.0231), and RMSE (0.0242), together with the largest COD value (0.9934). It also has the highest p -values and the lowest WEXT value (−0.3128). These results favor the IPGD among the competing models for the second dataset.

In addition to the numerical results, the graphical comparisons also support the superiority of the proposed model.

For the density fit, Figure 6 presents the estimated probability density functions of all competing models for the first dataset. The plot shows that the fitted density of the IPGD follows the shape of the histogram more closely

Table 8: Parameter estimates, standard errors, negative log-likelihood values, and information criteria for the competing models fitted to the two datasets.

Model	$\widehat{E}_1$	$SE(\widehat{E}_1)$	$\widehat{E}_2$	$SE(\widehat{E}_2)$	NLL	AIC	AICc	BIC	HQIC
Data I									
IPGD($\hat{\kappa}, \hat{\phi}$)	430.5087	8.4652	1.8521	0.0516	85.7254	175.4508	176.0221	177.8069	176.0758
IGD($\hat{\alpha}, \hat{\beta}$)	4.0694	2.1805	38.8591	9.8968	88.2103	180.4207	180.9921	182.7768	181.0457
GAD($\hat{\beta}, \hat{\alpha}$)	0.24774	0.0719	5.6915	1.5781	86.9446	177.8892	178.4607	180.2454	178.5143
WD($\hat{\kappa}, \hat{\lambda}$)	2.3096	0.3356	26.0285	2.3695	88.8910	181.7820	182.3534	184.1381	182.4070
GD($\hat{\kappa}$)	0.1306	0.0153	-	-	89.1017	180.2034	180.3853	181.3815	180.5160
RD($\hat{\zeta}$)	17.8672	1.7885	-	-	89.3341	180.6683	180.8501	181.8463	180.9808
CJD($\hat{\theta}$)	0.1270	0.0151	-	-	90.0331	182.0662	182.2480	183.2442	182.3787
LD($\hat{\theta}$)	0.0837	0.0121	-	-	92.8833	187.7666	187.9484	188.9446	188.0791
XLD($\hat{\theta}$)	0.0807	0.0117	-	-	93.5479	189.0959	189.2777	190.2739	189.4084
Data II									
IPGD($\hat{\kappa}, \hat{\phi}$)	827.3746	5.9459	1.6683	0.0294	208.2477	420.4955	420.7682	424.1958	421.8879
IGD($\hat{\alpha}, \hat{\beta}$)	8.7226	1.6045	75.1176	4.3450	210.3244	424.6489	424.9216	428.3492	426.0413
GAD($\hat{\beta}, \hat{\alpha}$)	0.0773	0.0162	3.8654	0.7582	214.4539	432.9078	433.1805	436.6081	434.3003
WD($\hat{\kappa}, \hat{\lambda}$)	1.7959	0.1767	56.7176	4.3988	219.0020	442.0040	442.2767	445.7043	443.3965
GD($\hat{\kappa}$)	0.0600	0.0051	-	-	215.2139	432.4278	432.5167	434.2780	433.1241
RD($\hat{\zeta}$)	41.4688	3.4246	-	-	219.6309	441.2619	441.3508	443.1120	441.9581
CJD($\hat{\theta}$)	0.0592	0.0050	-	-	215.9941	433.9882	434.0771	435.8383	434.6844
LD($\hat{\theta}$)	0.0393	0.0041	-	-	219.6914	441.3828	441.4717	443.2330	442.0791
XLD($\hat{\theta}$)	0.0385	0.0040	-	-	220.2949	442.5898	442.6787	444.4399	443.2860

Table 9: Goodness-of-fit statistics and weighted extropy values for the competing models fitted to the two datasets.

Model	KS	KS-pv	AD	AD-pv	CVM	CVM-pv	RMSE	WEXT	COD
Data I									
IPGD	0.0710	0.9990	0.1627	0.9976	0.0210	0.9967	0.0286	-0.3473	0.9900
IGD	0.1252	0.8021	0.6199	0.6276	0.0874	0.6546	0.0527	-0.3076	0.9575
GAD	0.1404	0.6799	0.4032	0.8443	0.0569	0.8383	0.0464	-0.3292	0.9730
WD	0.1438	0.6515	0.7474	0.5190	0.1084	0.5485	0.0609	-0.2886	0.9447
GD	0.1556	0.5541	0.8998	0.4132	0.1353	0.4399	0.0649	-0.2343	0.9203
RD	0.1514	0.5883	0.8582	0.4397	0.1219	0.4907	0.0619	-0.2500	0.9329
CJD	0.1778	0.3882	1.1275	0.2962	0.1759	0.3209	0.0754	-0.2182	0.8832
LD	0.2399	0.1062	1.9678	0.0962	0.3413	0.1033	0.1089	-0.1782	0.6704
XLD	0.2527	0.0774	2.1379	0.0778	0.3761	0.0829	0.1147	-0.1708	0.6119
Data II									
IPGD	0.0669	0.9845	0.1564	0.9981	0.0231	0.9936	0.0242	-0.3128	0.9934
IGD	0.0835	0.8982	0.4686	0.7781	0.0606	0.8124	0.0330	-0.2988	0.9854
GAD	0.1312	0.3935	1.0519	0.3307	0.1661	0.3450	0.0577	-0.2685	0.9562
WD	0.1412	0.3058	1.7577	0.1256	0.2695	0.1651	0.0717	-0.2245	0.9146
GD	0.1191	0.5174	1.1721	0.2782	0.1654	0.3468	0.0553	-0.2344	0.9526
RD	0.1745	0.1143	2.0665	0.0847	0.3651	0.0892	0.0856	-0.2500	0.8917
CJD	0.1321	0.3852	1.2998	0.2323	0.1820	0.3059	0.0579	-0.2264	0.9460
LD	0.1978	0.0507	2.4242	0.0546	0.3769	0.0828	0.0842	-0.1829	0.8471
XLD	0.2049	0.0387	2.5736	0.0456	0.4055	0.0695	0.0876	-0.1787	0.8300

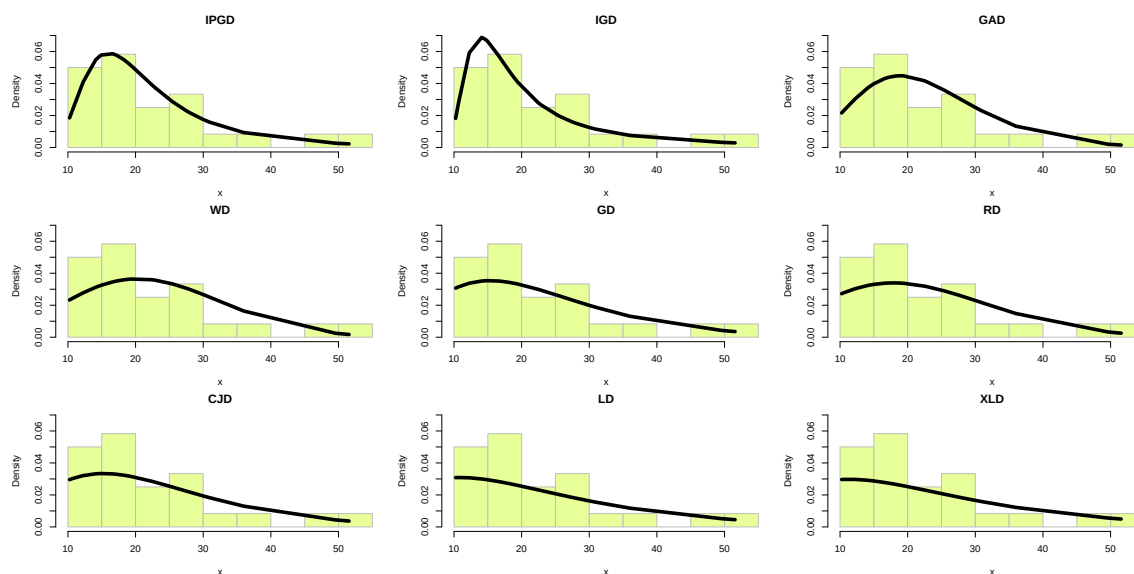


Fig. 6: Estimated PDFs of the competing models for the first dataset.

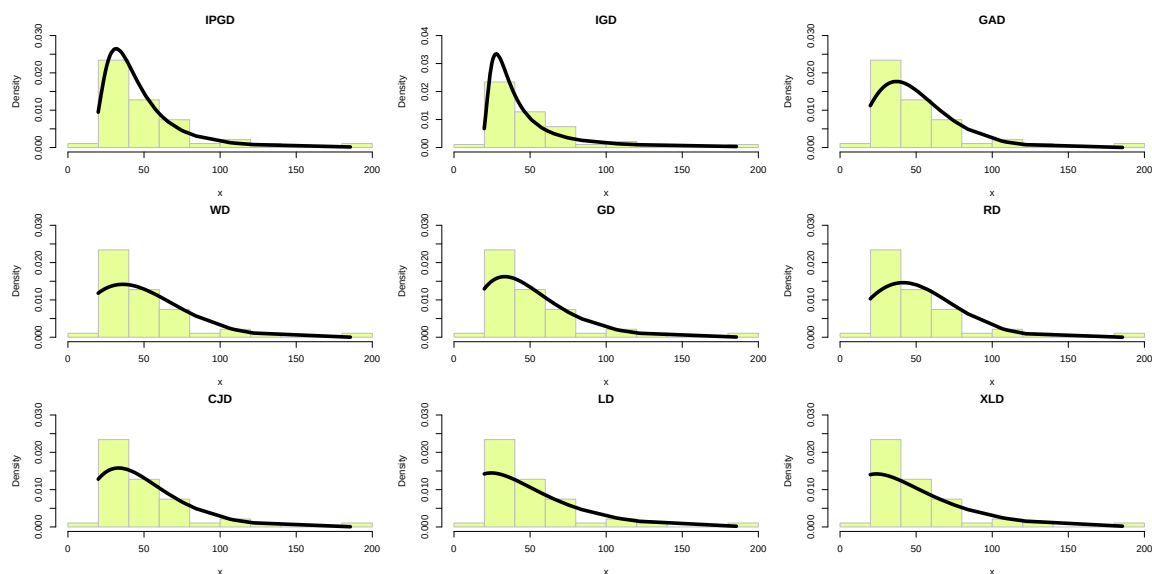


Fig. 7: Estimated PDFs of the competing models for the second dataset.

than the other models. It captures the right-skewed pattern and the peak of the data very well.

Similarly, Figure 7 shows the estimated probability density functions for the second dataset. The IPGD again gives the closest agreement with the histogram shape. It better captures the strong right skewness and the heavier tail behaviour observed in the flood discharge data.

For the cumulative fit, Figure 8 compares the empirical cumulative distribution function with the fitted cumulative distribution functions for the first dataset. The fitted CDF of the IPGD lies closest to the empirical CDF

over almost the entire data range. The closeness between the two curves indicates that the proposed model describes the cumulative behaviour of the data more accurately than the other models.

The same behaviour is seen in Figure 9 for the second dataset. The fitted CDF of the IPGD almost overlaps with the empirical CDF. This graphical evidence agrees with the goodness-of-fit statistics and confirms the better performance of the IPGD.

The probability-probability plots also give further support. Figure 10 presents the P-P plots for the first

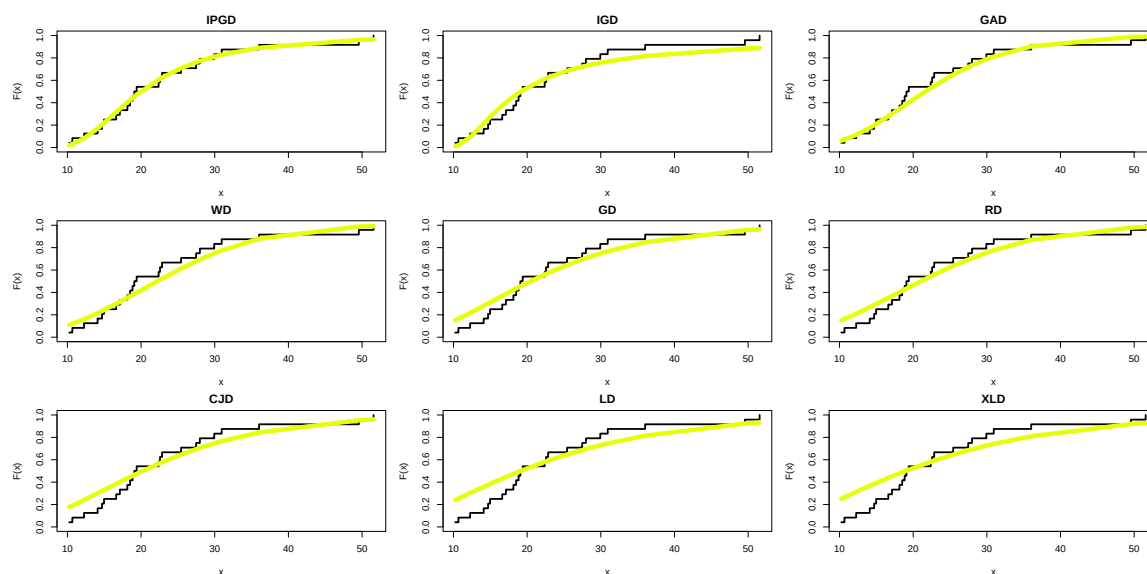


Fig. 8: Plots comparing the empirical CDF with the fitted CDFs of all models for the first dataset.

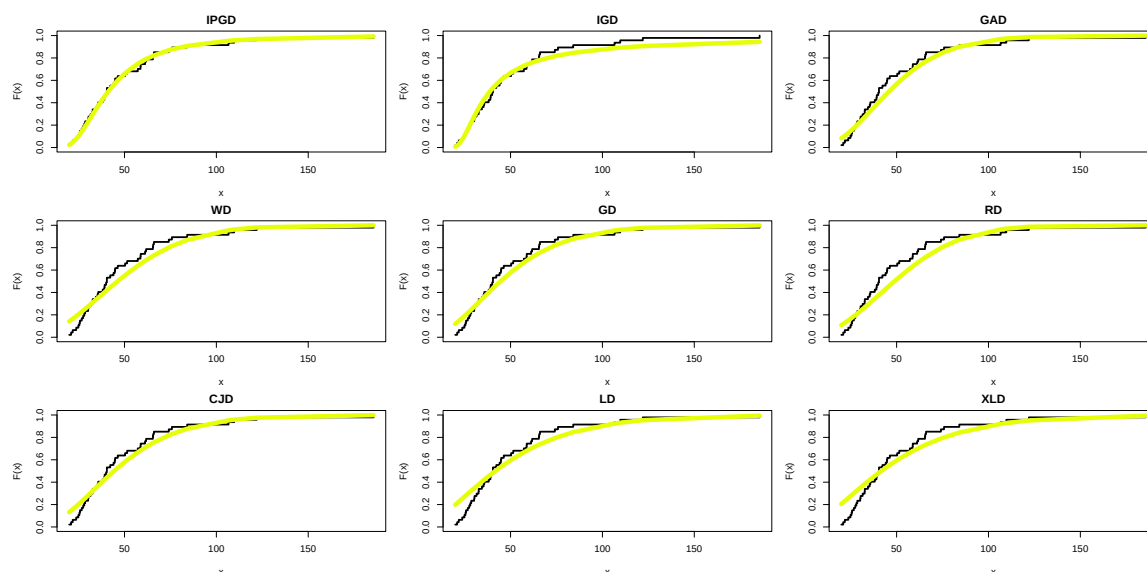


Fig. 9: Plots comparing the empirical CDF with the fitted CDFs of all models for the second dataset.

dataset. The points for the IPGD are more closely aligned with the diagonal reference line than those of the competing models. This close alignment indicates that the theoretical probabilities from the IPGD agree well with the empirical probabilities.

Likewise, Figure 11 shows the P-P plots for the second dataset. The IPGD produces points that lie close to the diagonal reference line, supporting the numerical results.

The graphical comparisons in Figures 6–11 support the numerical results in Tables 8 and 9. Among the

models considered, the IPGD provides the best overall fit for both datasets based on the reported information criteria and goodness-of-fit statistics.

7 Conclusion

A new two-parameter distribution called the Inverse Power Garhy distribution (IPGD) was proposed in this study. The model was obtained by applying the inverse power transformation to the Garhy distribution. The

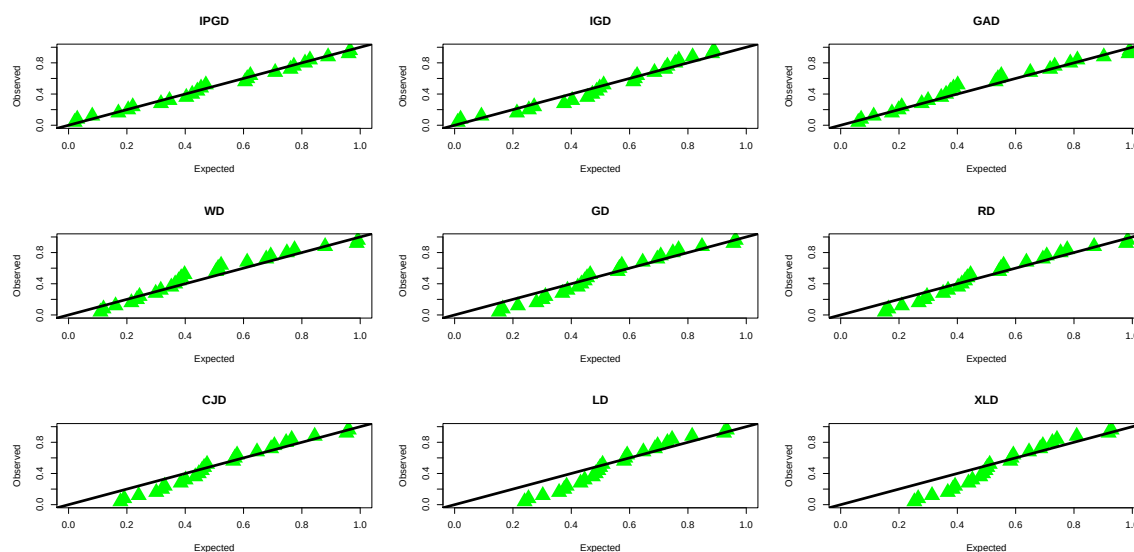


Fig. 10: P–P plots of the fitted models for the first dataset

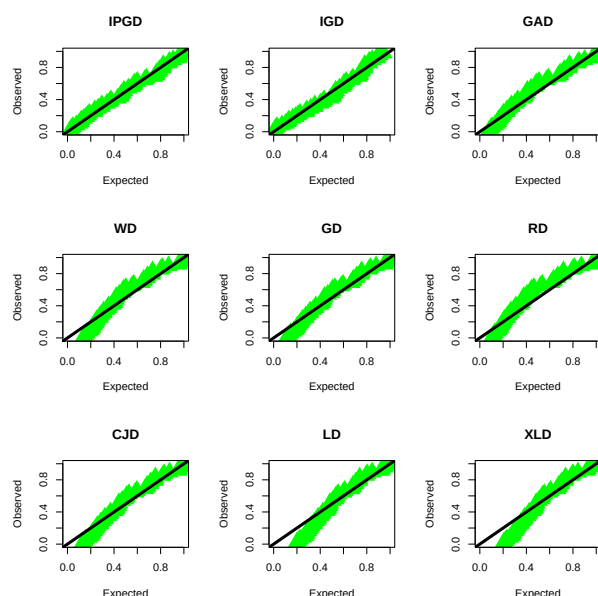


Fig. 11: P–P plots of the fitted models for the second dataset

proposed distribution can model decreasing and inverted bathtub-shaped hazard functions. Several statistical properties of the distribution were derived. The model parameters were estimated using fifteen classical estimation methods. A simulation study was conducted to examine the finite-sample performance of the estimators. The proposed distribution was applied to flood and engineering datasets. The results showed that it provided

a better fit than the Garhy distribution and several existing models. The IPGD may be useful in engineering, environmental science, reliability analysis, and related fields. Future studies may consider applications of the proposed distribution to other types of data and the development of further extensions.

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Data Availability Statement

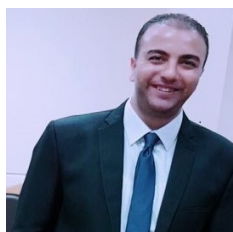
The data that support the findings of this study are available within the article.

Conflicts of Interest

The authors declare no conflict of interest.

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