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Self Similarity of Sousa Ramos's Trees and Mira's Boxes Within the Boxes

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Abstract: This work consists on the theory of embedded boxes structure [7], from the point of view of the theory of symbolic dynamics [6], tree structure Ramos [8], [9], [10] and star product [1] . This merger allowed the outcome-applications defined on the real line and a better understanding of the structure of the bifurcation, which allows us to characterize and understand the dynamics behind an application within the chaotic dynamical systems.

Keywords: discrete dynamical systems, symbolic dynamics, star-product, Sousa Ramos's Trees

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction and motivation

José Sousa Ramos and Christian Mira were good friends in life but also in their interests in dynamical systems. Although they have never published a common paper they have worked in very similar problems and solved them from different points of view. This paper is dedicated to the unification of some of their results concerning discrete dynamical systems in the interval in order to clarify the common results. Concretely, we prove the equivalence between the Tree Structure and the Boxes Within Boxes Theory in the case of an unimodal map. We introduce both theories: tree structure and star product from Sousa Ramos's Ph.D. Thesis in section 2 and Boxes Within Boxes theory of Mira in section 3. Then, in section 4, we establish the connection between these two theories showing their equivalence. At the end we present some examples to illustrate the procedure.

2 Star-Product in Sousa Ramos's trees

This section is devoted to the introduction of the $*$-product (star product) and the self-similarity structure of Sousa Ramos's trees.

2.1 Symbolic Dynamics

Let f_a be a one parameter family of unidimensional quadratic endomorphisms, which can always be reduced, by a linear change of coordinates, to one of the forms $x_{k+1} = x_k^2 - a$ or $x_{k+1} = 4x_k b(1 - x_k)$ for k integer and a, b real numbers. The periodic orbits of period k are given by the solutions of $f_a^k(x) = x$. According to Sousa Ramos [8] it was Myrberg who, in 1958, initiate the systematic study of periodic orbits of period k and the bifurcations that generate them. For a given value of k, there is a number N_k of k-periodic orbits which grows rapidly with k. Those orbits differ from each other through symbolic sequences. It was again Mirberg, using a symbolic 2-letter alphabet, who did it for the first time, putting the initial condition $x_0 = 0$ in the critical point of the map. Those results were continued by Metropolis, Stein, Gumowski-Mira, Milnor-Thurston, Guckenheimer among others.

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Let $I=[0,1]$ be the unitary interval in the real line. Let us fix a unimodal family of maps $f_b : I \to I$, which we denote by $\mathscr{C}$. We say that $f_b \in \mathscr{C}$ if and only if the following properties hold:

1. $f_b \in C^3(I)$ with $f_b(0) = f_b(1) = 0$;
2. f_b has a local maximum in the critical point $x = c$. The map is strictly increasing in $[0,c]$ and strictly decreasing in $[c,1]$;
3. The Schwarzian derivative of f_b is negative in $I \setminus \{c\}$;
4. f_b is differentiable with respect to b, $f_0(x) = 0$ and $f_1(c) = 1$.

Sousa Ramos introduced this family of maps with the aim of characterizing the bifurcations which occur while varying the parameter b.

In what follows, let $f_b \in \mathscr{C}$. Let $I_1 = [0,c]$ and $I_2 = [c,1]$ be the intervals of monotonicity of f_b

For each value of the parameter b we obtain an orbit $\{x_j : j \in \mathbb{N}\}$ of c, obtained by the iterative process $x_0 = c; \quad x_j = f_b^j(c) = f_b(x_{j-1}), \quad j = 1,2,\ldots$. To each such orbit we associate a sequence of symbols $S = S_0 S_1 \ldots S_j É$, where $S_0 = C$ and for $j = 1,2,\ldots,$

$$S_j = \begin{cases} L & \text{if } f_b^j(c) < c \\ C & \text{if } f_b^j(c) = c \\ R & \text{if } f_b^j(c) > c \end{cases}.$$

The set of all such sequences gives rise to the set $\Sigma_C \subset \Sigma_3 = \{L,C,R\}^{\mathbb{N}}$. A sequence $S \in \Sigma_C$ is called admissible. For a given sequence $S = CS_1 \ldots S_k \cdots \in \Sigma_C$ we call $S^{[k]} = S_1 \ldots S_k$ a k-block of S. Denote by σ the shift-operator acting on Σ_3. For each sequence $CS_1 \cdots \in \Sigma_3$, we call $\sigma(S)$, (after [6]) the kneading sequence. If the itinerary of the critical point is periodic of period k, the correspondent kneading sequence is also periodic and and can be represented by $S^{(k)} = S_1 \ldots S_{k-1} C$, i.e., the repeating block of $\sigma(S)$. Instead, we can write $S^{(k)} = S^{[k-1]}C$, where $S^{[k-1]}$ is composed only by symbols R or L. The sequence with length 1, $S^{(1)} = bC$ corresponds to the situation where the critical point, c is a fixed point, $c = f(c)$, b (from blank) is used to denote $S^{[0]}$. In the set of all kneading sequences, $\Sigma_M \subset \Sigma_3$, define an order relation through the R-parity of the sequence, that is, the parity of the number of R-symbols in S. Let S and P be two symbolic sequences, let i be an integer such that $S_i \neq P_i$ and $S_j = P_j$ for all $j < i$. If the R-parity of the block $S_1 \ldots S_{i-1} = P_1 \ldots P_{i-1}$ is even, the order relation in the symbols is given by $L <_R C <_R R$. If the R-parity of the block $S_1 \ldots S_{i-1} = P_1 \ldots P_{i-1}$ is odd, the order relation in the symbols is given by $R <_R C <_R L$. In both cases we say that $S < P$ if $S_i <_R P_i$. If such i does not exist, then $P = S$. This order relation induces an order relation in Σ_C as follows. If S and P belong to Σ_C, we say that $S < P$ if and only if $\sigma(S) < \sigma(P)$. Note that all admissible sequences in Σ_M are maximal with respect to the order defined above, because $\sigma^m(S) < S$ for all $m > 1$.

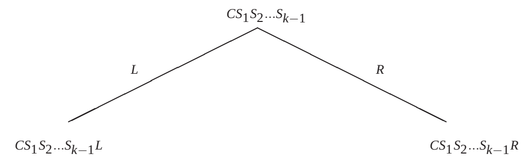

Fig. 1: Ramification of the Kneading sequences's tree of the unimodal maps if the R-parity is even.

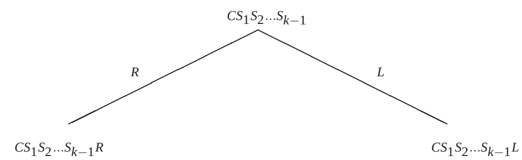

Fig. 2: Ramification of the tree of the unimodal maps if the R-parity is odd.

2.2 Tree structure

In this section we present the tree structure of symbolic sequences introduced by Sousa Ramos (see [8],[9], [10], [11]). According to [8], to each $(\Sigma_C, <_R)$ we associate a tree structure as follows. In each vertex is a sequence of symbols $CS^{[k-1]} = CS_1 \ldots S_{k-1}$, for some $k \in \mathbb{N}$. Each branching of the tree corresponds to a new iteration, in such a way that, to each new vertex we associate a symbol R or L, according to the place, right or left, where the iteration of the critical point falls. Thus, the order of ramification of each vertex can be described by Fig. 1 and Fig. 2.

Using these rules we can describe the tree of symbolic itineraries of the critical point, $CS^{[k-1]}$, the tree $\mathscr{A}_C$. The tree of kneading sequences $\mathscr{A}_M$ is obtained shifting each vertex, $\mathscr{A}_M = \sigma(\mathscr{A}_C)$ and can be seen in Fig.3.

These trees obey to the order relation defined in the sequences, that is, in each level, the sequences are ordered from the left to the right. But not just in each level, the projection in an imaginary line on the ground of the tree, is also ordered from left to the right.

2.3 On the star-product and self similarity of the tree structure

To introduce the $*$-product as a composition law we will follow [1], where a internal similarity law of the family of admissible sequences has been proven. This internal similarity will be observed in the tree structure.

Let $P = P_1 P_2 \ldots P_{p-1}$ and $Q = Q_1 Q_2 \ldots Q_{n-1}$ with $P_i \in \{R,L\}, Q_i \in \{R,L\}$, be two finite sequences of

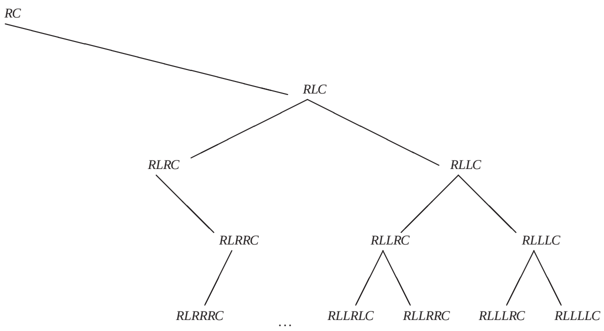

Fig. 3: Kneading sequences's tree $\mathscr{A}_M$.

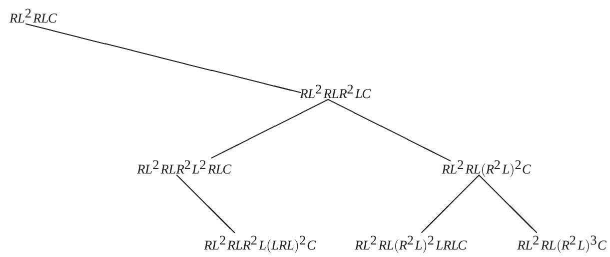

Fig. 4: Subtree $\mathscr{A}_{RLC}$.

symbols (allowed or not). Let

$$\overline{P_i} = \begin{cases} L & \text{if } P_i = R \\ R & \text{if } P_i = L \end{cases}.$$

Definition 1.*If $p > 1$, define*
$Q * P = QP_1QP_2Q...QP_{p-1}Q$, if the number of R symbols in Q is even;

$Q * P = Q\overline{P}_1Q\overline{P}_2Q...Q\overline{P}_{p-1}Q$, if the number of R symbols in Q is odd
If $p = 1$, that is, $P = b$, define $Q * P = Q$.

Example 1.$R * RL^n = RLR^{2n+1}$

Example 2.$RLR * RL^n = RLR^3(LR)^{2n+1}$.

Example 3.$RL * RLR = RLLRLRRLLRL = RL^2RLR^2L^2RL$.

We list below some properties of the $*$-mapping which can be seen in [1].

1. The commutativity does not hold in general. For example $R * RL \neq RL * R$;
2. The associativity holds, $(P * Q) * S = P * (Q * S)$, for all sequences P, Q, S;
3. If P and Q are allowed sequences, then so is $P * Q$. The period of $P * Q$ is the product of the periods of p and Q;
4. If P, Q_1 and Q_2 are allowed sequences and $Q_1 < Q_2$, then $P * Q_1 < P * Q_2$.
5. Let the allowed sequences P, Q_1, Q_2, Q_1' and Q_2' satisfy $P * Q_1 = Q_1' < Q_2' = P * Q_2$. Then, corresponding to any allowed Q' with $Q_1' < Q' < Q_2'$ one can always find an allowed sequence Q such that $Q' = P * Q$.
6. For any allowed sequence Q, one has

$$b < Q < Q * Q < Q * Q * Q \ldots$$

and

$$RL^n > Q * RL^n > Q * Q * RL^n < Q * Q * Q * RL^n > \ldots;$$

7. Let $P_1 = b$, $P_2 = RL^n$ and P and Q any allowed sequences, $P_1 < P$, $Q < P_2$. Set $P_i' = Q * P_i$, $i = 1, 2$; $P' = Q * P$. Then, to every P corresponds a P'. Conversely, for every P' with $Q * P_1 = P_1' < P' < P_2' = Q * P_2$ corresponds a P with $P_1 < P < P_2$, $P' = Q * P$.

The Property 7, above, is the analogous, in the "tree language" of Sousa Ramos, of the following theorem. It establishes that each vertex $S^{(k)} \in \mathscr{A}_M$ is associated to a subtree $\mathscr{A}_{S^{(k)}}$ which is isomorphic to $\mathscr{A}_M$. This subtree contains all admissible sequences of $\mathscr{A}_M$ that are located between the two extremal branches of the subtree. The subtree $\mathscr{A}_{S^{(k)}}$ is given by

$$\mathscr{A}_{S^{(k)}} = S^{[k-1]} * \mathscr{A}_M.$$

Each subtree $\mathscr{A}_{S^{(k)}}$ as defined above have two extremal branches: the minimal extremal branch $(S^{[k-1]} * L^{\infty})$ and the maximal branch $(S^{[k-1]} * RL^{\infty})$.

Theorem 1.*(Sousa Ramos et al, 86, [4]) Each subtree $\mathscr{A}_{S^{(k)}}$ consists of all admissible sequences in the tree $\mathscr{A}_M$, located between the extremal branches of the subtree.*

The proof of this theorem is supported by two lemmas, proved in [8].

Lemma 1.*(Lemma 4 of [8]) Between the minimal branch $(S^{[k-1]} * L^{\infty})$ and the central branch $(S^{[k-1]} * C)^{\infty}$ there are no admissible sequences.*

Lemma 2.*(Lemma 5 of [8]) Between the central branch $(S^{[k-1]} * C)^{\infty}$ and the maximal extremal branch $(S^{[k-1]} * RL^{\infty})$ all the sequences have the form $(S^{[k-1]} * PC)$, where $PC \in \mathscr{A}_M$.*

We can see in Figure 4 the subtree $\mathscr{A}_{RLC}$

3 Box-Within-A-Box Structure

The "box-within-a-box" bifurcation structure is described in [7] for a one dimensional unimodal smooth map, but it may be considered at the basis of the bifurcation mechanisms of any smooth one-dimensional multimodal map. This type of structure will be denoted BB thereafter.

The objective of this section is to explain the bifurcation structures "box-within-a-box" that are observed in certain types of recurrences, as a result of the variation of a parameter. The results given in this section are primarily drawn from references [7].

We will focus on recurrences of the form

$$X_{n+1} = F(X_n, \lambda) \tag{1}$$

where X_n is a vector of $\mathbb{R}$ and a λ real parameter. The function F is of class C^p, $p > 1$ on $\mathbb{R}$.

The "boxes" of the considered structure are intervals of the real parameter corresponding to the existence of an attractor (periodic orbit or invariant set) related to a given order. Boxes of different kinds can be defined. A box of the 1-st kind of a k-cycle opens (or starts) at the fold bifurcation giving rise to a couple of k-cycles (i.e., two cycles of period k), one attracting and one repelling, while a box of the 2nd kind of a k-cycle starts at a flip bifurcation of a $k-$cycle, giving rise to an attracting $2k-$cycle.

The closure of a box of the 2nd kind corresponds to the first homoclinic bifurcation of the k-cycle whose flip bifurcation started the box. Just before the bifurcation, $2k-$cyclic attracting invariant intervals exist (inside which the dynamics seem chaotic), whose immediate basins (considering the map T^{2k}) are such that two consecutive ones are bounded by the same periodic point; that is, these immediate basins may be coupled into pairs of two consecutive ones, which are cyclic for the map T^k , and separated by a repelling point of the k-cycle whose flip bifurcation started the box. At the bifurcation value closing the box of 2nd kind and after, k-cyclic invariant intervals are attracting, that is, the closure of a box of the 2nd kind causes the transition from 2k- to k-cyclic attracting invariant intervals (inside which the dynamics seem chaotic), and the new immediate basins are the reunion, by pairs, of the old immediate basins.

The closure of a box of the 1st kind of a k-cycle occurs at the homoclinic bifurcation of the k-cycle born repelling at the fold bifurcation which started the box. Just before the bifurcation, k-cyclic attracting invariant intervals exist, whose immediate basins, considering the map T^k, are such that any two of them are not consecutive. Each point on the boundary of an immediate basin is a limit point of disjoint components of the total basins. The k immediate basins are separated by connected components of the total basins, which are intertwined in a chaotic way.

Soon after the bifurcation, the k intervals are no longer invariant, and a wider invariant absorbing interval generally exists, with complex dynamics, which include not only the old k intervals, but also some components of their old total basins. Such an absorbing interval includes generally an attracting cycle of high period, and the computed trajectory seems chaotic into this invariant interval. That is, this bifurcation causes (apparently) a sudden increase of the chaotic (in a nonstrict sense) attracting set, from k disjoint intervals into a unique interval which includes all the previous ones. Soon after the bifurcation the iterated points of a generic trajectory visits more often the old k invariant intervals and less frequently the remaining parts, so that we have regions with high density of iterated points (related to the old k intervals) and regions with low density (related to the old regions of immediate basins and fractal components of the old total basins).

3.1 Box-within-a-box structure type Quadratic (BBQ)

The BB structures are observed in the case of recurrences of the form:

$$x_{n+1} = f(x_n, \lambda) \tag{2}$$

where f is a function, defined on an interval I of $\mathbb{R}$ that we assume here, at least, class C^3 (f belongs to $C^3(I)$). λ is a real parameter. In this subsection we consider a function of the actual variable having one extremum (unimodal).

Before explaining in detail the BBQ structure, we recall the different types of bifurcations that will be involved,

a) For a parameter value noted $\lambda_{(k)_0}$, two cycles of order k appear with their multiplier equal to 1, one becomes attractive for $\lambda > \lambda_{(k)_0}$ and the other becomes repulsive. This bifurcation is called a fold bifurcation and is denoted B1.

b) For a value of parameter noted $\lambda_{k,2^i}$, a cycle of order k exchange stability, its multiplier is equal to -1, giving rise to a cycle of order 2k, which takes the stability of order k cycle. This bifurcation corresponds to a doubling of the period, it is called a flip bifurcation and is denoted B2.

3.2 Example of logistic map

Logistic map is considered, in order to give a clearer description of the box-within-a-box structure.

Consider the function, defined on an interval I of $\mathbb{R}$, of the form:

$$x_{n+1} = f(x_n, \lambda) = x_n^2 - \lambda. \tag{3}$$

The set of all possible values for (3) bifurcation occurs when $\lambda \in \left[-\frac{1}{4}, 2\right]$. This interval is denoted by Ω_1 and corresponds to the first box of the structure BBQ , which contains all other boxes.

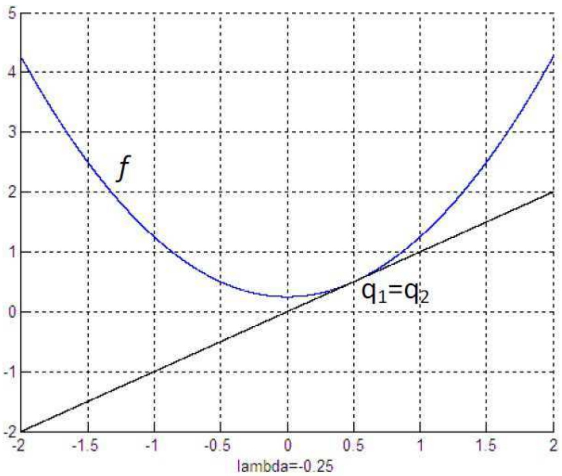

Fig. 5: The bifurcation B1 in the case of the logistic map. Two fixed points q_1 ans q_2 appear, one attractive, the other one repulsive.

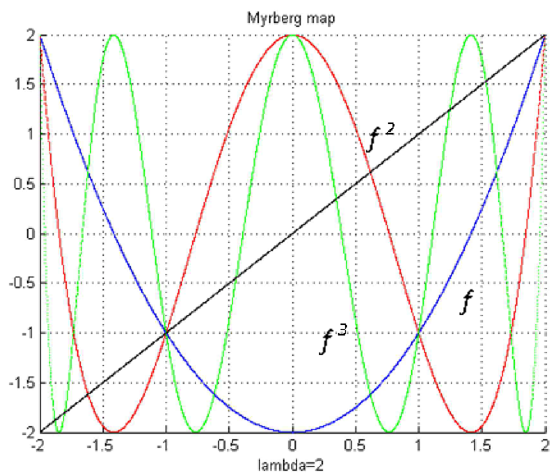

Fig. 6: $\lambda = \lambda_1^*$, $q_1 = C_1$ and the segment CC_1 is a chaotic segment. On the Figure, one can see the graph of f^2 in red and of f^3 in green.

3.2.1 Description of first box Ω_1.

The lower limit of this interval $\lambda_{(1)_0} = -\dfrac{1}{4}$, corresponds to the appearance of the two fixed points, one repulsive q_1, another attractive q_2, by a bifurcation B1 (see Fig. 5).

The upper limit of the interval $\lambda_1^* = 2$, corresponds to the merger of the fixed point q_1 with the critical point of rank one C_1 (Fig. 6). The segment CC_1 is a chaotic invariant segment, a chaotic attractor exists. All cycles of all orders exist and are repulsive; all these points are located on the segment CC_1 and are dense on CC_1.

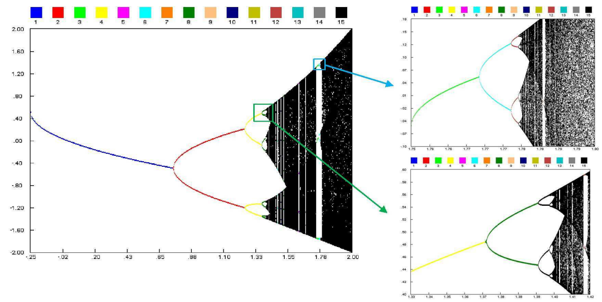

Fig. 7: Bifurcation diagram of the logistic map.

For each value of $\lambda \in \Omega_1 = \left[-\dfrac{1}{4}, 2\right]$ an attractive set (order k cycle or chaotic invariant segment, which may optionally be cyclic) exists on the interval CC_1, certain values of λ correspond to bifurcations B1 or B2.

For $\lambda > \lambda_1^*$ no more bifurcation occurs. Infinitely many repulsive cycles of all orders exist. Each initial condition, chosen on the real axis, leads to an iterated sequence that converges to the point at infinity (divergence).

3.2.2 Bifurcations diagram of the map (3) in the plane (λ, x).

Before explaining the structure of the following boxes, contained in the first box Ω_1, we obtain, through numerical simulations, the following diagram (fig. 7), drawn in the plane (λ, x). This bifurcation diagram allows to visualize the attractor obtained for each value of $\lambda \in \left[\lambda_{(1)_0}, \lambda_1^*\right] = \left[-\dfrac{1}{4}, 2\right]$.

We can observe the existence of periodic orbits of different orders, the phenomena of period doubling and behaviors that seem disorganized and correspond to the chaos on one or more intervals of variable x for a fixed value of λ. This behavior inside the first box Ω_1, is encoded by the structure box-within-box and will be explained describing the other boxes. We will detail this description in the next subsection.

3.3 Description of boxes Ω_k

We observe a succession of bifurcations B2, from fixed point q_2, for values λ noted λ_{2^i}, $i = 1, 2, \ldots,$. This sequence of bifurcations is called period doubling cascade and corresponds to the appearance of cycles of order 2^i, $i = 1, 2, \ldots$.

This sequence of (λ_{2^i}), $i = 1, 2, \ldots$ has a value of accumulation 1.401155.... This value is denoted by $\lambda_{(1)_s}$, is located in the box Ω_1, and is the limit value for which the chaos exists. From this value, there is, indeed, an infinity of order 2^i cycles which are inside CC_1. For $\lambda = \lambda_{(1)_s}$ a chaotic attractor of Cantor type exists in the interval CC_1. With this value of accumulation we split the box Ω_1.

3.3.1 Intervals ω_1 and Δ_1.

The interval $\left]\lambda_{(1)_0}, \lambda_{(1)_s}\right[$ is denoted ω_1 and the interval $\left]\lambda_{(1)_s}, \lambda_1^*\right[$ is denoted by Δ_1 (see Fig. 8) . ω_1 contains no odd order cycle, only cycles of order of power of 2; the odd order cycles will appear in the interval Δ_1, beyond the value $\lambda^*_{2^i}$, defined below.

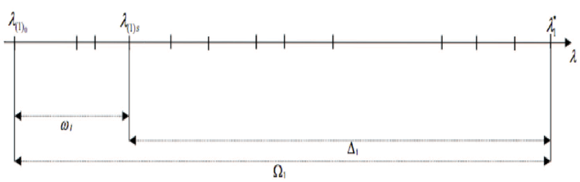

Fig. 8: Scheme of the structure of the box Ω_1.

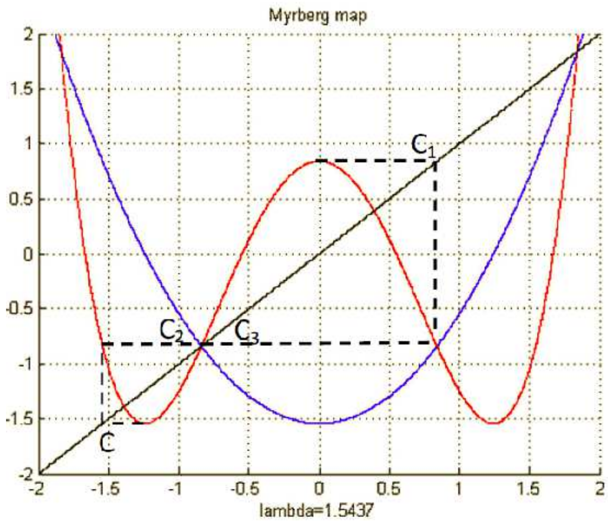

Fig. 9: $\lambda = \lambda^*_{2^i}$, the rank-2 critical point, C_2, merges with the fixed point q_2, from which the period doubling cascade is issued. The segment CC_1 is chaotic.

3.3.2 Cyclic chaotic segment

For $\lambda = \lambda^*_{2^i}$, a value which corresponds to the corresponds to the merger of the critical point C_2^i with a point of the cycle of order 2^{i-1}, which is destabilized by a bifurcation B2 (for $\lambda = \lambda_{2^i}$), a cyclic chaotic invariant segment of order 2^{i-1} is observed (for $\lambda = \lambda^*_{2^1}$ see Figs. 9 and 10).

The bifurcations of this type occur in the reverse order of doublings bifurcations period ($\lambda^*_{2^{i+1}} < \lambda^*_{2^i}$) and $\lambda_{(1)_s}$ is also a point of accumulation of $\lambda^*_{2^i}, i \to +\infty$.

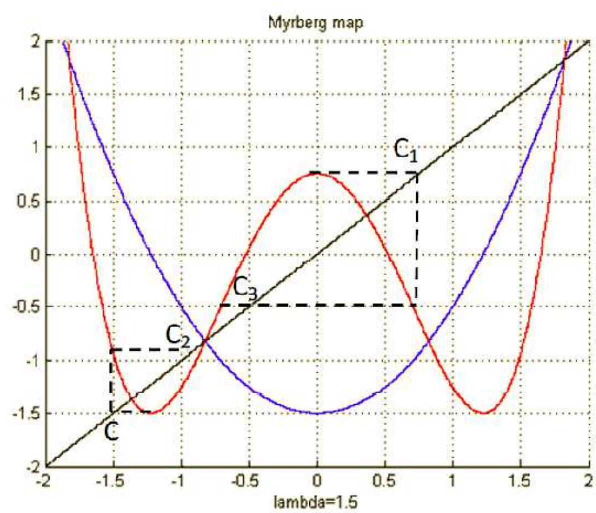

Fig. 10: $\lambda < \lambda^*_{2^i}$, the two segments CC_2 and C_3C_1 constitute an order 2 cyclic absorbing segment, which contains either an order $2i$ cyclic chaotic segment, either an order $2i$ periodic orbit

3.3.3 Description of the box Ω_3.

The study of f^3 allows us to highlight the same phenomena to f^3 than that of f. It is possible to set an interval (box) $\Omega_3 = \left[\lambda_{(3)_0}, \lambda_3^*\right]$ contained in the interval Ω_1 and which has the same structure and the same properties as Ω_1. However, for the transformation f^3 instead of f, Ω_3 corresponds to the birth of two cycles of order 3, an attractive, the other repulsive, by a bifurcation B1 (Fig. 11).

λ_3^* corresponds to the merger of critical points C_1, C_2 and C_3 with the three points of the order cycle 3, born repulsive for $\lambda = \lambda_{(3)_0}$, (see Fig. 12, Fig. 13). $\lambda = \lambda_3^*$,

Ω_3 splits in the same way than Ω_1 into two intervals ω_3 and Δ_3;

ω_3 contains all bifurcation values for doubling period obtained for f^3, that is to say, corresponding to appearance of cycles of order 3.2^i, $i = 1, 2, ...,$

Δ_3 contains the values of bifurcation that will make appear cycles of order $3.m$, $m = 3, 4, 5, ...$

The value $\lambda_{(3)_s}$, the accumulation value for doublings period f^3, separates the two intervals ω_3 and Δ_3. Values

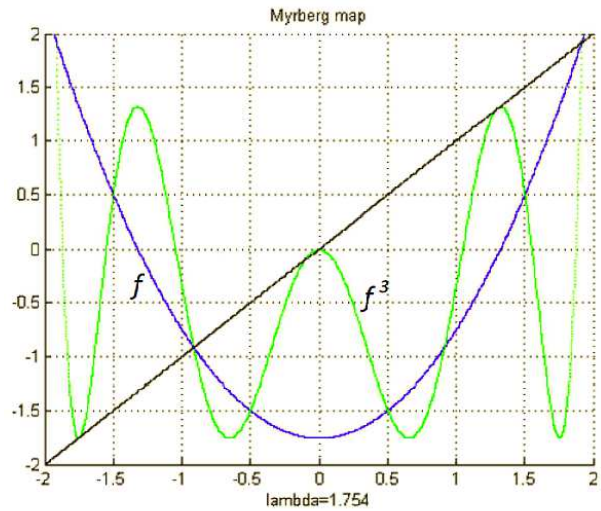

Fig. 11: $\lambda = \lambda_{(3)_0}$, two order 3 cycles appear by a bifurcation B1.

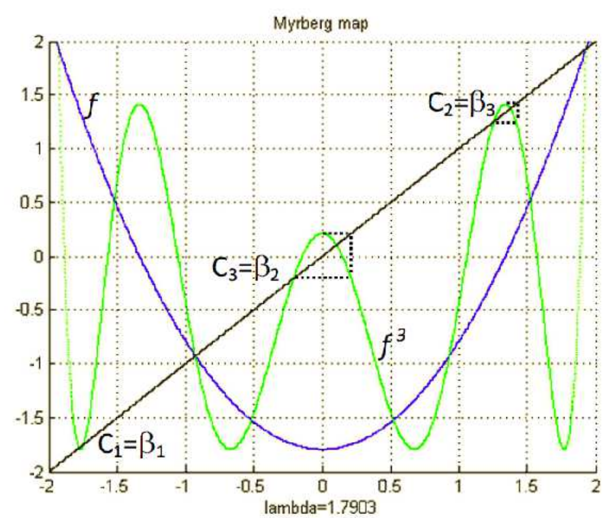

Fig. 12: $\lambda = \lambda_3^*$, the critical points C_1, C_3 and C_2 respectively merge with the points of the order 3 cycle born repulsive for $\lambda = \lambda_{(3)_0}$

$\lambda_{(3)_s}$and λ_3^*, have the same properties for f^3, as $\lambda_{(1)_s}$and λ_1^* for f; in particular for λ_3^*, a cyclic chaotic invariant set of order 3 is within the interval CC_1 (see Fig. 13).

3.3.4 Description of boxes Ω_k^j.

A similar study of transformations f^k can highlight the existence of boxes $\Omega_k^j = \left[\lambda_{(k)_0}^j, \lambda_k^{*j}\right]$ all having the same

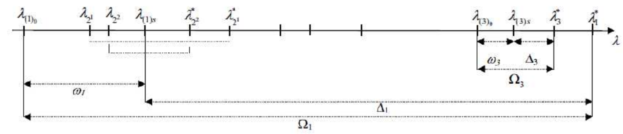

Fig. 13: Scheme of the structure of the box Ω_3.

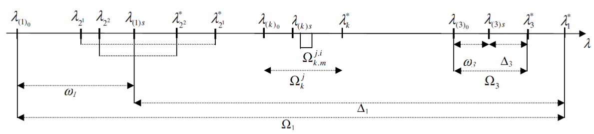

Fig. 14: Scheme of the box-within-box bifurcation structure.

structure of Ω_1 and Ω_3; period doubling cascades, accumulation value $\lambda_{(k)_s}^j$, intervals ω_k^j and Δ_k^j. This justifies the fractal nature of the structure of bifurcations BBQ and the term "box-within-a-box".

The emergence of a box Ω_k^j is linked to the existence of k extrema of f^k that will cross the diagonal $x_{n+1} = x_n$ in the plane (x_n, x_{n+1}) (bifurcation B1). When the value of k increases, the number of extrema of f^k increases, it is then possible to obtain several bifurcations B1 for f^k, which cause the existence of several boxes associated with f^k, the index j corresponds a numbering of the boxes to distinguish. For example, there are three boxes Ω_5^j, $j = 1,2,3$.

The Fig. 14 outlines this structure - box-within-box.

4 Box-Within-a-box bifurcation structure and Sousa Ramos's trees

The aim of this section is to characterize the "box-within-a-box" bifurcation structure using the techniques of symbolic dynamics for unimodal maps, always based on the kneading theory of Milnor and Thurston [6], the formalism of symbolic dynamics developed by J. Sousa Ramos [8], [9] and the $*$-product, work of Derrida, Gervois, and Pomeau [1].

As was done in the previous section, we will consider

$$x_{n+1} = f(x_n, \lambda) = f_\lambda(x_n)$$

under the same conditions given in section (3.1).

Thus, using the tree structure developed by Sousa Ramos (2.2), the order relation established in the symbolic sequences set (2.1) and the $* - product$ (2.3), we define $\mathscr{A}_P$ as the set of parameters λ, for which the map f_λ has a kneading sequence $S^{[k-1]}C \in \Sigma_M$.

Definition 2.*Let $\prec$ be the order relation induced in $\mathscr{A}_P$ by*

$$\lambda_{S^{(k)}} \prec \lambda_{Q^{(q)}} \textit{ iff } S^{(k)} <_R Q^{(q)}.$$

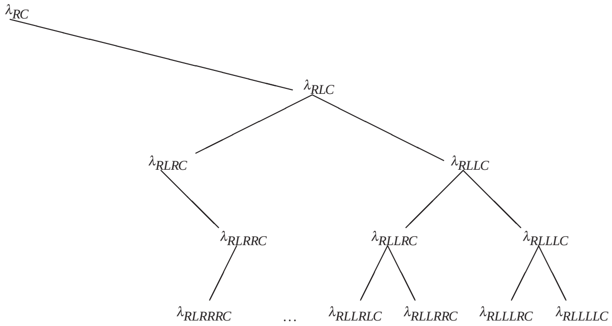

Fig. 15: Tree $\mathscr{A}_P$.

for any k and q positive integer.

With this order relation, we can represent $\mathscr{A}_P$ using Sousa Ramos's tree structure (fig. 15), establishing a connection between the symbolic sequence order and a order of the parameters on the set $\mathscr{A}_P$.

As was stablished in lemma 2 of section 2.3, between the sequence $\left(S^{[k-1]} * C\right)^\infty$ and $S^{[k-1]} * RL^\infty$ are all the sequences of the form $S^{[k-1]} * PC$ where $PC \in \mathscr{A}_M$. For instance, suppose that $S^{[k-1]} = R$ and $P = R$ in this case we have $S^{[k-1]} * PC = RLRC$, which is a periodic kneading sequence of period 4. Or, if $P = RL$ then we have $S^{[k-1]} * PC = RLRRRC$, which is a periodic kneading sequence of period 6.

Motivated by this examples, in a direct way we have the following result:

Proposition 1. *Let $S^{[k-1]}C$ be periodic kneading sequence of period k, and let $PC \in \mathscr{A}_M$ such that P is a symbolic sequence with $n-1$ symbols. Then:*

$$S^{[k-1]} * PC = S^{[k-1]}P_1 S^{[k-1]}P_2 \ldots S^{[k-1]}P_{n-1}S^{[k-1]}C$$

and, if the R-parity of $S^{[k-1]}$ is even,

$$S^{[k-1]} * PC = S^{[k-1]}\overline{P}_1 S^{[k-1]}\overline{P}_2 S^{[k-1]}\overline{P}_{n-1}S^{[k-1]}C,$$

if the R-parity of $S^{[k-1]}$ is odd. In both cases it corresponds to periodic kneading sequences of period $k \times n$.

Definition 3. *We define $\mathscr{A}_P\left(S^{[k-1]}C\right)$ as the set of parameters λ, for which the map f_λ has a kneading sequence in the tree $S^{[k-1]} * \mathscr{A}_M$.*

Proposition 2. *Let $\lambda \in \mathscr{A}_P\left(S^{[k-1]}C\right)$. Then the orbit of the critical point of the map f_λ is periodic of period p such that $p \bmod (k) = 0$.*

Proof. The result comes directly from the previous proposition and definition of the set $S^{[k-1]} * \mathscr{A}_M$.

Definition 4. *Let $\lambda_{S^{(k)}}$ be the parameter value such that the itinerary of the critical point under the map $f_{\lambda_{S^{(k)}}}$ is given by the symbolic sequence $S^{(k)}$.*

Proposition 3. *For the parameter value $\lambda_{S^{[k-1]}*RL^\infty}$ the itinerary of the critical point will fall into a eventually periodic orbit. The periodic sub-orbit of the eventually periodic orbit fall on repulsive fixed points of the map $f^{[k]}$.*

Proof. Without loss of generality, let assume that the sequence $S^{[k-1]}$ has an odd R-parity. Thus we have the sequence

$$S^{[k-1]} * RL^\infty = S^{[k-1]}L(S^{[k-1]}R)^\infty$$

which is a eventually periodic orbit.

Thus, using symbolic dynamics and the $*-$product, we have the following relation with the embedded boxes Ω_k (see section 3.3):

$$\Omega_2 = [\lambda_{RC}, \lambda_{R*RL^\infty}]$$
$$\Omega_3 = [\lambda_{RLC}, \lambda_{RL*RL^\infty}]$$
$$\ldots$$
$$\Omega_k = \left[\lambda_{S^{[k-1]}C}, \lambda_{S^{[k-1]}*RL^\infty}\right]$$

The number of embedded boxes of type Ω_k, for a given symbolic kneading sequence $S^{[k-1]}C$ is directly related to the number of periodic admissible kneading sequences of dimension k. For this reason we are working in the set Σ_M. For example, if we consider kneading sequences of period 5, from the condition of admissibility (see section 2.1), we only must consider the kneading sequences $RLLLC$, $RLRRC$ and $RLLRC$. For instace, the sequence $RRLLC$ is not admissible because $RLLCR >_R RRLLC$. Thus, we have three sets of embedded boxes of type Ω_5. Consider now the $\mathscr{A}_P\left(S^{[k-1]}C\right)$ tree. Noting that the parameter values $\lambda \in \mathscr{A}_P\left(S^{[k-1]}C\right)$ appear ordered according to the order relation of the respective kneading symbolic sequence, we have in $\mathscr{A}_P\left(S^{[k-1]}C\right)$ tree a structure that allows us to completely characterize the sets ω_k and Δ_k such that $\omega_k \cup \Delta_k = \Omega_k$.

Example 4. Let g_λ be the unimodal map given by

$$g_\lambda(x) = \lambda - x^2.$$

This map has the same qualitative characteristics as the map in the section 3.2 . For reasons of compatibility with the symbolic dynamics and tree structure established in section 3, we will consider the map g_λ.
Let $\lambda \in [\lambda_{RC}, \lambda_{RLR^\infty}]$, we obtain the bifurcation diagram in figure 16.

In figure 16 note that for the last value of $\lambda \in [\lambda_{RC}, \lambda_{RLR^\infty}]$, we obtain only a point in the bifurcation

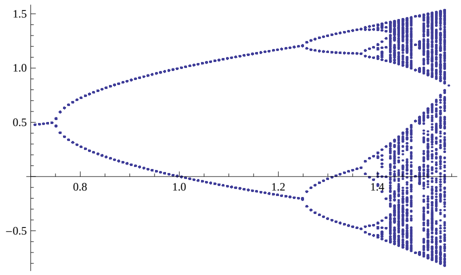

Fig. 16: Bifurcation diagram for $\lambda \in [\lambda_{RC}, \lambda_{RLR^\infty}]$

diagram. That is because for the parameter value $\lambda = \lambda_{RLR^\infty}$ there is a merger between the orbits of the critical points of $f^{[2]}$ map into a single repulsive fixed point of the map $f^{[2]}$. This situation is represented in figure 17.

Let $\lambda \in \mathscr{A}_P\left(S^{[k-1]}C\right)$. If we want to determine the values of the parameter λ, for which we obtain a periodic kneading sequence of period k, we could do this by solving an equation obtained from the kneading sequence. Let

$$R(y) = \sqrt{\lambda - y}$$
$$L(y) = -\sqrt{\lambda - y}$$

be two inverse functions, obtained from the function f. In order to better understand our intentions about this two maps let consider the following cases: If the critical point is a fixed point then we only have to solve $R(0) = 0$;. In this case the solution is trivial, ie, $\lambda = 0$. If the orbit of the critical point is of period two, RC, we must solve the equation

$$\lambda = \sqrt{\lambda - 0} = R(0),$$

which admissible solution is $\lambda = 1$; If the orbit of the critical point is of period 3, RLC, we must solve the equation

$$\lambda = \sqrt{\lambda + \sqrt{\lambda - 0}} = R \circ L(0)$$

which admissible solution will be $\lambda = 1.75488$.

Thus, to obtain the numerical value of λ_{RLR^∞}, and because we have an eventually periodic orbit, we must solve the equation

$$a - \left(a - a^2\right)^2 = \sqrt{a - \left(a - (a - a^2)^2\right)},$$

which admissible solution is $\lambda \simeq 1.54369$. In figure 17 we can see the orbit of the critical point of the map $g_{1.54369}$, and its symbolic sequence RLR^∞.

To obtain the numerical value of $\lambda_{RL*RL^\infty} = \lambda_{RLL(RLR)^\infty}$, and because we have an eventually periodic orbit, we must solve the equation

$$a - (a - a^2)^2 = \sqrt{a + \sqrt{a - \sqrt{a - a(1 - a(1 - a(1 - a)^2)^2)}}}$$

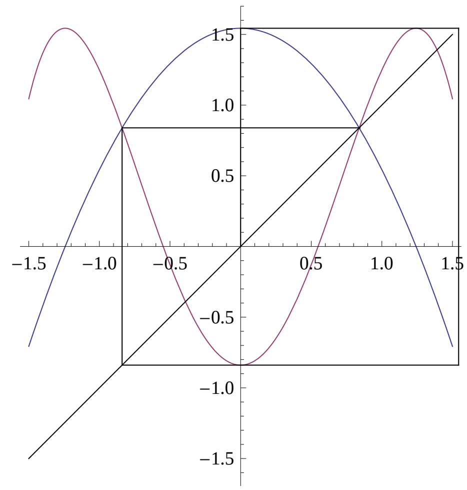

Fig. 17: The map $g_{1.54369}$ and $g_{1.54369}^{[2]}$, the line $y = x$ and the orbit of critical point of $g_{1.54369}$

which the admissible solution is $\lambda \simeq 1.79033$. Because the orbit is eventually periodic, we have to go looking for the value from which we obtain the periodic block of the orbit. In this case, this value is given by $a - (a - a^2)^2$. In figure 18 we can see the orbit of the critical point of the map $g_{1.79033}$, and its symbolic sequence $RLL(RLR)^\infty$.

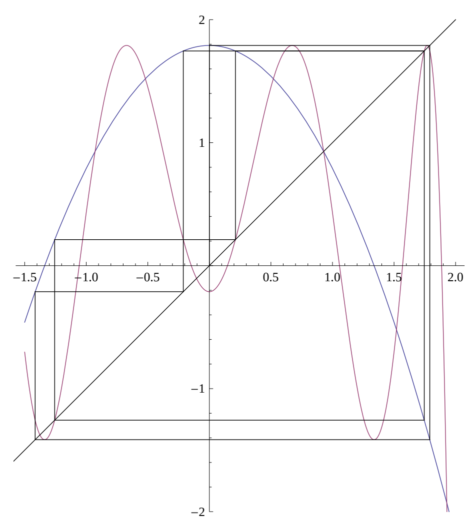

Fig. 18: The map $g_{1.79033}$ and $g_{1.79033}^{[3]}$, the line $y = x$ and the orbit of critical point of $g_{1.79033}$

*Example 5.*Let us consider the tent map

$$h_\lambda(x) = 1 - \lambda|x|.$$

The aim is to verify the merging of the orbits when we apply the star product. Here the boxes Ω_k for $k > 2$ is set with a single point, which is the value of the parameter λ for which the map h_λ have a kneading sequence.

With the aim of looking for the set containing all orbits of period $2p$, we must consider the maximal sequence $RC * RL^\infty$. From [2] we have the following condition that kneading sequence must verify:

$$1 = q_\delta(\lambda) = \sum_{i=0}^{+\infty} \frac{(-1)^i \delta_0\delta_1 \ldots \delta_i}{\lambda^{i+1}}.$$

where $\delta = S^{[k-1]}C$ such that $\delta_i = -1$ if $\delta_i = L$, $\delta_i = +1$ if $\delta_i = R$ and $\delta_i = \pm 1$ if $\delta_i = C$. If we consider the sequence

$$\delta = RLR^\infty$$

such that $R = +1$ and $L = -1$, and considering the sum of a convergent geometric series with ratio $\left(\frac{-1}{a}\right)$ we have

$$1 = \frac{1}{\lambda} + \frac{1}{\lambda(\lambda+1)}.$$

The admissible solution is $\lambda = 1.4142$. Thus we have our first box where we have the orbits of period $2p$

$$\Omega_2 = [1, 1.4142].$$

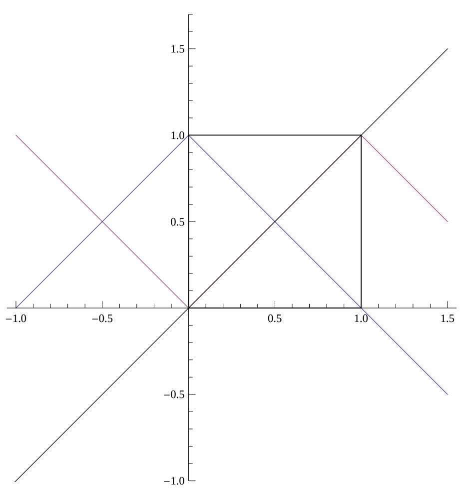

Fig. 19: The map $h_{1.4142}$ and $h^{[2]}_{1.4142}$, the line $y = x$ and the orbit of critical point of $h_{1.4142}$

Now the merging situation. If we want to get the box where we have the orbits of period $3p$, we must consider, for the minimum extreme, the sequence

$$\delta_l = (RLC)^\infty$$

and for the maximum extreme, the sequence

$$\delta_u = RLC * RL^\infty$$

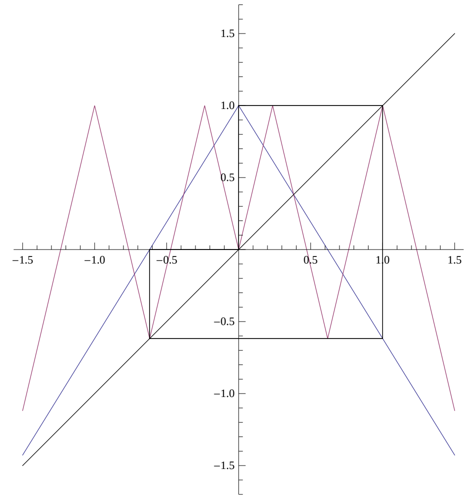

Fig. 20: The map $h_{1.61803}$ and $h^{[3]}_{1.61803}$, the line $y = x$ and the orbit of critical point of $h_{1.61803}$

But here,the set Ω_3 has only one point,

$$a = 1.61803.$$

For this parameter value we have all orbits of period $3p$. The next figure shows the plot of the tent map $T_a(x)$ and the map $T_a^3(x)$ for $a \simeq 1.61803$.

5 Conclusions

Using the kneading theory of Milnor and Thurston [6], the formalism of symbolic dynamics developed by J. Sousa Ramos [8], [9], [10], [4] and the work of Derrida, Gervois, and Pomeau about the $*$-product [1], we characterize the embedded boxes structure, work done by C. Mira [7], in an accurate manner, because we can fully characterize the lambda values in the set of parameters where all bifurcations occur in unimodal maps, making only use of symbolic computation, without using approximate values. In a future work, is our goal to build these tools into maps defined in higher dimensions.

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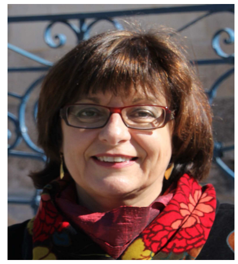

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