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A note about properties of exponential divisors

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Abstract: The aim of this paper is to present some properties about the arithmetic functions which use exponential divisors.Among these properties we study the equation $\tau(n)=\tau^{(e)}(n)\tau^{*}(n)$, we show that $\sqrt{n}\leq\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}$, for all $n\geq 1$ and other inequalities related to these multiplicative arithmetic functions.

Keywords: exponential divisor, arithmetic function

1. Introduction

The present paper contains some results regarding the inequalities related to several multiplicative arithmetic functions which use exponential divisors and exponential unitary divisors. But, first, we present several classes of divisors, which we will use in this paper.

Let n be a positive integer number. Starting from the set of divisors of n were defined two main multiplicative arithmetic functions, namely, $\sigma(n)$ is the sum of the divisors of n and $\tau(n)$ is the number of divisors of n.

In [2], E. Cohen changed the notion of *block-factor* which was introduced by R. Vaidyanathaswamy in [14], by the notion of *unitary divisor*, which means that: a divisor d of n is a unitary divisor when $\left(d,\frac{n}{d}\right)=1$. In this case, we note by $\sigma^{*}(n)$ the sum of the unitary divisors of n and by $\tau^{*}(n)$ the number of unitary divisors of n, which is, in fact, the number of the square-free divisors of n. Several inequalities inequalities between these arithmetic functions are given below.

S. Sivaramakrishnan and C. S. Venkataraman (see e.g. [8]), show that $\sqrt{n}\leq\frac{\sigma(n)}{\tau(n)}$, for every $n\geq 1$. J. Sándor and L. Tóth in [8] studied the same type of inequality for the unitary divisors, thus, they proved that $\sqrt{n}\leq\frac{\sigma^{*}(n)}{\tau^{*}(n)}$, for any $n\geq 1$. For other arithmetic inequalities see the books [1], [7], [8], [9].

Therefore, we will show, in this paper, the same type of the inequality for the arithmetic functions which use exponential divisors.

The class of *exponential divisors* was introduced by M. V. Subbarao in [10], in the following way: d is said to be an exponential divisor (or e-divisor) of $n=p_1^{a_1}...p_r^{a_r}>1$, if $d=p_1^{b_1}...p_r^{b_r}>1$, where $b_i|a_i$, for any $1\leq i\leq r$.
A variety of results related to the exponential divisors are given in many sources, such as: [9, 10, 11, 12].

N. Minculete and L. Tóth in [13] presented some properties of the arithmetic functions which use exponential unitary divisors or e-unitary divisors of $n=p_1^{a_1}...p_r^{a_r}>1$, if $d=p_1^{b_1}...p_r^{b_r}>1$, where b_i is an unitary divisor of a_i, so $\left(b_i,\frac{a_i}{b_i}\right)=1$, for any $1\leq i\leq r$.

Studying the various relations of the multiplicative arithmetic functions defined by different types of the divisors, we remarked a series of inequalities established between these ones. Among these, we have, from [4] and [5], the following

$$\tau^{(e)}(n)+\tau^{*}(n)\leq\tau(n)+1, \quad (1)$$

$$\sigma^{(e)}(n)+\sigma^{*}(n)\leq\sigma(n)+n, \quad (2)$$

and

$$\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}\geq\gamma(n)\cdot\frac{\tau^{(e)}(n)+1}{2}\geq\gamma(n). \quad (3)$$

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It is easy to see that, we have

$$\sigma^{(e)*}(n) \le \sigma^{(e)}(n) \le \sigma(n), \tag{4}$$

for any $n \ge 1$.

2. Main results

We will give below other inequalities for the above arithmetic functions to get a better characterization of the connections between them.

Theorem 2.1. a) Equation

$$\tau(n) = \tau^{(e)}(n)\tau^*(n), \tag{5}$$

has solution $n = 1$ and an infinity of solutions of the form $n = \prod_{\substack{p|n \\ a\in\{1,3\}}} p^a$.

b) There are an infinite number n so that we have

$$\tau(n) > \tau^{(e)}(n)\tau^*(n), \tag{6}$$

for $n = \prod_{p|n} p^a$, where $a \notin \{2,4,6\}$ and at least one prime number p of the decomposition of n has a exponent unequal to 1 and 3;

c) There are an infinite number n so that we have

$$\tau(n) < \tau^{(e)}(n)\tau^*(n), \tag{7}$$

for $n = \prod_{p|n} p^a$, where $a \in \{1,2,3,4,6\}$ and at least one prime number p of the decomposition of n has a exponent unequal to 1 and 3.

Proof. a) For $n = 1$ we can see, immediately, that the equation is verified. For $n = p^a$, where p is a prime number, the equation becomes $\tau(p^a) = \tau^{(e)}(p^a)\tau^*(p^a)$, which is equivalent to the equation

$$\frac{a+1}{2} = \tau(a), \tag{8}$$

for every $a \ge 1$.

This atypical equation leads to the idea search an upper limit or lower, as a function of variable a, which may be easier to handle. In [9], Sierpinski showed that $\tau(a) < 2\sqrt{a}$, for any $a \ge 1$, which means that the equation (8) becomes $a+1 < 4\sqrt{a}$, it follows that $a \in (7-4\sqrt{3}, 7+4\sqrt{3}) \cap \mathbb{N} = \{1,2,3,...,13\}$.

Substituting these values in equation (8), we deduce $a \in \{1,3\}$, and considering the fact that the functions $\tau, \tau^{(e)}$ and τ^* are multiplicative, we have

$$\tau(n) = \tau\left(\prod_{\substack{p|n \\ a\in\{1,3\}}} p^a\right) = \prod_{\substack{p|n \\ a\in\{1,3\}}} \tau(p^a) = \prod_{\substack{p|n \\ a\in\{1,3\}}} (a+1) =$$

$$= \prod_{\substack{p|n \\ a\in\{1,3\}}} 2\tau(a) = \tau^*\left(\prod_{\substack{p|n \\ a\in\{1,3\}}} p^a\right)\tau^{(e)}\left(\prod_{\substack{p|n \\ a\in\{1,3\}}} p^a\right) =$$

$$\tau^*(n)\tau^{(e)}(n).$$

b) To prove these requirements we will study the values of a, for which the inequality $\frac{a+1}{2} > \tau(a)$ holds, the idea is based on equality (8).

From Langford's inequality (see e.g. [7]), we have $\frac{a+1}{2} \ge \frac{\sigma(a)}{\tau(a)}$, for any $a \ge 1$, which means that,using relation (8), we have to say under what conditions $\frac{\sigma(a)}{\tau(a)} \ge \tau(a)$, so, we must find the values of a for which the inequality $\sigma(a) = \tau^2(a)$ holds.

But, G. Mincu and L. Panaitopol studied in [3] the equation $\sigma(a) = \tau^2(a)$, obtaining solutions 1 and 3. How can we use this result in our approach is easy to see.

In demonstration of this equation we see, from [3], that $\frac{\sigma(a)}{\tau^2(a)} \ge \frac{13}{9} > 1$, for $a \ge 5$ and odd, it follows the inequality $\frac{a+1}{2} > \tau(a)$.

For $a = 2^m$, with $m \ge 4$, we have $\frac{\sigma(a)}{\tau^2(a)} > 1$, so $\frac{a+1}{2} > \tau(a)$. If $a \in \{2,4,8\}$, then implies $\frac{\sigma(a)}{\tau(a)} < \tau(a)$. But, for $a = 8$, we get $\tau(8) = 4 < \frac{9}{2}$, so $\frac{a+1}{2} > \tau(a)$, and for $a \in \{2,4\}$, we will obtain $\frac{a+1}{2} < \tau(a)$.

Case remains to be studied is $a = 2^m \cdot 3$. Therefore, we have the relation $\frac{\sigma(2^m\cdot 3)}{\tau^2(2^m\cdot 3)} = \frac{\sigma(2^m)}{\tau^2(2^m)} > 1$, for every $m \ge 4$. By checking for $m \in \{1,2,3\}$, we deduce that only $a = 6$, we have $\frac{a+1}{2} < \tau(a)$.

In other words, we obtain $\frac{a+1}{2} > \tau(a)$, for any $a \ne 1,2,3,4,6$, and for $a \in \{2,4,6\}$ we deduce $\frac{a+1}{2} < \tau(a)$.

In conclusion, if we take $n = \prod_{p|n} p^a$, where $a \notin \{2,4,6\}$ and at least one prime number p of the decomposition of n has a exponent unequal to 1 and 3, then we find

$$\tau(n) > \tau^{(e)}(n)\tau^*(n),$$

and if we take $n = \prod_{p|n} p^a$, where $a \in \{1,2,3,4,6\}$ and at least one prime number p of the decomposition of n has a exponent unequal to 1 and 3, then we obtain

$$\tau(n) < \tau^{(e)}(n)\tau^*(n).$$

□

Remark 2.2. If we combine the two forms of n, then we deduce that there are situations when $\tau(n) > \tau^{(e)}(n)\tau^*(n)$ and there are situtions when the reverse inequality holds. Therefore, for $n = p^2q^5$ we have $\tau(n) = 18 > 16 = \tau^{(e)}(n)\tau^*(n)$, and for $n = p_1^2p_2^2q^5$ we get $\tau(n) = 54 < 64 = \tau^{(e)}(n)\tau^*(n)$.

Proposition 2.3. For any $n \geq 1$, the inequality

$$\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \tau^{(e)}(n) + \gamma(n) - 1, \quad (9)$$

holds.

Proof. Case $n = 1$, implies equality in the relation (9). For every number $n \geq 2$, we deduce the inequality $(\tau^{(e)}(n) - 1)(\gamma(n) - 2) \geq 0$, which is equivalent to $\tau^{(e)}(n)\gamma(n) + 2 \geq 2\tau^{(e)}(n) + \gamma(n)$, so $\tau^{(e)}(n)\gamma(n) + \gamma(n) \geq 2\tau^{(e)}(n) + 2\gamma(n) - 2$, which means that $\frac{(\tau^{(e)}(n)+1)\gamma(n)}{2} \geq \tau^{(e)}(n) + \gamma(n) - 1$. According to inequality (3), it follows inequality (9).

Theorem 2.4. For every $n \geq 1$, there is the following inequality

$$\sigma^{(e)}(n) \leq n + n\ln\tau^{(e)}(n). \quad (10)$$

Proof. For $n = 1$, the inequality is true.
If $n > 1$, then the smallest exponential divisor of $n = p_1^{a_1}p_2^{a_2}...p_r^{a_r} > 1$ is $p_1p_2...p_r = \gamma(n)$. Next, we will replace the exponential exponential divisors of n with similar values, but higher than they. If the largest exponential divisor of n is n, the next exponential divisor of n, in descending order, is less than $\frac{n}{2}$. Therefore, if $\gamma(n) = d_1, d_2, ..., d_s = n$ are the exponential divisors of n, then it is easy to see that $d_j \leq \frac{n}{j}$, for all $i = \overline{1,s}$, where $s = \tau^{(e)}(n)$. Consequently, we obtain

$$\sigma^{(e)}(n) = d_s + d_{s-1} + d_{s-2} + ... + d_1 \leq n + \frac{n}{2} + \frac{n}{3} + ... + \frac{n}{s} =$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{s}\right) < n(1 + \ln s),$$

so, we deduce inequality (10).

Theorem 2.5. For any $n \geq 1$, there is the following inequality

$$\sqrt{n} \leq \frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)}. \quad (11)$$

Proof. If $n = 1$, we get equality in (11). Let's consider $n \geq 2$. For $n = p$, where p is a prime number, we have $\sqrt{p} \leq p$, which is true.

Next, we consider the case when $n = p^a$, with $a \geq 2$, because, we are interested in the number of divisor a. If a is a prime number, then the number a has only two divisors, so inequality (11) becomes $2\sqrt{p^a} \leq p^a + p$, which is true, because $0 \leq (\sqrt{p^a} - 1)^2 + p - 1$. If a is a compose number, then $a \geq 4$ and we will write the divisors of a in ascending order, in following way, $1 = d_1 < d_2 < ... < d_{s-1} < d_s = a$, where $s = \tau(a) \geq 3$, $d_2 \geq 2$ and $d_{s-1} \leq a - 2$. One can easily show that $p^a \geq p^{a-1} + p^{a-2} + ... + p$, for all $a \geq 1$ and for every prime number p.

We note that among the numbers 1,2,3,...,$a-2$ are the numbers $a - d_{s-1}, a - d_{s-2}, ..., a - d_2$, wich means that

$$\sigma^{(e)}(p^a) = p^a + p^{d_{s-1}} + p^{d_{s-2}} + ... + p^{d_2} + p \geq$$
$$p^{a-1} + p^{a-2} + ... + p + p^{d_{s-1}} + p^{d_{s-2}} + ... + p^{d_2} + p \geq$$
$$p^{a-1} + p^{a-d_2} + ... + p^{a-d_{s-2}} + p^{a-d_{s-1}} + p + p^{d_{s-1}} + p^{d_{s-2}} +$$
$$... + p^{d_2} + p \geq 2(s-1)\sqrt{p^a} \geq s\sqrt{p^a} = \tau(a)\sqrt{p^a} =$$
$$\tau^{(e)}(p^a)\sqrt{p^a}, \text{because, we know that } s \geq 2.$$

Hence $\sigma^{(e)}(p^a) \geq \tau^{(e)}(p^a)\sqrt{p^a}$, for every prime number p and for all $a \geq 1$.

By using the fundamental theorem of arithmetic, for $n > 1$, we have the decomposition in prime factors of n given as $n = p_1^{a_1}p_2^{a_2}...p_r^{a_r}$, from where, making the product for all prime factors in the above inequality, and according to the functions $\sigma^{(e)}(n)$, $\tau^{(e)}(n)$ and $\sqrt{n}$ are multiplicative, we deduce the inequality from the statement.

□

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