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# Extensions of Riemann-Liouville Fractional Integral Inequalities on Time Scales

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**Abstract:** In this article, we present some inequalities involving Riemann-Liouville Fractional Integrals and establish some extensions of the integrals on time scales. Also, we establish some results through the application of arithmetic-geometric mean inequality.

**Keywords:** AM-GM inequality, extension,fractional integral inequality, time sacles.

## 1 Introduction

An arbitrary non-empty closed subset of the real numbers is called a time scale and it is mostly designated by $\mathbb{T}$. The area was exposed to mathematicians by Hilger in 1988 [16]. Hilger only came out with Delta $(\Delta)$ derivatives, Nabla $(\nabla)$ derivatives and their anti-derivatives initially. After the introduction by Hilger, Time Scale Calculus has became familiar with researchers and made a lot of advancement in the research fraternity. The discovery of time scales is a big gain to researchers. In the area of analysis for instance, Time Scale Calculus gives a resourceful tool to merge continuous and discrete problems in one theory. When we chose $\mathbb{T}=\mathbb{R}$, then the theory of time scale turns to real analysis and when $\mathbb{T}=\mathbb{Z}$, it turns to discrete analysis.

The purpose of this paper is to extend Riemann-Liouville Fractional Integral inequalities on time scales and also present some inequalities involving the integrals on time scales. The other parts of this papers are arranged in the following order. Part 2 is the preliminary section which is made up of basic concepts in time scales and some useful definitions. Part 3, presented the results and discussions section of this paper and part 4 contains the conclusion section.

## 2 Preliminaries

Here we state some key definitions in the area of time scales, details are the cited reference.

**Definition 21** *[4, 10]Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$ and $\sigma: \mathbb{T} \to \mathbb{T}$. Then the forward jump operator is defined as*

$$\sigma(t) = \inf\{s\colon s \in T, s > t\}.$$

**Definition 22** *[4, 10, 17] Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$ and $\rho: \mathbb{T} \to \mathbb{T}$. Then the backward jump operator is defined as*

$$\rho(t) = \sup\{s\colon s \in T, s < t\}.$$

Let $\mathbb{T}=\mathbb{R}$. Then $\sigma(t)=t$. Also, let $\mathbb{T}=\mathbb{Z}$. Then $\sigma(t)=t+1$. Correspondingly, let $\mathbb{T}=\mathbb{R}$, then $\rho(t)=t$. Let $\mathbb{T}=\mathbb{Z}$, then $\rho(t)=t-1$.

**Definition 23** *[3]Let $t \in \mathbb{T}$, then $t$ is called right dense if $\sigma(t)=t$. It is called right scattered if $\sigma(t)>t$.*

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**Definition 24** *[3]Let $t \in \mathbb{T}$, then $t$ is called left dense if $\rho(t) = t$. It is called left scattered if $\rho(t) < t$.*

**Definition 25** *Let $t \in \mathbb{T}$. Suppose $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is right-dense.*

**Definition 26** *Let $t \in \mathbb{T}$. Suppose $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is left-dense.*

Points that are both right-dense and left-dense at the same time are referred as dense.

**Definition 27** *[12] A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense (rd)-continuous, provided it is continuous at all right -dense(rd) points in $\mathbb{T}$ and its left-sided limits are finite at the left-dense points in $\mathbb{T}$. All right-dense (rd)-continuous functions are represented by $\mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$.*

**Definition 28** *[7,25] Suppose $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^{\Bbbk}$, then $f^{\triangle}(t)$ (if only it exist) with the property that, for any $\varepsilon > 0$ there exist a neighbourhood $\bigcup$ of $t$, thus $\bigcup = (t - \sigma, t + \sigma) \backslash \mathbb{T}$ for some $\sigma > 0$ so that*

$$[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \leq \varepsilon \, |\sigma(t) - s| \, , \; \forall \; s \in \bigcup. \tag{1}$$

**Definition 29** *[11] Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^{\Bbbk}$ then the following are valid.*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t), \qquad \alpha \quad \text{is constant.}$$

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

**Definition 210** *[10,11] A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta anti-derivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{k}$, the delta indefinite and definite integrals are defined respectively by the following equations.*

$$\int f(t)\Delta t = F(t) + C, \tag{2}$$

$$\int_a^b f(s)\Delta s = F(b) - F(a). \tag{3}$$

**Definition 211** *[10,11] A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla anti-derivative of $g : \mathbb{T} \to \mathbb{R}$ provided $G^{\nabla}(t) = g(t)$ for all $t \in \mathbb{T}_k$, the nabla indefinite and definite integrals are defined respectively by the following equations.*

$$\int g(s)\nabla s = G(b) - G(a). \tag{4}$$

$$\int_a^b g(s)\nabla s = G(b) - G(a). \tag{5}$$

When $\mathbb{T}$ has a left-scattered maximum $M$, then it is $\mathbb{T}^k = \mathbb{T} - M$. Also, when $\mathbb{T}$ has a right-scattered minimum $m$, then it is $\mathbb{T}_k = \mathbb{T} - m$.

**Definition 212** *[26] Let $a_1, a_2 \in \mathbb{T}$, and $f^{\Delta}, g^{\Delta}$ be left-dense continuous functions. Then*

$$\int_{a_1}^{a_2} f(t)g^{\Delta}(t)\Delta t = (fg)(a_2) - (fg)(a_1) - \int_{a_1}^{a_2} f^{\Delta}(t)g(\rho(t))\Delta t \tag{6}$$

is the time scale integration by parts.

**Definition 213** *[8] Let $t$ be any arbitrary point in $\mathbb{T}$. Then for any given function $f$ defined on $\mathbb{T}$ is $\Delta$-integrable from $t$ to $\sigma(t)$ and is expressed as*

$$\int_t^{\sigma(t)} f(x)\Delta x = (\sigma(t) - t)f(t). \tag{7}$$

**Definition 214** *[8,26]The graininess function $\mu : \mathbb{T} \longrightarrow [0, \infty)$ is defined by*

$$\mu(t) = \sigma(t) - t. \tag{8}$$

**Lemma 21** *[11] Let $t, \sigma(t) \in \mathbb{T}$. if $f, g \in \mathbb{C}_{rd}(\mathbb{T}, \mathbb{R})$, then*

$$\int_t^{\sigma(t)} |f(\tau)g(\tau)|\Delta\tau \leq \left[\int_t^{\sigma(t)} |f(\tau)|^p \Delta\tau\right]^{\frac{1}{p}} \left[\int_t^{\sigma(t)} |g(\tau)|^q \Delta\tau\right]^{\frac{1}{q}}, \tag{9}$$

where $q > 1$, $\frac{1}{q} + \frac{1}{p} = 1$.

**Definition 215** *[21] Let $a, b \in \mathbb{T}, a < b, h : \mathbb{T} \to \mathbb{R}$ and $\alpha \in [0, 1]$. Then diamond-$\alpha$ integral (or $\Diamond_\alpha$-integral) of $h$ from $a$ to $b$ (or on $[a, b]_{\mathbb{T}}$) is defined by*

$$\int_a^b h(t)\Diamond_\alpha t = \alpha \int_a^b h(t)\Delta t + (1 - \alpha)\int_a^b h(t)\nabla t, \tag{10}$$

*provided $h$ is delta and nabla integrable on $[a, b]_{\mathbb{T}}$.*

**Lemma 22** *[5] Let $f(\tau), g(\tau)$ be two $\mathbb{C}_{rd}$ functions, $\sigma(t), t \in \mathbb{T}$ and $0 < \frac{f(\tau)}{g(\tau)} \le \varphi$. Then we have*

$$\mu(\tau)f(\tau) \le \varphi^{\frac{-1}{q}} \left[\int_t^{\sigma(t)} |f(\tau)| \diamond \alpha\tau\right]^{\frac{1}{p}} \left[\int_t^{\sigma(t)} |g(\tau)| \diamond \alpha\tau\right]^{\frac{1}{q}} \quad (11)$$

for $0 \le \alpha \le 1$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The left and right Riemann-Liouville integrals (R-L I) discovered around 1826 to 1882 by G.F.B. Riemann (1826-1866) and J. Liouville (1809-1882) [2,9] are defined respectively by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (12)$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad (13)$$

where $f$ is a continuous function on the interval $[a,b]$ and $\Gamma$ is the classical Euler gamma function.

Riemann-Liouville integrals are said to have been derived from the well known Cauchy's formula.

**Definition 216** *[6] Let $f \in L_1(a,b)$, then the Cauchy formula is given by*

$$\int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_1} f(t)dtdx_1 \dots dx_{(n-1)} = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt \quad (14)$$

for $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$.
There are many extensions or generalizations of (12) and (13) which include the following:

**Definition 217** *[13] Let $\beta \in \mathbb{R}(\beta) > 0$, then the left and right sided fractional conformable integral operators are defined respectively as follows;*

$${}_a^{\beta}\mathbb{J}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right)^{\beta-1} (t-a)^{\alpha-1} f(t)dt, \quad \theta > t \quad (15)$$

and

$${}_a^{\beta}\mathbb{J}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha}\right)^{\beta-1} (b-t)^{\alpha-1} f(t)dt, \quad b > t. \quad (16)$$

Also see [1,23].

**Definition 218** *[24] Let $f$ be integrable function, then*

$${}_a^{\beta}\mathbb{J}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{x^{\alpha} - t^{\alpha}}{\alpha}\right)^{\beta-1} t^{\alpha-1} f(t)dt, b > t, \quad (17)$$

for all $\alpha, \beta > 0$ and $\Gamma$ is the Euler Gamma function.

**Definition 219** *[2,15] Riemann-Liouville $k$-fractional integrals are defined as*

$$\mathbb{J}_{k,a^+}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a, \quad \mathbb{R}(\alpha) > 0 \quad (18)$$

and

$$\mathbb{J}_{k,b^-}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad b > x, \quad \mathbb{R}(\alpha) > 0, \quad (19)$$

where $\Gamma_k$ is the $k$ -gamma function.

**Definition 220** *Let $k > 0$, then $\Gamma_k$ function is defined as*

$$\Gamma_k(x) = \lim_{x \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad x \in C \backslash kZ^-. \quad (20)$$

**Definition 221** *For $x \in \mathbb{C}$ with $\mathbb{R}(x) > 0$, then the $k$ -gamma function, $\Gamma_k(x)$ is given by the integral*

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt, \quad (21)$$

for $t, k > 0$.

**Definition 222** *[14,18] Let $f \in L_1[0;\infty)$. Then the generalized Riemann - Liouville fractional integral $\mathbb{I}^{\alpha,k} f(x)$ of order $\alpha, k \ge 0$ is defined as*

$$\mathbb{I}^{\alpha,k} f(x) = \frac{(k+1)^{k+1}}{\Gamma(\alpha)} \int_0^x \left(x^{k+1} - t^{k+1}\right)^{\alpha} t^k f(t) dt \qquad (22)$$
$$\mathbb{I}^0 f(x) = f(x).$$

Where $\Gamma$ is the gamma function.

**Remark 21** *Let $k=0$, then equation (22) becomes the known classical Riemann - Liouville fractional integral.*

More on the Riemann-Liouville integrals can be found in [19,20,22].

**Definition 223** *Let $p_i, q_i$ be real positive numbers. Then the Arithmetic-Geometric Mean Inequality is defined as*

$$\frac{p_i + q_i}{2} \geq \sqrt{p_i q_i}, \qquad (23)$$

where equality occurs when $p_i = q_1$.

**Definition 224** *Let $x : x \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}$. Then we have*

$$f^{\Delta}(x) = f^{\nabla}(x) = f'(x). \qquad (24)$$

## 3 Results and Discussion

We begin the results and discussion with the following definition.

**Definition 31** *Let $f(\tau)$ be continuous and delta integrable on $[t, \sigma(t)]_{\mathbb{T}}$. Then we define the left and right Riemann-Liouville conformable fractional integral operator as*

$$\Delta_{a^+}^{\alpha,\lambda} f(\rho) = \frac{1}{\Gamma_k(\lambda)} \int_t^{\theta} \left(\frac{\rho^{\alpha\beta^{-1}} - \tau^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1} \times \qquad (25)$$
$$\tau^{\frac{\alpha}{\beta}-1} f(\tau) \Delta\tau, \quad \theta > t$$

and

$$\Delta_{b^-}^{\alpha,\lambda} f(\rho) = \frac{1}{\Gamma_k(\lambda)} \int_{\theta}^{\sigma(t)} \left(\frac{\rho^{\alpha\beta^{-1}} - \tau^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1} \times \qquad (26)$$
$$\tau^{\frac{\alpha}{\beta}-1} f(\tau) \Delta\tau, \quad \sigma(t) > \theta.$$

For all $\alpha, \beta \in (0, \infty)$, $\lambda, \eta \neq k$, $k > 0$, $\lambda, \eta > 0$, $\rho > \tau$ and $t, \theta, \sigma(t) \in \mathbb{T}$.

**Remark 31** *For $\beta = k = 1$ in (25) and (26) yields*

$$\Delta_{a^+}^{\alpha,\lambda} f(\rho) = \frac{1}{\Gamma(\lambda)} \int_t^{\theta} \left(\frac{\rho^{\alpha} - \tau^{\alpha}}{\alpha}\right)^{\lambda-1} \tau^{\alpha-1} f(\tau) \Delta\tau, \qquad (27)$$
$$\theta > t,$$

$$\Delta_{b^-}^{\alpha,\lambda} f(\rho) = \frac{1}{\Gamma(\lambda)} \int_{\theta}^{\sigma(t)} \left(\frac{\rho^{\alpha} - \tau^{\alpha}}{\alpha}\right)^{\lambda-1} \tau^{\alpha-1} f(\tau) \Delta\tau, \qquad (28)$$
$$\sigma(t) > \theta.$$

**Remark 32** *Suppose $\alpha = \beta = 1$ and $k = 1$ in (25) and (26), then the following Riemann - Liouville fractional integral operators on time scales are obtained.*

$$\Delta_{a^+}^{\lambda} f(\rho) = \frac{1}{\Gamma(\lambda)} \int_t^{\theta} (\rho - \tau)^{\lambda-1} f(\tau) \Delta\tau, \quad \theta > t. \qquad (29)$$

$$\Delta_{b^-}^{\lambda} f(\rho) = \frac{1}{\Gamma(\lambda)} \int_{\theta}^{\sigma(t)} (\rho - \tau)^{\lambda-1} f(\tau) \Delta\tau, \quad \sigma(t) > \theta. \qquad (30)$$

**Lemma 31** *Let $f, g, h\colon \mathbb{T} \to \mathbb{R}$ be delta integrable functions such that $0 \leq \frac{f(\rho)}{h(\psi)} \leq m \leq \frac{f(\xi)}{h(\Psi)}$ and $0 \leq g(\rho) \leq m \leq g(\xi)$, then*

$$f(\xi)g(\xi)h(\psi) - f(\xi)g(\rho)h(\psi) - f(\rho)g(\xi)h(\Psi) + \qquad (31)$$
$$f(\rho)g(\rho)h(\Psi) \geq 0,$$

for all $\xi > \rho$, $\psi > \Psi$ and $\xi, \rho, \psi, \Psi \in \mathbb{R}$.

*Proof.* Given that

$$m - \frac{f(\rho)}{h(\psi)} \geq 0,$$

and

$$\frac{f(\xi)}{h(\Psi)} - m \geq 0.$$

Then

$$\frac{f(\xi)}{h(\Psi)} - \frac{f(\rho)}{h(\psi)} \geq 0. \qquad (32)$$

Similarly

$$m - g(\rho) \geq 0,$$

and

$$g(\xi)-m\geq 0.$$

Then

$$g(\xi)-g(\rho)\geq 0. \tag{33}$$

From (32) and (33) we obtain

$$\frac{f(\xi)}{h(\Psi)}g(\xi)-\frac{f(\xi)}{h(\Psi)}g(\rho)-\frac{f(\rho)}{h(\psi)}g(\xi)+\frac{f(\rho)}{h(\psi)}g(\rho)\geq 0$$

as required.

**Remark 33** *Putting* $h(\Psi)=h(\psi)$ *for all* $\Psi,\psi\in[t,\sigma(t)]_{\mathbb{T}}$*, we obtain*

$$f(\xi)g(\xi)-f(\xi)g(\rho)-f(\rho)g(\xi)+f(\rho)g(\rho)\geq 0. \tag{34}$$

**Lemma 32** *Let* $:\mathbb{T}\to\mathbb{R}$ *be delta-integrable function on* $[0,\xi]_{\mathbb{T}}$*. Then*

$$\int_0^\xi\left(\frac{\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}u^{\alpha^{-1}\beta-1}\Delta u=\frac{k(\alpha^{-1}\beta)^{\frac{\lambda}{k}}\xi^{(2\alpha\beta^{-1})\frac{\lambda}{k}-1}}{\lambda}, \tag{35}$$

for all $\alpha,\beta\in(0,\infty)$, $k,\lambda>0$, $\lambda\neq k$ and $\xi>u$.

*Proof.* Clearly

$$\int_0^\xi\left(\frac{\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha^{-1}\beta-1}\Delta u=(\alpha^{-1}\beta)^{\frac{\lambda}{k}-1}\int_0^\xi\left(\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}u^{\frac{\alpha}{\beta}-1}\Delta u,$$

$$\int_0^\xi\left(\frac{\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha^{-1}\beta-1}\Delta u=(\alpha^{-1}\beta)^{\frac{\lambda}{k}-1}\xi^{(\alpha\beta^{-1})\frac{\lambda}{k}-1}\int_0^\xi\left(1-\left(\frac{u}{\xi}\right)^{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}u^{\frac{\alpha}{\beta}-1}\Delta u.$$

Let $\mu=\left(\frac{u}{\xi}\right)^{\alpha\beta^{-1}}$ and by Definition 224 we have

$$\int_0^\xi\left(\frac{\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha^{-1}\beta-1}\Delta u=(\alpha^{-1}\beta)^{\frac{\lambda}{k}-1}\xi^{(\alpha\beta^{-1})\frac{\lambda}{k}-1}\frac{\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\int_0^1(1-\mu)^{\frac{\lambda}{k}-1}\Delta\mu.$$

Also, let $v=1-\mu$, thus

$$\int_0^\xi\left(\frac{\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha^{-1}\beta-1}\Delta u=(\alpha^{-1}\beta)^{\frac{\lambda}{k}-1}\xi^{(\alpha\beta^{-1})\frac{\lambda}{k}-1}\frac{\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\int_1^0-(v)^{\frac{\lambda}{k}-1}\Delta v,$$

$$\int_0^\xi\left(\frac{\xi^{\alpha\beta^{-1}}-u^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha^{-1}\beta-1}\Delta u=\frac{k(\alpha^{-1}\beta)^{\frac{\lambda}{k}}\xi^{(\alpha\beta^{-1})\frac{\lambda}{k}}}{\lambda}.$$

**Theorem 33** *Let* $f,g,h\colon\mathbb{T}\to\mathbb{R}$ *be delta integrable functions on* $[t,\sigma(t)]_{\mathbb{T}}$ *such that* $0\leq\frac{f(\rho)}{h(\psi)}\leq m\leq\frac{f(\xi)}{h(\Psi)}$ *and* $0\leq g(\rho)\leq m\leq g(\xi)$*. Then*

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)\leq\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t). \tag{36}$$

*Proof.* Multiplying through (34) by $\frac{1}{\xi^{1-\alpha\beta^{-1}}\Gamma_k(\lambda)}\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}$ and then integrating the result with respect to $\xi$ over $[0,t]_{\mathbb{T}}$ and then applying Definition 31, Definition 224 and Lemma 32 we have

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)-\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)-\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)\geq 0.$$

Simplifying gives

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)\leq\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t). \tag{37}$$

.

**Corollary 34** *Let* $f,g,h\colon\mathbb{T}\to\mathbb{R}$ *be delta integrable functions on* $[t,\sigma(t)]_{\mathbb{T}}$*. Then*

$$\left(\Delta_k^{\lambda}fg\right)(t)\geq\frac{\Gamma_k(\lambda+1)}{kt^{\frac{2\lambda}{k}-2}}\left(\Delta_k^{\lambda}f\right)(t)\left(\Delta_k^{\lambda}g\right)(t), \tag{38}$$

for all $\lambda>1$, $k>1$.

*Proof.* Substituting $\beta=\alpha=1$ into (37) gives the result.

**Theorem 35** *Let Let $f,g,h\colon \mathbb{T}\to\mathbb{R}$ be delta integrable functions on $[t,\sigma(t)]_{\mathbb{T}}$ such that $0\le\frac{f(\rho)}{h(\psi)}\le m\le\frac{f(\xi)}{h(\Psi)}$ and $0\le g(\rho)\le m\le g(\xi)$. Then*

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}g\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}f\right)(t)\times\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)\le\frac{1}{4}(\left(\Delta_k^{\alpha\beta^{-1},\eta}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}fg\right)(t))^2.$$

*Proof.* Multiplying through (31) by $\frac{1}{\xi^{1-\alpha\beta^{-1}}\Gamma_k(\lambda)}\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}$ and then integrating with respect to $\xi$ over $[0,t]_{\mathbb{T}}$ and also applying Definition 31, Definition 224 and Lemma 32 we have

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)h(\psi)-\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)g(\rho)h(\psi)-f(\rho)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)h(\Psi)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)f(\rho)g(\rho)h(\Psi)\ge 0. \tag{39}$$

Further multiplying through (39) by $\frac{1}{\rho^{1-\alpha\beta^{-1}}\Gamma_k(\eta)}\left(\frac{t^{\alpha\beta^{-1}}-\rho^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\eta}{k}-1}$ and then integrating with respect to $\rho$ over $[0,t]_{\mathbb{T}}$ yields

$$\left(\Delta_k^{\alpha\beta^{-1},\eta}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)h(\psi)-\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}g\right)(t)h(\psi)-\left(\Delta_k^{\alpha\beta^{-1},\eta}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)h(\Psi)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}fg\right)(t)h(\Psi)\ge 0. \tag{40}$$

Thus

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}g\right)(t)h(\psi)+\left(\Delta_k^{\alpha\beta^{-1},\eta}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)h(\Psi)\le\left(\Delta_k^{\alpha\beta^{-1},\eta}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)h(\psi)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}fg\right)(t)h(\Psi). \tag{41}$$

Applying (23) to (41) yields

$$\left(\Delta_k^{\alpha\beta^{-1},\eta}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(\;\; \left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}fg\right.$$
$$\ge 2\left(\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}g\right)(t)h(\psi)\left(\Delta_k^{\alpha\beta^{-1},\eta}f\right)(t)\times\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)\right.$$

Let $h(\psi)=h(\Psi)=1$, we have

$$\left(\Delta_k^{\alpha\beta^{-1},\eta}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}fg\right)(t)\ge 2\left(\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}g\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}f\right)(t)\times\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)\right)^{\frac{1}{2}}, \tag{42}$$

and this further yields

$$\left(\Delta_k^{\alpha\beta^{-1},\lambda}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}g\right)\times(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}f\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}g\right)(t)\le\frac{1}{4}(\left(\Delta_k^{\alpha\beta^{-1},\eta}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\lambda}fg\right)(t)+\left(\Delta_k^{\alpha\beta^{-1},\lambda}\right)(t)\left(\Delta_k^{\alpha\beta^{-1},\eta}fg\right)(t))^2.$$

**Theorem 36** *Let Let $f,g,h\colon \mathbb{T}\to\mathbb{R}$ be delta integrable functions on $[t,\sigma(t)]_{\mathbb{T}}$ such that $0\le\frac{f(\rho)}{h(\psi)}\le m\le\frac{f(\xi)}{h(\Psi)}$ and $0\le g(\rho)\le m\le g(\xi)$. Then*

$$\frac{\left(\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fg\right)(t)}{\left(\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fh\right)(t)}\le\frac{\left(\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}g\right)(t)}{\left(\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}\right)h(t)} \tag{43}$$

*for all $t>\xi,\rho$.*

*Proof.* From Lemma 31 we have

$$f(\xi)h(\xi)g(\rho)-f(\xi)h(\rho)g(\xi)-f(\rho)h(\xi)g(\rho)+f(\rho)h(\rho)g(\xi)\ge 0. \tag{44}$$

Multiplying through (44) by $\frac{1}{\xi^{1-\alpha\beta^{-1}}\Gamma_k(\lambda)}\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}$ and then integrating the result with respect to $\xi$ over $[0,t]_{\mathbb{T}}$ and then applying (25) and Lemma 32 we have

$$\frac{1}{\Gamma_k(\lambda)}\int_0^t\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha\beta^{-1}-1}\times f(\xi)h(\xi)g(\rho)$$
$$-\frac{1}{\Gamma_k(\lambda)}\int_0^t\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha\beta^{-1}-1}\times f(\xi)h(\rho)g(\xi)$$
$$-\frac{1}{\Gamma_k(\lambda)}\int_0^t\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha\beta^{-1}-1}f\times(\rho)h(\xi)g(\rho)$$
$$+\frac{1}{\Gamma_k(\lambda)}\int_0^t\left(\frac{t^{\alpha\beta^{-1}}-\xi^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}\xi^{\alpha\beta^{-1}-1}f\times(\rho)h(\rho)g(\xi)\geq 0.$$

This further yields

$$\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fh(t)g(\rho)-\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fg(t)h(\rho)- f(\rho)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}h(t)g(\rho)+ f(\rho)h(\rho)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}g(t)\geq 0. \tag{45}$$

Again multiplying through (45) by $\frac{1}{\rho^{1-\alpha\beta^{-1}}\Gamma_k(\lambda)}\left(\frac{t^{\alpha\beta^{-1}}-\rho^{\alpha\beta^{-1}}}{\alpha\beta^{-1}}\right)^{\frac{\lambda}{k}-1}$ and integrating with respect to $\rho$ over $[0,t]_{\mathbb{T}}$ we have

$$\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fh(t)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}g(t)-\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fg(t)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}h(t)- \Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fg(t)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}h(t)+ \Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fh(t)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}g(t)\geq 0. \tag{46}$$

Thus

$$\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fg(t)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}h(t)\leq\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}fh(t)\Delta_{k,a^+}^{\alpha\beta^{-1},\lambda}g(t). \tag{47}$$

## 4 Conclusion

In this paper, we defined the left and right Riemann-Liouville conformable fractional integral operator on time scale. With that definition, we presented some extensions of fractional integral inequalities on time scales by applying Riemann-Liouville fractional integral and AM-GM inequality.


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