On Soft Separation Axioms via Fuzzy $\alpha$-Open Soft Sets

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Abstract: In the present paper, we continue the study of the properties of fuzzy $\alpha$-open (closed) soft sets, which were first introduced in [1]. Also, we investigate the concepts of fuzzy $\alpha$-soft interior (closure), fuzzy $\alpha$-continuous (open) soft functions and fuzzy $\alpha$-separation axioms which are important for further research on fuzzy soft topology. In particular we study the relationship between fuzzy $\alpha$-soft interior fuzzy $\alpha$-soft closure, which are basic for further research on fuzzy soft topology and will fortify the footing of the theory of fuzzy soft topological space. Further, we study the properties of fuzzy soft $\alpha$-regular spaces and fuzzy soft $\alpha$-normal spaces. Moreover, we show that if every fuzzy soft point $f_s$ is fuzzy $\alpha$-closed soft set in a fuzzy soft topological space $(X, \tau, E)$, then $(X, \tau, E)$ is fuzzy soft $\alpha$-$T_1$– (resp. $T_2$–) space.

Keywords: Soft set, Fuzzy soft set, Fuzzy soft topological space, Fuzzy $\alpha$-soft interior, Fuzzy $\alpha$-soft closure, Fuzzy $\alpha$-open soft, Fuzzy $\alpha$-closed soft, Fuzzy $\alpha$-continuous soft functions, Fuzzy soft $\alpha$-separation axioms, Fuzzy soft $\alpha$-$T_i$-spaces $(i = 1, 2, 3, 4)$, Fuzzy soft $\alpha$-regular, Fuzzy soft $\alpha$-normal.

1 Introduction

In real life situation, the problems in economics, engineering, social sciences, medical science etc. do not always involve crisp data. So, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy set, intuitionistic fuzzy set, rough set, bipolar fuzzy set, i.e. which we can use as mathematical tools for dealings with uncertainties. But, all these theories have their inherent difficulties. The reason for these difficulties Molodtsov [35] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties which is free from the above difficulties. In [35,36], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [33], the properties and applications of soft set theory have been studied increasingly [7,29,36]. Xiao et al. [46] and Pei and Miao [39] discussed the relationship between soft sets and information systems. They showed that soft sets are a class of special information systems. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [4,6,10,12,21,27,31,32,33,34,36,37,49]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [11].

Recently, in 2011, Shabir and Naz [42] initiated the study of soft topological spaces. They defined soft topology as a collection $\tau$ of soft sets over $X$. Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft nbhd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Min in [45] investigate some properties of these soft separation axioms. In [19], Kandil et al. introduced some soft operations such as semi open soft, pre open soft, $\alpha$-open soft and $\beta$-open soft and investigated their properties in detail. Kandil et al. [26] introduced the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. The notion of soft ideal was initiated for the first time by Kandil et al. [22]. They also introduced the concept of soft local function. These concepts are discussed with a view to find

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new soft topologies from the original one, called soft topological spaces with soft ideal \((\mathcal{X}, \tau, \mathcal{E}, \mathcal{I})\). Applications to various fields were further investigated by Kandel et al. [20,21,23,24,25,28]. The notion of supra soft topological spaces was initiated for the first time by El-sheikh and Abd El-latif [14]. They also introduced new different types of subsets of supra soft topological spaces and study the relations between them in detail. The notion of b-open soft sets was initiated for the first time by El-sheikh and Abd El-latif [13], which is extended by Abd El-latif et al. in [3]. Maji et al. [31] initiated the study involving both fuzzy sets and soft sets. In [9] the notion of fuzzy soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then, many scientists such as X. Yang et al. [38] defined the notion of a mapping on classes of fuzzy soft sets, which is a fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Chang [12] introduced the concept of fuzzy topology on a set \(X\) by axiomatizing a collection \(\mathcal{T}\) of fuzzy subsets of \(X\). [43] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [41] gave the definition of fuzzy soft topology over the initial universe set. Some fuzzy soft topological properties based on fuzzy semi (resp. \(\beta\)-, pre) open soft sets, were introduced in [2, 17, 18, 27].

In the present paper, we investigate some new properties of fuzzy \(\alpha\)-open soft sets and fuzzy \(\alpha\)-closed soft sets, which were first introduced in [1]. Also, we study the notions of fuzzy \(\alpha\)-soft interior, fuzzy \(\alpha\)-soft closure and fuzzy \(\alpha\)-separation axioms. Further, we study the properties of fuzzy soft \(\alpha\)-regular spaces and fuzzy soft \(\alpha\)-normal spaces. Moreover, we show that if every fuzzy soft point \(f_e\) is fuzzy \(\alpha\)-closed soft set in a fuzzy soft topological space \((\mathcal{X}, \mathcal{E}, \mathcal{E})\), then \((\mathcal{X}, \mathcal{E}, \mathcal{E})\) is fuzzy soft \(\alpha\)-\(T_1\) (resp. \(T_2\)) space.

\section*{2 Preliminaries}

In this section, we present the basic definitions and results of fuzzy soft set theory which will be needed in the paper.

\textbf{Definition 2.1.}[48] A fuzzy set \(A\) of a non-empty set \(X\) is characterized by a membership function \(\mu_A : X \rightarrow [0, 1] = I\) whose value \(\mu_A(x)\) represents the "degree of membership" of \(x\) in \(A\) for \(x \in X\). We denote family of all fuzzy sets by \(I^X\). If \(A, B \in I^X\), then some basic set operations for fuzzy sets are given by Zadeh [48], as follows:

\begin{enumerate}
\item \(A \leq B \iff \mu_A(x) \leq \mu_B(x) \quad \forall x \in X\).
\item \(A = B \iff \mu_A(x) = \mu_B(x) \quad \forall x \in X\).
\item \(C = A \cup B \iff \mu_C(x) = \mu_A(x) \vee \mu_B(x) \quad \forall x \in X\).
\item \(D = A \cap B \iff \mu_D(x) = \mu_A(x) \wedge \mu_B(x) \quad \forall x \in X\).
\item \(M = A^c \iff \mu_M(x) = 1 - \mu_A(x) \quad \forall x \in X\).
\end{enumerate}

\textbf{Definition 2.2.}[35] Let \(X\) be an initial universe and \(E\) be a set of parameters. Let \(P(X)\) denote the power set of \(X\) and \(A\) be a non-empty subset of \(E\). A pair \((f, \alpha)\) denoted by \(F_\alpha\) is called a soft set over \(X\), where \(f\) is a mapping given by \(f : A \rightarrow P(X)\). In other words, a soft set over \(X\) is a parameterized family of subsets of the universe \(X\). For a particular \(e \in A\), \(f(e)\) may be considered the set of \(e\)-approximate elements of the soft set \((f, \alpha)\) and if \(e \notin A\), then \(F(e) = \{\phi\}\).

\textbf{Definition 2.3.}[31] Let \(A \subseteq E\). A pair \((f, \alpha)\), denoted by \(F_\alpha\), is called fuzzy soft set over \(X\), where \(f\) is a mapping given by \(f : A \rightarrow I^X\) defined by \(f_a(e) = \mu_A(e)\), where \(\mu_A(e) = 0\) if \(e \notin A\) and \(\mu_A(e) \neq 0\) if \(e \in A\), where \(\mu_A(x) = 0 \forall x \in X\). The family of all these soft sets over \(X\) denoted by \(SS(X)_\alpha\).

\textbf{Definition 2.4.}[40] Let \(\mathcal{S}\) be a collection of fuzzy soft sets over a universe \(X\) with a fixed set of parameters \(E\), then \(\mathcal{S} \subseteq SS(X)_E\) is called fuzzy soft topology on \(X\) if

(1) \(\emptyset, \mathcal{S} \subseteq \emptyset \in \mathcal{S}\), where \(\emptyset(e) = \emptyset\) and \(\emptyset(e) = \mathcal{S}\) \(\forall e \in E\),

(2) the union of any members of \(\mathcal{S}\) belongs to \(\mathcal{S}\),

(3) the intersection of any two members of \(\mathcal{S}\) belongs to \(\mathcal{S}\).

The triplet \((X, \mathcal{S}, \mathcal{E})\) is called fuzzy soft topological space over \(X\). Also, each member of \(\mathcal{S}\) is called fuzzy open soft in \((X, \mathcal{S}, \mathcal{E})\). We denote the set of all open soft sets by \(FOS(X, \mathcal{S}, \mathcal{E})\), or \(FOS(X)\).

\textbf{Definition 2.5.}[40] Let \((X, \mathcal{S}, \mathcal{E})\) be a fuzzy soft topological space. A fuzzy soft set \(f_A\) over \(X\) is said to be fuzzy closed soft set in \(X\), if its relative complement \(f_A^c\) is fuzzy open soft set. We denote the set of all fuzzy closed soft sets by \(FCS(X, \mathcal{S}, \mathcal{E})\), or \(FCS(X)\).

\textbf{Definition 2.6.}[38] Let \((X, \mathcal{S}, \mathcal{E})\) be a fuzzy soft topological space and \(f_A \in SS(X)_E\). The fuzzy soft closure of \(f_A\), denoted by \(Fcl(f_A)\) is the intersection of all fuzzy closed soft super sets of \(f_A\), i.e.,

\[Fcl(f_A) = \bigcap\{h_D : h_D \text{ is fuzzy closed soft set and } f_A \subseteq h_D\}\]

The fuzzy soft interior of \(g_B\), denoted by \(Fin(f_A)\) is the fuzzy soft union of all fuzzy open soft subsets of \(f_A\), i.e.,

\[Fin(f_A) = \bigcup\{h_D : h_D \text{ is fuzzy open soft set and } h_D \subseteq g_B\}\]

\textbf{Definition 2.7.}[30] The fuzzy soft set \(f_A \in SS(X)_E\) is called fuzzy soft point if there exist \(x \in X\) and \(e \in E\) such that \(\mu_A(x) = \alpha (0 < \alpha \leq 1)\) and \(\mu_A^e(y) = 0\) for each \(y \in X - \{x\}\), and this fuzzy soft point is denoted by \(x^e_{f_A}\) or \(f_e\).

\textbf{Definition 2.8.}[30] The fuzzy soft point \(x^e_{f_A}\) is said to be belonging to the fuzzy soft set \((g_A)\), denoted by \(x^e_{f_A} \in (g_A)\), if for the element \(e \in A\), \(\alpha \leq \mu_{g_A}^e(x)\).

\textbf{Theorem 2.1.}Mahanta2012f Let \((X, \mathcal{S}, \mathcal{E})\) be a fuzzy soft topological space and \(f_e\) be a fuzzy soft point. Then, the following properties hold:

\begin{enumerate}
\item If \(f_e \in g_A\), then \(f_e \in g_A^e\).
\item \(f_e \in g_A^e \Rightarrow f_e^c \in g_A^e\).
\end{enumerate}
Every non-null fuzzy soft set $f_A$ can be expressed as the union of all the fuzzy soft points belonging to $f_A$.

**Definition 2.9.**[30] A fuzzy soft set $g_B$ in a fuzzy soft topological space $(X, \mathcal{S}, E)$ is called fuzzy soft neighborhood of the fuzzy soft point $x_0^E$ if there exists a fuzzy open soft set $h_C$ such that $x_0^E \in h_C \subseteq g_B$. A fuzzy soft set $g_B$ in a fuzzy soft topological space $(X, \mathcal{S}, E)$ is called fuzzy soft neighborhood of the soft set $f_A$ if there exists a fuzzy open soft set $h_C$ such that $f_A \subseteq h_C \subseteq g_B$. The fuzzy soft neighborhood system of the fuzzy soft point $x_0^E$, denoted by $N_2(x_0^E)$, is the family of all its fuzzy soft neighborhoods.

**Definition 2.10.**[30] Let $(X, \mathcal{S}, E)$ be a fuzzy soft topological space and $Y \subseteq X$. Let $h_B^Y$ be a fuzzy soft set over $(Y, E)$ such that $h_B^Y : Y \to T$ such that $h_B^Y(e) = \mu_{h_B}^Y(e)$, $\mu_{h_B}^Y(x) = \begin{cases} 1 & x \in Y, \\ 0, & x \notin Y. \end{cases}$

Let $\mathcal{T}_Y = \{h_B^Y \cap g_B : g_B \in \mathcal{S}\}$, then the fuzzy soft topology $\mathcal{T}_Y$ on $(Y, E)$ is called fuzzy soft subspace topology for $(Y, E)$ and $(\mathcal{T}_Y, \mathcal{S})$ is called fuzzy soft subspace of $(X, \mathcal{S}, E)$.

**Definition 2.11.**[38] Let $FSS(Y)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over $X$ and $Y$, respectively. Let $u : X \to Y$ and $p : E \to K$ be mappings. Then the map $f_{pu} : FSS(Y)_E \to FSS(Y)_K$ such that,

1. If $f_A \in FSS(Y)_E$. Then the image of $f_A$ under the fuzzy soft mapping $f_{pu}$ is the fuzzy soft set over $Y$ defined by $f_{pu}(f_A)(y)$, where $\forall k \in p(E), \forall y \in Y$, $f_{pu}(f_A)(y) = \begin{cases} \bigvee_{u(x)=y} [\forall (p(e) \in k)(f_A(e))] & \text{if } x \in u^{-1}(y), \\ 0 & \text{otherwise}. \end{cases}$

2. If $g_B \in FSS(Y)_K$, then the pre-image of $g_B$ under the fuzzy soft mapping $f_{pu}$ is the fuzzy soft set over $X$ defined by $f_{pu}^{-1}(g_B)$, where $\forall e \in p^{-1}(k), \forall x \in X$, $f_{pu}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & \text{for } p(e) \in B, \\ 0 & \text{otherwise}. \end{cases}$

The fuzzy soft mapping $f_{pu}$ is called surjective (resp. injective) if $p$ and $u$ are surjective (resp. injective), also it is said to be constant if $p$ and $u$ are constant.

**Definition 2.12.**[38] Let $(X, \mathcal{T}_1, E)$ and $(Y, \mathcal{T}_2, K)$ be two fuzzy soft topological spaces and $f_{pu} : FSS(Y)_E \to FSS(Y)_K$ be a fuzzy soft mapping. Then $f_{pu}$ is called

1. Fuzzy continuous soft if $f_{pu}^{-1}(g_B) \in \mathcal{T}_1 \forall (g_B) \in \mathcal{T}_2$.
2. Fuzzy open soft if $f_{pu}(g_A) \in \mathcal{T}_2 \forall (g_A) \in \mathcal{T}_1$.

**Theorem 2.2.**[5] Let $FSS(X)_E$ and $FSS(Y)_K$ be two families of fuzzy soft sets. For the fuzzy soft function $f_{pu} : FSS(X)_E \to FSS(Y)_K$, the following statements hold,

(a) $f_{pu}^{-1}((g,B)_c) = (f_{pu}^{-1}(g,B))_c \forall (g,B) \in FSS(Y)_K$.
(b) $f_{pu}((f_{pu}^{-1}((g,B)))) = (g,B) \forall (g,B) \in FSS(Y)_K$. If $f_{pu}$ is surjective, then the equality holds.
(c) $(f,A) \subseteq f_{pu}^{-1}(f((f,A))) \forall (f,A) \in FSS(X)_E$. If $f_{pu}$ is injective, then the equality holds.
(d) $f_{pu}(\emptyset) = \emptyset, f_{pu}(1_E) = 1_K$. If $f_{pu}$ is surjective, then the equality holds.
(e) $f_{pu}^{-1}(1_K) = 1_E$ and $f_{pu}^{-1}(\emptyset) = \emptyset_E$.
(f) If $(f,A) \subseteq (g,A)$, then $f_{pu}(f,A) \subseteq f_{pu}(g,A)$.

**Definition 2.13.**[30] Let $(X, \mathcal{T}, E)$ be a fuzzy soft topological space. A fuzzy soft separation of $1_E$ is a pair of non null proper fuzzy open soft sets $g_B, h_C$ such that $g_B \cap h_C = 0_E$ and $1_E = g_B \cup h_C$.

**Definition 2.14.**[30] A fuzzy soft topological space $(X, \mathcal{T}, E)$ is said to be fuzzy soft connected if and only if there is no fuzzy soft separations of $X$. Otherwise, $(X, \mathcal{T}, E)$ is said to be fuzzy soft disconnected space.

**Definition 2.16.**[27] Two fuzzy soft sets $f_A$ and $g_B$ are said to be disjoint, denoted by $f_A \cap g_B = 0_E$, if $A \cap B = \emptyset$ and $\mu_A^E \cap \mu_B^E = 0 \forall e \in E$.

**Theorem 2.3.**[27] Let $(X, \mathcal{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then,

1. $f_A \in FSSOS(X)_E$ if and only if $Fcl(f_A) = Fcl(Fint(f_A))$.
2. If $g_B \in \mathcal{T}_E$, then $g_B \cap Fcl(f_A) \subseteq Fcl(g_B \cap g_B)$.

**Definition 2.15.**[19] Let $(X, \mathcal{T}, E)$ be a soft topological space and $F_E \in SS(X)_E$. If $F_E \subseteq int(cl(int(F_A)))$, then $F_A$ is called $\alpha$-open soft set. We denote the set of all $\alpha$-open soft sets by $\alpha OS(X, \mathcal{T}, E)$, or $\alpha OS(X)$ and the set of all $\alpha$-closed soft sets by $\alpha CS(X, \mathcal{T}, E)$, or $\alpha CS(X)$.

### 3 Fuzzy $\alpha$-open (closed) soft sets

In this section, we move one step forward to investigate new properties of the notions of fuzzy $\alpha$-open soft sets, fuzzy $\alpha$-closed soft sets [1] and study various properties and notions related to these structures.

**Definition 3.1.**[1] Let $(X, \mathcal{T}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. If $f_A \subseteq Fint(Fcl(Fint(f_A)))$, then $f_A$ is called fuzzy $\alpha$-open soft set. We denote the set of all fuzzy $\alpha$-open soft sets by $\alpha OS(X, \mathcal{T}, E)$, or $\alpha OS(X)$ and the set of all fuzzy $\alpha$-open soft sets by $\alpha OS(X, \mathcal{T}, E)$, or $\alpha OS(X)$.

**Theorem 3.1.** Let $(X, \mathcal{T}, E)$ be a fuzzy soft topological space and $f_A \in FOS(X)_E$. Then
(1) Arbitrary fuzzy soft union of fuzzy $\alpha$-open soft sets is fuzzy $\alpha$-open soft.
(2) Arbitrary fuzzy soft intersection of fuzzy $\alpha$-closed soft sets is fuzzy $\alpha$-closed soft.

**Proof.**

(1) Let $\{ (f, A) : j \in J \} \subseteq F\alpha OS(X)$. Then, $\forall j \in J$, $(f, A) \subseteq \text{Fint}(\text{Fcl}(\text{Fint}(f, A)))$. It follows that, $\bigcup_j (f, A) \subseteq \bigcup_j \text{Fint}(\text{Fcl}(\text{Fint}(f, A)))$. Hence, $\text{Fint}(\text{Fcl}(f, A)) = \text{Fint}(\text{Fcl}(\text{Fint}(f, A))) = \text{Fint}(\text{Fcl}(\text{Fint}(\text{Fcl}(f, A))))$.

(2) By a similar way.

**Theorem 3.2.** Let $(X, \mathcal{E}, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in F\alpha OS(X)$ if and only if $\text{Fcl}(f_A) = \text{Fint}(\text{Fcl}(\text{Fint}(f_A)))$.

**Proof.** Immediate.

**Definition 3.2.** Let $(X, \mathcal{E}, E)$ be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_A \in \text{Fint}(X)_E$. Then, $f_A$ is said to be a fuzzy soft interior point of $f_A$ if $\exists g_B \in F\alpha OS(X)$ such that $f_A \subseteq g_B \subseteq f_A$. The set of all fuzzy $\alpha$-soft interior points of $f_A$ is called the fuzzy $\alpha$-soft interior of $f_A$ and is denoted by $\text{Faint}(f_A)$.

**Proof.** Immediate.

**Example 3.1.** Let $X = \{ a, b, c \}$, $\mathcal{E} = \{ e_1, e_2, e_3 \}$ and $A, B, C, D \subseteq E$ where $A = \{ e_1, e_2 \}$, $B = \{ e_2, e_3 \}$, $C = \{ e_1, e_3 \}$ and $D = \{ e_2 \}$. Let $\mathcal{E} = \{ \{e_1, e_2\}, \{e_1, e_3\}, \{e_1\}, \{e_2\} \}$. Then, $f_{A_B} = \{ \{a\}, \{a, b\} \} \in F\alpha OS(X)$.

**Theorem 3.3.** Let $(X, \mathcal{E}, E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy $\alpha$-interior operator, denoted by $\text{Faint}$.

(1) $\text{Faint}(1_E) = 1_E$ and $\text{Faint}(0_E) = 0_E$.

(2) $\text{Faint}(f_A) \subseteq f_A$.

(3) $\text{Faint}(f_A)$ is the largest fuzzy $\alpha$-open soft set contained in $f_A$.

(4) If $f_B \subseteq g_B$, then $\text{Faint}(f_A) \subseteq \text{Faint}(g_B)$.

(5) $\text{Faint}\{\text{Faint}(f_A)\} = \text{Faint}(f_A)$.

(6) $\text{Faint}(f_A) \cup \text{Faint}(g_B) \subseteq \text{Faint}\{f_A \cup (g_B)\}$.

(7) $\text{Faint}\{f_A \cap (g_B)\} \subseteq \text{Faint}(f_A) \cap \text{Faint}(g_B)$.

**Proof.** Obvious.

**Theorem 3.4.** Let $(X, \mathcal{E}, E)$ be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then, the following properties are satisfied for the fuzzy $\alpha$-closure operator, denoted by $\text{Fcl}$.

(1) $\text{Fcl}(1_E) = 1_E$ and $\text{Fcl}(0_E) = 0_E$.

(2) $\text{Fcl}(f_A) \subseteq \text{Fcl}(f_A)$.

(3) $\text{Fcl}(f_A)$ is the smallest fuzzy $\alpha$-closed soft set contains $f_A$.

(4) If $f_A \subseteq g_B$, then $\text{Fcl}(f_A) \subseteq \text{Fcl}(g_B)$.

(5) $\text{Fcl}(\text{Fcl}(f_A)) = \text{Fcl}(f_A)$.

(6) $\text{Fcl}(f_A) \cup \text{Fcl}(g_B) \subseteq \text{Fcl}\{f_A \cup (g_B)\}$.

(7) $\text{Fcl}\{f_A \cap (g_B)\} \subseteq \text{Fcl}(f_A) \cap \text{Fcl}(g_B)$.

**Proof.** Immediate.

**Lemma 3.1.** Every fuzzy open (resp. closed) soft set in a fuzzy soft topological space $(X, \mathcal{E}, E)$ is fuzzy $\alpha$-open (resp. closed) soft.

**Proof.** Let $f_A \in FOS(X)$. Then, $\text{Fint}(f_A) = f_A$. Since $f_A \subseteq F\alpha OS(X)$, then $f_A \subseteq \text{Fint}(\text{Fcl}(f_A))$. Thus, $f_A \in F\alpha OS(X)$.

**Remark 3.2.** The converse of Lemma 3.1 is not true in general as shown in the following example.

**Example 3.2.** Let $f_A = \{ \{a\}, \{a, b\} \}$ where $f_A \subseteq F\alpha OS(X)$. Then, $f_A \subseteq \text{Fint}(\text{Fcl}(f_A))$. Hence, $f_A \in F\alpha OS(X)$.

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4 Fuzzy $\alpha$-continuous soft functions

In [4], Karal et al. defined the notion of a mapping on classes of fuzzy soft sets, which is fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Kandil et al. [26] introduced some types of soft function in soft topological spaces. Here, we introduce the notions of fuzzy $\alpha$-soft function in fuzzy soft topological spaces and study its basic properties.

**Definition 4.1.** Let $(X, \mathcal{T}_1, E)$, $(Y, \mathcal{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu}: FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft function. Then, the function $f_{pu}$ is called:

1) Fuzzy $\alpha$-continuous soft if $f_{pu}^{-1}(g_B) \in F\alpha OS(Y) \forall g_B \in \mathcal{T}_2$.
2) Fuzzy $\alpha$-open soft if $f_{pu}(g_A) \in F\alpha OS(Y) \forall g_A \in \mathcal{T}_1$.
3) Fuzzy $\alpha$-closed soft if $f_{pu}(f_{pu}(g_A)) \in F\alpha OS(Y) \forall g_A \in \mathcal{T}_1$.
4) Fuzzy $\alpha$-irresolute soft if $f_{pu}^{-1}(g_B) \in F\alpha OS(Y) \forall g_B \in F\alpha OS(Y)$.
5) Fuzzy $\alpha$-irresolute open soft if $f_{pu}(g_A) \in F\alpha OS(Y) \forall g_A \in F\alpha OS(Y)$.
6) Fuzzy $\alpha$-irresolute closed soft if $f_{pu}(f_{pu}(g_A)) \in F\alpha OS(Y) \forall g_A \in \mathcal{T}_1$.

**Example 4.1.** Let $X = Y = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $A \subseteq E$ where $A = \{e_1, e_2\}$. Let $f_{pu} : (X, \mathcal{T}_1, E) \to (Y, \mathcal{T}_2, K)$ be the constant soft mapping where $\mathcal{T}_1$ is the indiscrete fuzzy soft topology and $\mathcal{T}_2$ is the discrete fuzzy soft topology such that $u(x) = a \forall x \in X$ and $p(e) = e_1 \forall e \in E$. Let $f_{pu}$ be fuzzy soft set over $Y$ defined as follows:

\[
\mu_{f_{pu}}^{e_1} = \{a_{0.1}, b_{0.5}, c_{0.4}\}, \mu_{f_{pu}}^{e_2} = \{a_{0.6}, b_{0.2}, c_{0.3}\}.
\]

Then $f_{pu} \in \mathcal{T}_2$. Now, we find $f_{pu}^{-1}(f_A)$ as follows:

\[
\begin{align*}
\mu_{f_{pu}^{-1}(f_A)}^{e_1} &= f_A(p(e_1))(u(a)) = f_A(e_1)(a) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{e_2} &= f_A(p(e_2))(u(b)) = f_A(e_2)(a) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{e_3} &= f_A(p(e_3))(u(c)) = f_A(e_3)(a) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{f_A(e_1)(b)} &= f_A(p(e_1))(u(b)) = f_A(e_1)(b) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{f_A(e_2)(c)} &= f_A(p(e_2))(u(c)) = f_A(e_2)(c) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{f_A(e_3)(a)} &= f_A(p(e_3))(u(a)) = f_A(e_3)(a) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{f_A(e_1)(b)} &= f_A(p(e_1))(u(b)) = f_A(e_1)(b) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{f_A(e_2)(c)} &= f_A(p(e_2))(u(c)) = f_A(e_2)(c) = 0.6, \\
\mu_{f_{pu}^{-1}(f_A)}^{f_A(e_3)(a)} &= f_A(p(e_3))(u(a)) = f_A(e_3)(a) = 0.6.
\end{align*}
\]

Hence, $f_{pu}^{-1}(f_A) \notin F\alpha OS(X)$. Therefore, $f_{pu}$ is not fuzzy $\alpha$-continuous soft function. On the other hand, if we consider $\mathcal{T}_1$ is the discrete fuzzy soft topology. In this case, $f_{pu}$ will be fuzzy $\alpha$-continuous soft function.

**Theorem 4.1.** Every fuzzy continuous soft function is fuzzy $\alpha$-continuous soft function.

**Proof.** Obvious.

**Corollary 3.1.** Let $(X, \mathcal{T}_1, E)$ be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then, $f_A \in F\alpha CS(X)$ if and only if $f_A = f_{pu} \cup Fcl(Fint(Fcl(f_{pu}))$.

**Theorem 4.2.** Let $(X, \mathcal{T}_1, E)$, $(Y, \mathcal{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu}$ be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then, the following are equivalent:

1) $f_{pu}$ is a fuzzy $\alpha$-continuous soft function.
2) $f_{pu}^{-1}(h_B) \in F\alpha CS(X) \forall h_B \in FCS(Y)$.
3) $f_{pu}(F\alpha cl(g_A)) \subseteq Fcl_{\mathcal{T}_2}(f_{pu}(g_A)) \forall g_A \in FSS(X)_E$.
4) $F\alpha cl(f_{pu}^{-1}(h_B)) \subseteq f_{pu}^{-1}(Fcl_{\mathcal{T}_2}(h_B)) \forall h_B \in FSS(Y)_K$.
5) $f_{pu}^{-1}(Fint_{\mathcal{T}_2}(h_B)) \subseteq F\alpha int(f_{pu}^{-1}(h_B)) \forall h_B \in FSS(Y)_K$.

**Proof.** Immediate from Lemma 3.1.

**Theorem 4.3.** Let $(X, \mathcal{T}_1, E)$ and $(Y, \mathcal{T}_2, K)$ be fuzzy soft topological spaces and $f_{pu}$ be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then, the following are equivalent:

1) $f_{pu}$ is a fuzzy $\alpha$-continuous soft function.
2) $f_{pu}^{-1}(h_B) \in F\alpha CS(X) \forall h_B \in FCS(Y)$.
3) $f_{pu}(F\alpha cl(g_A)) \subseteq Fcl_{\mathcal{T}_2}(f_{pu}(g_A)) \forall g_A \in FSS(X)_E$.
4) $F\alpha cl(f_{pu}^{-1}(h_B)) \subseteq f_{pu}^{-1}(Fcl_{\mathcal{T}_2}(h_B)) \forall h_B \in FSS(Y)_K$.
5) $f_{pu}^{-1}(Fint_{\mathcal{T}_2}(h_B)) \subseteq F\alpha int(f_{pu}^{-1}(h_B)) \forall h_B \in FSS(Y)_K$.

**Proof.**
Theorem 4.5. Let $f_{pu}: FSS(X)_E \rightarrow FSS(Y)_K$ be a fuzzy α-open soft function. If $k_D \in FSS(Y)_K$ and $l_C \in \mathcal{A}_F$ such that $f_{pu}^{-1}(k_D) \subseteq l_C$, then there exists $h_B \in \mathcal{F}_{AC}(Y)$ such that $h_D \subseteq h_B$ and $f_{pu}^{-1}(h_B) \subseteq k_D$.

Proof. Let $k_D \in FSS(Y)_K$ and $l_C \in \mathcal{A}_F$ such that $f_{pu}^{-1}(k_D) \subseteq l_C$. Then, $f_{pu}(l_C) \subseteq k_D$ by Theorem 2.2 where $l_C \in \mathcal{A}_F$. Since $f_{pu}$ is fuzzy α-open soft function, then $f_{pu}(l_C) \subseteq f_{pu}(k_D)$. Therefore, $h_B = f_{pu}(l_C)^\alpha$. Hence, $h_B \subseteq f_{pu}(k_D)$ and $f_{pu}^{-1}(h_B) = f_{pu}^{-1}(f_{pu}(l_C)^\alpha) \subseteq f_{pu}^{-1}(f_{pu}(k_D)) = f_{pu}^{-1}(k_D) \subseteq l_C$. This completes the proof.

Theorem 4.6. Let $(X, \mathcal{A}_F, E)$ and $(Y, \mathcal{A}_F, K)$ be fuzzy soft topological spaces and $f_{pu}$ be a soft function such that $f_{pu}: FSS(X)_E \rightarrow FSS(Y)_K$. Then, the following are equivalent:

1. $f_{pu}$ is a fuzzy α-open soft function.
2. $\forall h_B \in FSS(X)_E$, $f_{pu}(F_{\alpha}(h_B)) \subseteq F_{\alpha}(f_{pu}(h_B))$.

Proof. It follows immediately from Theorem 4.3.

5 Fuzzy soft α-separation axioms

Soft separation axioms for soft topological spaces were studied by Shabir and Naz [42]. Kandil et al. [26] introduced and studied the notions of soft α-separation axioms in soft topological spaces. Here, we introduce the notions of fuzzy soft α-separation axioms in fuzzy soft topological spaces and study some of its basic properties.

Definition 5.1. A fuzzy soft topological space $(X, \mathcal{A}_F, E)$ is said to be a fuzzy soft α-T$_0$-space if for every pair of distinct fuzzy soft points $f_a, g_b$ there exist a fuzzy soft open set containing one of the points but not the other.

Examples 5.1.

1. Let $X = \{a, b, c, d\}, E = \{e_1, e_2, e_3\}$ and $\mathcal{A}_F$ be the discrete fuzzy soft topology on $X$. Then, $(X, \mathcal{A}_F, E)$ is fuzzy soft α-T$_0$-space.

2. Let $X = \{a, b, c\}, E = \{e_1, e_2\}$ and $\mathcal{A}_F$ be the indiscrete fuzzy soft topology on $X$. Then, $(X, \mathcal{A}_F, E)$ is fuzzy soft α-T$_0$-space.
Theorem 5.4. Let \((X, \mathcal{S}, E)\) be a fuzzy soft topological space. If \((X, \mathcal{S}, E)\) is fuzzy soft \(\alpha\text{-T}_2\)-space, then for every pair of distinct fuzzy soft points \(f, g\), there exists a fuzzy \(\alpha\)-closed soft set \(S\) such that containing one of the fuzzy soft points \(f\) and \(g\), but not the other \(f\) or \(g\). Hence, \(f\) and \(g\) are distinct fuzzy soft points. 

Proof. Let \(f, g\) be two distinct fuzzy soft points. By assumption, there exists a fuzzy \(\alpha\)-open soft set \(S\) such that \(f \not\subseteq S\) and \(g \subseteq S\). Hence, \(f\) and \(g\) are distinct fuzzy soft points.

Theorem 5.5. A fuzzy soft subspace \((Y, \mathcal{S}_Y, E)\) of fuzzy soft \(\alpha\text{-T}_2\)-space \((X, \mathcal{S}, E)\) is fuzzy soft \(\alpha\text{-T}_2\)-space.

Proof. Let \(j, k\) be two distinct fuzzy soft points. By assumption, there exist disjoint fuzzy \(\alpha\)-open soft sets \(A\) and \(B\) such that \(j \not\subseteq A\) and \(k \subseteq B\). Hence, \(j\) and \(k\) are distinct fuzzy soft points.

Theorem 5.6. If every fuzzy soft point of a fuzzy soft topological space \((X, \mathcal{S}, E)\) is fuzzy \(\alpha\)-closed soft, then \((X, \mathcal{S}, E)\) is fuzzy soft \(\alpha\text{-T}_2\)-space.

Proof. It similar to the proof of Theorem 5.3.

Definition 5.4. Let \((X, \mathcal{S}, E)\) be a fuzzy soft topological space, \(h\) be a fuzzy \(\alpha\)-closed soft set, and \(g\) be a fuzzy soft point such that \(g \not\subseteq h\). If there exist disjoint fuzzy \(\alpha\)-open soft sets \(F\) and \(G\) such that \(h \cap F = G\), then \(g\) is a fuzzy \(\alpha\)-regular soft point.

Theorem 5.7. Let \((X, \mathcal{S}, E)\) be a fuzzy soft topological space and \(g\) be a fuzzy \(\alpha\)-closed soft set. If \(g\) is a fuzzy \(\alpha\)-regular soft point, then there exists a fuzzy \(\alpha\)-open soft set \(S\) such that \(g \subseteq S\).

Proof. From Theorem 2.1, by hypothesis, there exist disjoint fuzzy \(\alpha\)-open soft sets \(S\) and \(T\) such that \(g \subseteq S\). Hence, \(g\) is a fuzzy \(\alpha\)-closed soft set.

Theorem 5.8. Every fuzzy soft \(\alpha\text{-T}_2\)-space, in which every fuzzy soft point is fuzzy \(\alpha\)-closed, is fuzzy soft \(\alpha\text{-T}_2\)-space.

Proof. Let \(f, g\) be two distinct fuzzy soft points. By hypothesis, \(g\) is a fuzzy \(\alpha\)-closed soft set. From fuzzy soft regularity, there exist disjoint fuzzy \(\alpha\)-open soft sets \(S\) and \(T\) such that \(g \subseteq S\) and \(h \not\subseteq T\).

Theorem 5.9. A fuzzy soft subspace \((Y, \mathcal{S}_Y, E)\) of fuzzy soft \(\alpha\text{-T}_2\)-space \((X, \mathcal{S}, E)\) is fuzzy soft \(\alpha\text{-T}_2\)-space.

Proof. By Theorem 5.2, \((X, \mathcal{S}, E)\) is fuzzy soft \(\alpha\text{-T}_2\)-space. If \(g\) be a fuzzy \(\alpha\)-closed soft set, then \(g\) is fuzzy soft \(\alpha\)-regular soft point.

Definition 5.5. Let \((X, \mathcal{S}, E)\) be a fuzzy soft topological space and \(h, g\) be disjoint fuzzy \(\alpha\)-closed soft sets. If there exist disjoint fuzzy \(\alpha\)-open soft sets \(F\) and \(G\) such that \(h \subseteq F\) and \(g \subseteq G\), then \((X, \mathcal{S}, E)\) is called fuzzy soft \(\alpha\)-normal space.

Theorem 5.10. Let \((X, \mathcal{S}, E)\) be a fuzzy soft topological space. Then, the following are equivalent:

1. \((X, \mathcal{S}, E)\) is a fuzzy soft \(\alpha\)-normal space.
2. For every fuzzy \(\alpha\)-closed soft set \(h\) and fuzzy \(\alpha\)-open soft set \(g\), there exists a fuzzy \(\alpha\)-open soft set \(S\) such that \(h \cap S \subseteq g\).

Proof. (1) \(\Rightarrow\) (2) Let \(h\) be a fuzzy \(\alpha\)-closed soft set and \(g\) be a fuzzy \(\alpha\)-open soft set such that \(h \subseteq g\). Then, \(h\) is fuzzy \(\alpha\)-closed soft. It follows by (1), there exist disjoint fuzzy \(\alpha\)-open soft sets \(S\) and \(T\) such that \(h \subseteq S\) and \(S \cap T = \emptyset\).

(2) \(\Rightarrow\) (1) Let \(S\) and \(T\) be disjoint fuzzy \(\alpha\)-closed soft sets. Then, \(S\) and \(T\) are fuzzy \(\alpha\)-open soft sets.

Theorem 5.11. A fuzzy \(\alpha\)-closed soft subspace \((Y, \mathcal{S}_Y, E)\) of fuzzy soft \(\alpha\)-normal space \((X, \mathcal{S}, E)\) is fuzzy soft \(\alpha\)-normal.

Proof. Let \(g\) and \(h\) be disjoint fuzzy \(\alpha\)-closed soft sets. Then, \(h \subseteq g\). Hence, \(h\) is fuzzy \(\alpha\)-closed soft.

\(\square\)
\[ g_A = h_E \cap f_c \subseteq h_E \cap f_s \quad \text{and} \quad g_B = h_E \cap f_D \subseteq h_E \cap f_w. \]

Therefore, \((Y, \Sigma_2, E)\) is fuzzy soft \(\alpha\)-normal.

**Theorem 5.12.** Let \((X, \Sigma_1, E)\) and \((Y, \Sigma_2, K)\) be fuzzy soft topological spaces and \(f_{pu} : SS(X)_E \rightarrow SS(Y)_K\) be a fuzzy soft function which is bijective, fuzzy \(\alpha\)-irresolute soft and fuzzy \(\alpha\)-irresolute open soft. If \((X, \Sigma_1, E)\) is a fuzzy soft \(\alpha\)-normal space, then \((Y, \Sigma_2, K)\) is also a fuzzy soft \(\alpha\)-normal space.

**Proof.** Let \(f_A, g_B\) be disjoint fuzzy \(\alpha\)-closed soft sets in \(Y\). Since \(f_{pu}\) is fuzzy \(\alpha\)-irresolute soft, then \(f_{pu}^{-1}(f_A)\) and \(f_{pu}^{-1}(g_B)\) are fuzzy \(\alpha\)-closed soft set in \(X\) such that \(f_{pu}^{-1}(f_A) \cap f_{pu}^{-1}(g_B) = f_{pu}^{-1}[f_A \cap g_B] = f_{pu}^{-1}[\tilde{0}_K] = \tilde{0}_E\) from Theorem 2.2. By hypothesis, there exist disjoint fuzzy \(\alpha\)-open soft sets \(k_C\) and \(k_D\) in \(X\) such that \(f_{pu}^{-1}(f_A) \subseteq k_C\) and \(f_{pu}^{-1}(g_B) \subseteq h_D\). It follows that, \(f_A = f_{pu}[f_{pu}^{-1}(f_A)] \subseteq f_{pu}(k_C)\) and \(g_B = f_{pu}[f_{pu}^{-1}(g_B)] \subseteq f_{pu}(h_D)\) from Theorem 2.2 and \(f_{pu}(k_C) \cap f_{pu}(h_D) = f_{pu}[k_C \cap h_D] = f_{pu}[\tilde{0}_K] = \tilde{0}_K\) from Theorem 2.2. Since \(f_{pu}\) is fuzzy \(\alpha\)-irresolute open soft function. Then, \(f_{pu}(k_C), f_{pu}(h_D)\) are fuzzy \(\alpha\)-open soft sets in \(Y\). Thus, \((Y, \Sigma_2, K)\) is a fuzzy soft \(\alpha\)-normal space.

**6 Conclusion**

Since the authors introduced topological structures on fuzzy soft sets [9,16,43], so the \(\alpha\)-topological properties, which introduced by Kandil et al. [26], is generalized here to the fuzzy soft sets which will be useful in the fuzzy systems. Because there exists compact connections between soft sets and information systems [46,39], we can use the results deduced from the studies on fuzzy soft topological space to improve these kinds of connections. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

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**References**


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