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Innovative Fractional Applications Using NE Transform

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Abstract: In this work, we use the NE transform to address several interesting problems. We examine market equilibrium economic models, the falling body problem, and the gross domestic product model with fractional derivatives. We use specific applications to illustrate the effectiveness of the NE transform. We investigate different kernels on the model to show the effects of different fractional derivatives. This research contributes to Industry, Innovation and Infrastructure, through advancing mathematical methods applicable to industrial innovation modelling.

Keywords: Power-law kernel, NE transform, power law kernel, exponential decay kernel, Mittag-Leffler kernel, modelling.

1 Introduction

The mathematical notion of differentiation, traditionally restricted to integer orders, has undergone a profound conceptual expansion over the last three centuries [1,2,3,4,5]. While classical calculus was formally established by Isaac Newton and Gottfried Wilhelm Leibniz, the idea that differentiation could be extended to non-integer orders appeared soon after its birth [6]. Subsequent developments by mathematicians such as Leonhard Euler, Joseph Liouville, and Bernhard Riemann gradually shaped the theoretical framework of what is now known as fractional calculus [7],[8]. These early analytical investigations laid the groundwork for modern fractional operators and their diverse applications.

In contrast to classical derivatives, fractional derivatives inherently incorporate memory effects due to their integral-based formulations [9,10,11]. This non-local characteristic implies that the current state of a system depends not only on its instantaneous configuration but also on its entire past evolution. Such a feature makes fractional differential equations particularly suitable for modeling complex systems where hereditary properties and long-range temporal interactions play a significant role [12]. Consequently, fractional calculus has emerged as a powerful modeling tool in physics, engineering, biology, finance, human health, human disease, and control theory [14,15,13].

Over the past few decades, the application spectrum of fractional-order models has expanded considerably [16,17,18] Moreover, fractional-order dynamical systems exhibit richer qualitative behaviors compared to their integer-order counterparts, including complex attractors, bifurcation structures, synchronization phenomena, and chaotic oscillations [19]. These properties have stimulated growing interest in stability analysis, numerical approximation techniques, and control strategies for fractional systems.

Among the various definitions proposed in the literature, the Riemann–Liouville and Caputo operators remain the most extensively studied formulations [7],[8]. Although both definitions rely on fractional integrals, the Caputo derivative is often preferred in engineering applications since it allows the use of classical initial conditions expressed in integer-order derivatives. Nevertheless, classical fractional operators involve singular power-law kernels, which may limit their applicability in certain physical contexts.

To address this limitation, Michele Caputo and Mauro Fabrizio introduced a fractional derivative characterized by a non-singular exponential kernel in 2015 [20]. This formulation, known as the Caputo–Fabrizio derivative, avoids the strong singularity present in traditional operators while preserving non-local behavior. Shortly thereafter, Abdon Atangana

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and Dumitru Baleanu proposed a new class of fractional derivatives based on the generalized Mittag–Leffler function [21]. The Atangana–Baleanu operators combine non-singular kernels with non-local characteristics, offering improved flexibility in modeling memory-dependent processes. These operators have been successfully implemented in heat transfer models, groundwater flow simulations, and nonlinear dynamical analyses [21,22,23].

Overall, fractional calculus represents a natural and mathematically consistent generalization of classical calculus. Its ability to incorporate memory and hereditary effects has significantly enhanced the modeling accuracy of complex systems. Ongoing research continues to focus on theoretical refinements, qualitative analysis of fractional dynamical systems, and the development of robust numerical schemes to simulate their behavior efficiently.

Fractional calculus generalizes classical differentiation and integration to non-integer orders, providing a flexible framework for modeling systems with memory and hereditary effects. The connection between fractional operators and integral transforms arises because fractional integrals and derivatives can often be expressed as convolution-type integrals, which are naturally analyzed using Laplace, Fourier, or Mellin transforms [7]. For example, the Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1}f(\xi)d\xi, \quad t > a.$$

and its Laplace transform is given by $L[I^{\alpha}f(t)] = s^{-\alpha}F(s)$, where $F(s)$ is the Laplace transform of $f(t)$ [24]. In recent research, generalized integral transforms have been introduced that encompass and extend classical transforms; one notable example is the Jafari transform, a Laplace-type integral transform proposed for solving integral and differential equations, including those involving fractional operators [25]. Such generalized transforms not only unify existing methods but also provide greater flexibility in handling weighted and non-standard fractional problems, making integral transforms indispensable tools in both theoretical and applied fractional calculus.

In this study, second section presents certain mathematical fundamentals. We describe the applications of the suggested models in Caputo sense, CF sense, and AB sense in the section three. Subsequently, the solutions for the fractional gross domestic product model, economic model, and falling body problem are derived using the NE transform.

2 Preliminaries

Definition 1. *To present a new integral transformation as a generalization and unification of previous existing transformations for Laplace and exponential order functions, we consider:*

$$A = \left\{ f(t) | \exists M, k_1, k_2 > 0, \quad |f(t)| \leq M exp\left(\frac{|t|}{k_i}\right); \quad if t \in (-1)^i x[0,\infty[\right\}.$$

The constant M for a given function in A must be a positive value. The integral equation defines a new integral transform, indicated by the operator E(s,u) as [26]:

$$E(s,u) = N[f(t)] = \frac{1}{s}\int_0^{+\infty} e^{-st} f(ut)dt, \tag{1}$$

or

$$E(s,u) = N[f(t)] = \frac{1}{su}\int_0^{+\infty} e^{\frac{-st}{u}} f(t)dt. \tag{2}$$

If $f(t)$ is a piecewise continuous function over the interval $[0,\infty]$. It is known as the inverse integral transformation of the function $E(s,u)$.

$$N^{-1}[E(s,u)] = \frac{1}{2\Pi i}\int_{c-\infty}^{c+\infty} e^{st}E(us)u^2 sds,$$

or

$$N^{-1}[E(s,u)] = \frac{1}{2\Pi i}\int_{c-\infty}^{c+\infty} e^{\frac{st}{u}}E(s)sds = f(t).$$

For random two provided functions $h(t), k(t) \in A$, and for arbitrary constants a and b,

$$N[ah(t) + bk(t)] = aN[h(t)] + bN[k(t)], \tag{3}$$

is written. An integer order derivative's NE is expressed as follows:

$$N[\frac{df(t)}{dt}] = \frac{sE(u,s)}{u} - \frac{f(0)}{su}, \tag{4}$$

and for the n-order derivative, the NE is given as:

$$N\left[\frac{d^n f(t)}{dt^n}\right] = \frac{s^n E(s,u)}{u^n} - \frac{s^{n-2}}{u^n} f(0) - \frac{s^{n-3}}{u^{n-1}} f^{'}(0) \cdots - \frac{f^{(n-1)}(0)}{su}.$$

We assume that $f(t) \in F$ is the Laplace transform $F(s)$. Thus, the NE transform $E(s,u)$ of $f(t)$ is presented by:

$$E(s,u) = \frac{1}{us} F\left(\frac{s}{u}\right).$$

The table below lists NE integral transformations for a few functions.

$$\begin{pmatrix} \underline{f(t)} & \underline{N(f(t))} \\ 1 & \frac{1}{s^2} \\ t & \frac{u}{s^3} \\ e^{at} & \frac{1}{s(s-au)} \\ sin(at) & \frac{au}{s(s^2+a^2u^2)} \\ cos(at) & \frac{1}{(s^2+a^2u^2)} \\ t^{\alpha} & \frac{u^{\alpha}}{s^{\alpha+2}}\Gamma(n+1) \end{pmatrix}.$$

Definition 2. *The classical convolution product is provided by [26]:*

$$(\theta * \lambda) = \int_0^u \theta(u-t)\lambda(t)dt.$$

When we apply the NE transform to the classical convolution, we obtain:

$$N(f*g) = usF(s,u)G(s,u). \tag{5}$$

Definition 3. *The Caputo derivative is defined as follows [27]:*

$${}_a^{\mathscr{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)dx,$$

where $\alpha \in \mathbb{C}, \quad \Re e(\alpha) > 0,\ n = [\Re e(\alpha)] + 1.$

Lemma 1. *The following result is obtained by applying the NE transform to the Caputo fractional derivative:*

$$N[{}_a^{\mathscr{C}}D_t^{\alpha} f(t)] = \left(\frac{u}{s}\right)^{-\alpha}\left(N[f(t)] - \frac{f(a)}{s^2}\right). \tag{6}$$

Proof. We have

$${}_a^{\mathscr{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)dx,$$

using convolution, gives:

$${}_a^{\mathscr{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left[t^{n-\alpha-1} * f^{(n)}(t)\right].$$

If we implement the NE transform, we get:

$$\begin{aligned} N[{}_a^{\mathscr{C}}D_t^{\alpha} f(t)] &= \frac{1}{\Gamma(n-\alpha)} us \frac{u^{n-\alpha-1}}{s^{n+2-\alpha-1}} \Gamma(n-\alpha) N[f^{(n)}(t)] \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}} N[f^{(n)}(t)] \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}}\left(\frac{s^n N[f(t)]}{u^n} - \frac{s^{n-2}}{u^n} f(0) - \frac{s^{n-3}}{u^{n-1}} f^{'}(0) \cdots - \frac{f^{(n-1)}(0)}{su}\right). \end{aligned}$$

• For $n = 1$, we have:

$$N[{}^{\mathscr{C}}_{a}D^{\alpha}_{t}f(t)] = \left(\frac{u}{s}\right)^{-\alpha}\left(N[f(t)] - \frac{f(a)}{s^2}\right).$$

Definition 4. *The definition for the CF derivative is defined as [20]:*

$${}^{\mathscr{CF}}_{a}D^{\alpha}_{t}f(t) = \frac{m(\alpha)}{1-\alpha}\int_a^t e^{\frac{-\alpha}{1-\alpha}(t-x)}f'(x)dx,$$

where $\alpha \in \mathbb{C}, \quad \Re e(\alpha) > 0,\ n = [\Re e(\alpha)] + 1.$

Lemma 2. *The following result is acquired by applying the NE transform to the CF fractional derivative:*

$$N[{}^{\mathscr{CF}}_{a}D^{\alpha}_{t}f(t)] = \frac{m(\alpha)}{1-\alpha}\left(\left(s^2N[f(t)] - f(a)\right)\frac{1}{s\left(s - \frac{-\alpha}{1-\alpha}u\right)}\right). \tag{7}$$

Proof. We have

$${}^{\mathscr{CF}}_{a}D^{\alpha}_{t}f(t) = \frac{m(\alpha)}{1-\alpha}\int_a^t e^{\frac{-\alpha}{1-\alpha}(t-x)}f'(x)dx.$$

Then, we write:

$${}^{\mathscr{CF}}_{a}D^{\alpha}_{t}f(t) = \frac{m(\alpha)}{1-\alpha}\left[e^{\frac{-\alpha}{1-\alpha}t} * f'(t)\right].$$

When we use the NE transform, we obtain:

$$\begin{aligned} N[{}^{\mathscr{CF}}_{a}D^{\alpha}_{t}f(t)] &= \frac{m(\alpha)}{1-\alpha}us\left[N[f'(t)] * N[e^{\frac{-\alpha}{1-\alpha}t}]\right] \\ &= \frac{m(\alpha)}{1-\alpha}\left(\left(s^2N[f(t)] - f(a)\right)\frac{1}{s\left(s - \frac{-\alpha}{1-\alpha}u\right)}\right). \end{aligned}$$

Definition 5. *We define the Mittag-Leffler function* $E_{\alpha,\beta}(u)$ *as [21]:*

$$E_{\alpha,\beta}(u) = \sum_{j=0}^{\infty}\frac{u^j}{\Gamma(\alpha j + \beta)}, \quad u, \beta \in \mathbb{C}, \quad \Re e(\alpha) > 0.$$

Lemma 3. *The NE transformation of the Mittag-Leffler function is presented as follows:*

$$N[E_{\alpha}(-at^{\alpha})] = \frac{s^{\alpha}}{s^2(s^{\alpha} + au^{\alpha})},$$

and

$$N[1 - E_{\alpha}(-at^{\alpha})] = \frac{au^{\alpha}}{s^2(s^{\alpha} + au^{\alpha})}.$$

Proof. We know that:

$$N[E_{\alpha}(-at^{\alpha})] = N\left[\sum_{j=0}^{\infty}\frac{-at^{\alpha}}{\Gamma(\alpha j + 1)}\right] = \sum_{j=0}^{\infty}\frac{1}{s^2}\left(\frac{-au^{\alpha}}{s^{\alpha}}\right)^k = \frac{s^{\alpha}}{s^2(s^{\alpha} + au^{\alpha})},$$

and

$$N[1 - E_{\alpha}(-at^{\alpha})] = N[1] - N[E_{\alpha}(-at^{\alpha})] = \frac{1}{s^2} - \frac{s^{\alpha}}{s^2(s^{\alpha} + au^{\alpha})} = \frac{au^{\alpha}}{s^2(s^{\alpha} + au^{\alpha})}.$$

Definition 6. *The AB derivative is defined as [21]:*

$${}^{\mathscr{AB}}_{a}D^{\alpha}_{t}f(t) = \frac{AB(\alpha)}{1-\alpha}\int_a^t E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right)f'(x)dx,$$

where $\alpha \in \mathbb{C},\ \Re e(\alpha) > 0,\ n = [\Re e(\alpha)] + 1.$

Lemma 4. *The following result is obtained by applying the NE transform to the AB fractional derivative:*

$$N[{}^{\mathscr{A}\mathscr{B}}_{a}D_t^{\alpha} f(t)] = \frac{AB(\alpha)}{1-\alpha}\left(s^2 N[f(t)] - f(a)\right)\frac{s^{\alpha-2}}{(s^{\alpha} - \frac{\alpha}{1-\alpha}u^{\alpha})}. \tag{8}$$

Proof. We have

$${}^{\mathscr{A}\mathscr{B}}_{a}D_t^{\alpha} f(t) = \frac{AB(\alpha)}{1-\alpha}\int_a^t E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right) f^{'}(x)dx.$$

When we use the NE transform and convolution, we obtain:

$$N[{}^{\mathscr{A}\mathscr{B}}_{a}D_t^{\alpha} f(t)] = \frac{AB(\alpha)}{1-\alpha} N[f'(t) * E_{\alpha}(\frac{-\alpha}{1-\alpha})t^{\alpha}] = \frac{AB(\alpha)}{1-\alpha}\left(s^2 N[f(t)] - f(a)\right)\frac{s^{\alpha-2}}{(s^{\alpha} - \frac{\alpha}{1-\alpha}u^{\alpha})}.$$

3 Applications

3.1 Gross domestic product model

Gross domestic product (GDP) is a critical metric for analyzing the degree of economic advancement in a country or region. It has a considerable impact on the development of future macroeconomic goals and policies related to economic regulation. Predicting GDP has been a key field of study due to the rapid economic growth [28]. Let us consider the following equation:

$$f'(t) = k, \qquad f'(t) = kf(t).$$

Now, we will investigate the solutions of the fractional GDP model by the NE transform method.

3.1.1 GDP model with Caputo derivative

We start by looking at the GDP model using the power-law kernel:

$${}^{\mathscr{C}}_{a}D_t^{\ltimes} f(t) = k, \qquad {}^{\mathscr{C}}_{a}D_t^{\ltimes} f(t) = kf(t). \tag{9}$$

Utilizing the NE transform on both sides of the above model's first equation results in:

$$\left(\frac{u}{s}\right)^{-\alpha}\left(N[f(t)] - \frac{f(a)}{s^2}\right) = \frac{k}{s^2},$$

and then

$$N[f(t)] = \frac{ku^{\alpha}}{s^{\alpha+2}} + \frac{f(a)}{s^2}.$$

When we use the inverse NE transform, the solution is obtained as:

$$f(t) = f(a) + \frac{t^{\alpha}k}{\Gamma(\alpha+1)}. \tag{10}$$

The operations are done for the second equation of (9). Then we will have:

$$f(t) = f(a)E_{\alpha}(kt^{\alpha}).$$

3.1.2 GDP model with CF derivative

Using the exponential-decay kernel, we examine the GDP model as follows:

$$ {}_{a}^{\mathscr{CF}}D_t^{\ltimes} f(t)k, \qquad {}_{a}^{\mathscr{CF}}D_t^{\ltimes} f(t)kf(t). \tag{11} $$

The NE transform is applied to both sides of the first equation of Eqs. (11) will yield:

$$ N[{}_{a}^{\mathscr{CF}}D_t^{\ltimes} f(t)] = N[k]. $$

If we utilize the NE transform and Lemma (2), we write

$$ N[f(t)] = \frac{k(s(1-\alpha)+\alpha u)}{s^3 m(\alpha)} + \frac{f(a)}{s^2}. $$

Upon applying the inverse NE transform to the equation mentioned above, we will arrive at:

$$ f(t) = f(a) + \frac{k(1-\alpha+\alpha t)}{m(\alpha)}. \tag{12} $$

The operations are done for second equation of (9). We get the solution as:

$$ f(t) = \frac{m(\alpha)f(a)}{m(\alpha)-k(1-\alpha)} exp\left(\frac{-k\alpha t}{m(\alpha)-k(1-\alpha)}\right). $$

3.1.3 GDP model with AB derivative

The GDP model with the AB derivative is now examined as:

$$ {}_{a}^{\mathscr{AB}}D_t^{\ltimes} f(t) = k, \qquad {}_{a}^{\mathscr{AB}}D_t^{\ltimes} f(t)kf(t). \tag{13} $$

Utilizing the NE transform on both sides of first equation of Eq. (13) and applying Lemma (4), gives:

$$ \frac{AB(\alpha)}{1-\alpha}\left(s^2 N[f(t)] - f(a)\right)\frac{s^{\alpha-2}}{(s^\alpha - \frac{\alpha}{1-\alpha}u^\alpha)} = \frac{k}{s^2}, $$

and

$$ N[f(t)] = \frac{k(s^\alpha(1-\alpha)-\alpha u^\alpha)}{s^2 s^\alpha AB(\alpha)} + \frac{f(a)}{s^2}. $$

Using the inverse NE transform, yields:

$$ f(t) = f(a) + \frac{k(1-\alpha)}{AB(\alpha)} + \frac{k\alpha t^\alpha}{AB(\alpha)\Gamma(\alpha+1)}. \tag{14} $$

If similar operations are done to second equation of Eq. (13), one can readily find

$$ f(t) = \frac{AB(\alpha)f(a)}{AB(\alpha)-k(1-\alpha)} E_\alpha\left[\frac{k\alpha t^\alpha}{AB(\alpha)-k(1-\alpha)}\right]. $$

3.2 *Economic model*

Numerous economic models are offered with mathematical tools that demonstrate how to take market issues into account while aiming for equilibrium. Numerous applications pertaining to these models are visible. The perspective on expense time is predicated on the shadily competitive setting, where the request tends to be lower and at a variety of costs rather than a single charge. In a competitive market, the quantity of goods that retailers prepare and the quantity that buyers desire is equal. This is known as the competitive equilibrium. Request function $q_d(t)$ and provision function $q_s(t)$ may provide the requested and prepared amounts by [29]:

$$ q_d(t) = d_0 - d_1 f(t), \quad q_s(t) = -s_0 + s_1 f(t). $$

The cost of goods is defined by the calculations above. The positive constants d_0, s_0, d_1, s_1 also have an impact on the amount prepared and asked. The equilibrium cost is calculated as follows:

$$f^* = \frac{d_0 + s_0}{d_1 + s_1},$$

for $q_d(t) = q_s(t)$, when the requested and supplied amounts are equal. As a result, the cost tends to remain constant, and in this situation, there is neither a shortage nor a surplus. The expense timing equation is something we evaluate as

$$f'(t) = k(q_d + q_s),$$

where $k > 0$. Then, we have:

$$f'(t) + k(d_1 + s_1)f(t) = k(d_0 + s_0). \tag{15}$$

One can get the solution:

$$f(t) = \frac{d_0 + s_0}{d_1 + s_1} - \left[f(0) + \frac{d_0 + s_0}{d_1 + s_1} \right] exp(-k(d_1 + s_1)t).$$

We determine the $f(0)$ as the expense at the time $t = 0$. We describe q_d and q_s by:

$$q_d(t) = d_0 - d_1 f(t) + d_2 f'(t), \qquad q_s(t) = -s_0 + s_1 f(t) - s_2 f'(t).$$

We equalize $q_d(t)$, $q_s(t)$ and obtain:

$$f'(t) - \frac{d_1 + s_1}{d_2 + s_2} f(t) = -\frac{d_0 + s_0}{d_2 + s_2}. \tag{16}$$

One can get the solution:

$$f(t) = \frac{d_0 + s_0}{d_1 + s_1} - \left[f(0) + \frac{d_0 + s_0}{d_1 + s_1} \right] exp\left(\frac{d_1 + s_1}{d_2 + s_2} t \right).$$

3.2.1 Economic model with Caputo derivative

We have:

$${}_0^{\mathscr{C}}D_t^{\alpha} f(t) + k(d_1 + s_1)f(t) = k(d_0 + s_0), \qquad \alpha \in (0,1).$$

Applying the NE transform to the equation above yields the following results:

$$N[{}_0^{\mathscr{C}}D_t^{\alpha} f(t)] + k(d_1 + s_1)N[f(t)] = k(d_0 + s_0).$$

Using the NE transform of the Caputo fractional derivative, gives:

$$\left(\frac{u}{s}\right)^{-\alpha} \left(N[f(t)] - \frac{f(0)}{s^2} \right) + k(d_1 + s_1)N[f(t)] = \frac{k(d_0 + s_0)}{s^2},$$

and

$$N[f(t)] = \frac{k(d_0 + s_0)s^{-\alpha}}{s^2(u^{-\alpha} + s^{-\alpha}k(d_1 + s_1))} + \frac{u^{-\alpha} f(0)}{s^2(u^{-\alpha} + s^{-\alpha}k(d_1 + s_1))}.$$

By benefiting from the inverse NE transform and Lemma (3), we obtain:

$$f(t) = \frac{(d_0 + s_0)}{(d_1 + s_1)}[1 - E_\alpha(-k(d_1 + s_1)t^\alpha)] + f(0)E_\alpha(-k(d_1 + s_1)t^\alpha). \tag{17}$$

If we investigate Eq. (16) with the Caputo derivative, the solution can be expressed as:

$$f(t) = -\frac{(d_0 + s_0)}{(d_1 + s_1)}[1 - E_\alpha(\frac{d_1 + s_1}{d_2 + s_2} t^\alpha)] + f(0)E_\alpha(\frac{d_1 + s_1}{d_2 + s_2} t^\alpha).$$

3.2.2 Economic model with CF derivative

We take into consideration:

$$ {}_0^{\mathscr{CF}}D_t^{\alpha} f(t)+k(d_1+s_1)f(t)=k(d_0+s_0). $$

When the NE transform is applied to the equation above, we acquire:

$$ N[{}_0^{\mathscr{CF}}D_t^{\alpha} f(t)]+k(d_1+s_1)N[f(t)]=k(d_0+s_0). $$

Using the NE transform of the CF derivative, yields:

$$ N[f(t)]=\frac{k(d_0+s_0)}{\frac{m(\alpha)}{1-\alpha}s^2\frac{1}{s(s-\frac{-\alpha}{1-\alpha}u)}+k(d_1+s_1)}+\frac{\frac{m(\alpha)}{1-\alpha}f(0)\frac{1}{s(s-\frac{-\alpha}{1-\alpha}u)}}{\frac{m(\alpha)}{1-\alpha}s^2\frac{1}{s(s-\frac{-\alpha}{1-\alpha}u)}+k(d_1+s_1)}. $$

With the inverse NE transform, one can find:

$$ f(t)=N^{-1}\left[\frac{k(d_0+s_0)}{\frac{m(\alpha)}{1-\alpha}(s^2)\frac{1}{s(s-\frac{-\alpha}{1-\alpha}u)}+k(d_1+s_1)}\right]+N^{-1}\left[\frac{\frac{m(\alpha)}{1-\alpha}f(0)\frac{1}{s(s-\frac{-\alpha}{1-\alpha}u)}}{\frac{m(\alpha)}{1-\alpha}(s^2)\frac{1}{s(s-\frac{-\alpha}{1-\alpha}u)}+k(d_1+s_1)}\right], $$

and the solution is obtained as follows:

$$ \begin{aligned} f(t)=&\frac{m(\alpha)f(0)\,exp\left(\frac{\alpha k(d_1+s_1)t}{-m(\alpha)+(\alpha-1)k(d_1+s_1)}\right)}{m(\alpha)-(\alpha-1)k(d_1+s_1))}\\ &-\frac{m(\alpha)(d_0+s_0)(-1+exp\left(\frac{\alpha k(d_1+s_1)t}{-m(\alpha)+(\alpha-1)k(d_1+s_1)}\right)+(\alpha-1)k(d_1+s_1)}{(d_1+s_1)(-m(\alpha)+(\alpha-1)k(d_1+s_1))}. \end{aligned} \tag{18} $$

If we consider Eq. (16) with the CF derivative, the solution is found as follows:

$$ \begin{aligned} f(t)=&\frac{m(\alpha)f(0)(d_2+s_2)}{m(\alpha)(d_2+s_2)+(\alpha-1)d_1+(\alpha-1)s_1}\,exp\left(\frac{\alpha(d_1+s_1)t}{m(\alpha)(d_2+s_2)+(\alpha-1)d_1+(\alpha-1)s_1}\right)\\ &+\frac{(d_0+s_0)((\alpha-1)d_1+(\alpha-1)s_1-m(\alpha))(-1+exp\left(\frac{\alpha(d_1+s_1)t}{m(\alpha)(d_2+s_2)+(\alpha-1)d_1+(\alpha-1)s_1}\right)(d_2+s_2)}{(d_1+s_1)m(\alpha)(d_2+s_2)+(\alpha-1)d_1+(\alpha-1)s_1}. \end{aligned} $$

3.2.3 Economic model with the Mittag-Leffler kernel

We consider the following problem:

$$ {}_0^{\mathscr{AB}}D_t^{\alpha} f(t)+k(d_1+s_1)f(t)=k(d_0+s_0). $$

By applying the NE transform to the final equation, we obtain:

$$ N[{}_0^{\mathscr{AB}}D_t^{\alpha} f(t)]+k(d_1+s_1)N[f(t)]=k(d_0+s_0). $$

Thus

$$ N[f(t)]=\frac{k(d_0+s_0)(s^{\alpha}(1-\alpha)-\alpha u^{\alpha})+AB(\alpha)f(0)s^{\alpha-2}}{s^{\alpha}AB(\alpha)+k(d_1+s_1)(s^{\alpha}(1-\alpha)-\alpha u^{\alpha})}. $$

Using the inverse NE transform, gives:

$$ f(t)=N^{-1}\left[\frac{AB(\alpha)f(0)s^{\alpha-2}}{s^{\alpha}AB(\alpha)+k(d_1+s_1)(s^{\alpha}(1-\alpha)-\alpha u^{\alpha})}\right]+N^{-1}\left[\frac{k(d_0+s_0)(s^{\alpha}(1-\alpha)-\alpha u^{\alpha})}{s^{\alpha}AB(\alpha)+k(d_1+s_1)(s^{\alpha}(1-\alpha)-\alpha u^{\alpha})}\right]. $$

After that, the analytical solution is obtained as follows:

$$f(t) = \frac{AB(\alpha)f(0)}{AB(\alpha)+k(d_1+s_1)(1-\alpha)}E_\alpha\left(-\frac{\alpha k(d_1+s_1)t^\alpha}{AB(\alpha)+k(d_1+s_1)(1-\alpha)}\right) + \frac{(1-\alpha)k(d_0+s_0)}{AB(\alpha)+k(d_1+s_1)(1-\alpha)}E_\alpha\left(-\frac{\alpha k(d_1+s_1)t^\alpha}{AB(\alpha)+k(d_1+s_1)(1-\alpha)}\right) + \frac{(d_0+s_0)}{(d_1+s_1)}\left[1-E_\alpha(-\frac{\alpha k(d_1+s_1)t^\alpha}{AB(\alpha)+k(d_1+s_1)(1-\alpha)})\right].$$

Taking into consideration Eq. (16) with AB derivative, we have:

$$f(t) = \frac{AB(\alpha)f(0)(d_2+s_2)}{AB(\alpha)(d_2+s_2)-(\alpha-1)d_1+(\alpha-1)s_1}E_\alpha\left(\frac{\alpha(d_1+s_1)t^\alpha}{AB(\alpha)(d_2+s_2)-(\alpha-1)d_1+(\alpha-1)s_1}\right) - \frac{(\alpha-1)(d_0+s_0)}{AB(\alpha)(d_2+s_2)-(\alpha-1)d_1+(\alpha-1)s_1}E_\alpha\left(\frac{\alpha(d_1+s_1)t^\alpha}{AB(\alpha)(d_2+s_2)-(\alpha-1)d_1+(\alpha-1)s_1}\right).$$

3.3 Falling body problem

For models similar to those previously discussed, the authors usually replace the integer-order derivative with a fractional derivative in order to obtain fractional solutions. However, in terms of using physical models, this approach is not totally correct due to the need to maintain the dimensional fractional equation. For example, the fractional falling body problem was presented by the authors in [30] while preserving the dimension.

$$\frac{d}{dt} \to \frac{1}{\varphi^{1-\alpha}}\frac{d^\alpha}{dt^\alpha}, \quad 0<\alpha\le 1,$$

where φ exists, the dimension of seconds. Additionally, in [31,32], the problem of a falling body has been explored using fractional operators that incorporate an exponential kernel. In this part, we explore the falling body problem using Newton's second law, which states that a particle's acceleration is proportional to its mass and the net force acting on it. Consider *a* mass m object falling through the air in a gravitational field with a velocity $v(0)$ from *a* height h. Newton's second law enables us to derive [33].

$$\frac{mdv}{dt} + mkv = -mg. \tag{19}$$

Then, we get:

$$v(t) = -\frac{g}{k} + exp(-kt)\left(v(0)+\frac{g}{k}\right), \tag{20}$$

Now, the falling body problem is handled with Caputo, CF, and AB derivatives by the NE transform method.

3.3.1 The falling body problem with the power-law kernel

We consider:

$${}_0^{\mathscr{C}}D_t^\alpha v(t) + k\varphi^{1-\alpha}v(t) = -g\varphi^{1-\alpha}.$$

Using the NE transform gives:

$$N[{}_0^{\mathscr{C}}D_t^\alpha v(t)] + k\varphi^{1-\alpha}N[v(t)] = N[-g\varphi^{1-\alpha}].$$

By applying Lemma (1), one can write

$$\left(\frac{u}{s}\right)^{-\alpha}\left[N[v(t)] - \frac{v(0)}{s^2}\right] + k\varphi^{1-\alpha}N\left[v(t)\right] = \frac{-g\varphi^{1-\alpha}}{s^2},$$

and

$$N[v(t)] = \frac{v(0)}{s^2\left(1+\frac{k\varphi^{1-\alpha}u^\alpha}{s^\alpha}\right)} + \frac{-g\varphi^{1-\alpha}u^\alpha}{s^2\left(1+\frac{k\varphi^{1-\alpha}u^\alpha}{s^\alpha}\right)}.$$

Applying the inverse NE transform yields:

$$v(t)=v(0)E_\alpha(k\varphi^{1-\alpha}t^\alpha)-\frac{g}{k}\left[1-E_\alpha(k\varphi^{1-\alpha}t^\alpha)\right].$$

3.3.2 The falling body problem with the exponential-decay kernel

The falling body problem in the sense of CF is given as:

$${}_0^{\mathscr{CF}}D_t^\alpha v(t)+k\varphi^{1-\alpha}v(t)=-g\varphi^{1-\alpha},$$

where the initial velocity $v(0)=v_0$, g is the gravitational constant and k is the positive constant rate which shows the mass of the body.

When we apply the NE transform to the equation above, we obtain:

$$N\left[{}_0^{\mathscr{CF}}D_t^\alpha v(t)\right]+k\varphi^{1-\alpha}N[v(t)]=N\left[-g\varphi^{1-\alpha}\right].$$

Then, by using Lemma (2) we can write

$$\frac{m(\alpha)}{1-\alpha}\left(s^2N[v(t)]-v(0)\right)\frac{1}{s\left(s-\frac{-\alpha}{1-\alpha}u\right)}+k\varphi^{1-\alpha}N[v(t)]=\frac{-g\varphi^{1-\alpha}}{s^2},$$

and

$$N[v(t)]=\frac{m(\alpha)v(0)}{m(\alpha)+k\varphi^{1-\alpha}(1-\alpha)\left(s^2+\frac{k\varphi^{1-\alpha}\alpha us}{m(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}\right)}-\frac{g\varphi^{1-\alpha}}{k\varphi^{1-\alpha}+\frac{m(\alpha)s}{s(1-\alpha)+\alpha u}}.$$

If we apply the inverse NE transform to the last equation, we acquire:

$$\begin{aligned}v(t)=&\frac{m(\alpha)v(0)}{m(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}exp\left(\frac{-\alpha k\varphi^{1-\alpha}t}{m(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}\right)\\&-\frac{g\left[m(\alpha)+k\varphi^{1-\alpha}(1-\alpha)-m(\alpha)exp\left(\frac{-\alpha k\varphi^{1-\alpha}t}{m(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}\right)\right]}{km(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}.\end{aligned}$$

3.3.3 The falling body problem with the Mittag-Leffler kernel

The falling body problem in the sense of AB is given as:

$${}_0^{\mathscr{AB}}D_t^\alpha v(t)+k\varphi^{1-\alpha}v(t)=-g\varphi^{1-\alpha}, \tag{21}$$

If we use the NE transform, we get:

$$N\left[{}_0^{\mathscr{AB}}D_t^\alpha v(t)\right]+k\varphi^{1-\alpha}N[v(t)]=N\left[-g\varphi^{1-\alpha}\right].$$

Then:

$$\frac{AB(\alpha)}{1-\alpha}\left(s^2N[v(t)]-v(0)\right)\frac{s^{\alpha-2}}{\left(s^\alpha-\frac{\alpha}{1-\alpha}u^\alpha\right)}+k\varphi^{1-\alpha}N[v(t)]=\frac{-g\varphi^{1-\alpha}}{s^2},$$

$$N[v(t)]=\frac{-g\varphi^{1-\alpha}(s^\alpha(1-\alpha)-\alpha u^\alpha)+AB(\alpha)v(0)s^{\alpha-2}}{AB(\alpha)s^\alpha+k\varphi^{1-\alpha}(s^\alpha(1-\alpha)-\alpha u^\alpha)}. \tag{22}$$

Using the inverse NE transform, the analytical solution of Eq. (21) is obtained as follows:

$$\begin{aligned}v(t)=&\frac{AB(\alpha)v(0)}{AB(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\varphi^{1-\alpha}t^\alpha}{AB(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}\right)\\&-\frac{g\varphi^{1-\alpha}(1-\alpha)}{AB(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}E_\alpha\left(-\frac{\alpha k\varphi^{1-\alpha}t^\alpha}{AB(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}\right)\\&-\frac{g}{k}\left[1-E_\alpha\left(-\frac{\alpha k\varphi^{1-\alpha}t^\alpha}{AB(\alpha)+k\varphi^{1-\alpha}(1-\alpha)}\right)\right].\end{aligned}$$

Our findings in this paper can be be fruitfully used to connect the Jaynes–Cummings–type studies by viewing the NE (non-integer / nonlocal energy or nonlinear-operator) transform as a mathematical tool for introducing fractional dynamics and memory effects into quantum optical models [34]. For example, the two-atom dissipative-cavity and entanglement-survival results in Zhang et al. (2010) suggest that non-Markovian, long-lived coherence emerges from modified system–reservoir interactions; applying an NE transform with fractional-order operators could model such nonlocal dissipation more naturally and predict altered sudden-death timescales [35]. Similarly, the entropy and uncertainty analyses which probe entropy squeezing, entropic uncertainty and quasiprobability distributions in generalized or nonlinear Jaynes–Cummings settings — provide concrete observables (entropy measures, squeezing parameters, Wigner/Q-functions) that an NE-transformed fractional model would modify in characteristic ways [36]

In short, integrating an NE fractional framework into these Jaynes–Cummings variants offers a unified route to cap turememory, anomalous relaxation, and enhanced nonlinearity, yielding testable predictions for entanglement dyna mics,entropy squeezing and phase-space distributions [37,38].

4 Conclusion

The primary purpose of this study is to show that the NE transform method is one of the most important and straightforward approaches to solving differential equations and systems. In terms of fractional calculus theory, the new integral transform looks promising and can be applied to a variety of linear and nonlinear equations. Additionally, the effect of the Caputo derivative via the power-law kernel, the effect of the CF derivative via the exponential-decay kernel, and the effect of the AB derivative via the Mittag-Leffler kernel have been investigated in this paper.

Availability of data and materials

All data that was used is included in the research.

Competing interests

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Authors' contributions

All authors have contributed, read, and approved the manuscript.

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Ethics approval and consent to participate

Not applicable.

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