The Number of Symmetric Colorings of the Dihedral Group $D_p$

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Abstract: We compute the number of symmetric $r$-colorings and the number of equivalence classes of symmetric $r$-colorings of the dihedral group $D_p$, where $p$ is prime.

Keywords: Dihedral group, symmetric coloring, optimal partition, Möbius function, lattice of subgroups

1 Introduction

The symmetries on a group $G$ are the mappings $G \ni x \mapsto gx^{-1}g \in G$, where $g \in G$. This is an old notion, which can be found in the book [4]. It has also interesting relations to Ramsey theory and to enumerative combinatorics [2, 7].

Let $G$ be a finite group and let $r \in \mathbb{N}$. An $r$-coloring of $G$ is any mapping $\chi : G \to \{1, \ldots, r\}$. The group $G$ naturally acts on the colorings. For every coloring $\chi$ and $g \in G$, the coloring $\chi g$ is defined by

$$\chi g(x) = \chi(xg^{-1}).$$

Let $[\chi]$ and $\text{St}(\chi)$ denote the orbit and the stabilizer of a coloring $\chi$, that is,

$$[\chi] = \{\chi g : g \in G\} \quad \text{and} \quad \text{St}(\chi) = \{g \in G : \chi g = \chi\}.$$

As in the general case of an action, we have that

$$|[\chi]| = |G : \text{St}(\chi)| \quad \text{and} \quad \text{St}(\chi g) = g^{-1}\text{St}(\chi)g.$$

Let $\sim$ denote the equivalence on the colorings corresponding to the partition into orbits, that is, $\chi \sim \varphi$ if and only if there exists $g \in G$ such that $\chi(xg^{-1}) = \varphi(x)$ for all $x \in G$.

Obviously, the number of all $r$-colorings of $G$ is $r^{|G|}$. Applying Burnside’s Lemma [1, 1, §3] shows that the number of equivalence classes of $r$-colorings of $G$ is equal to

$$\frac{1}{|G|} \sum_{g \in G} r^{|G: \langle g \rangle|},$$

where $\langle g \rangle$ is the subgroup generated by $g$.

A coloring $\chi$ of $G$ is symmetric if there exists $g \in G$ such that

$$\chi(gx^{-1}g) = \chi(x)$$

for all $x \in G$. That is, if it is invariant under some symmetry. A coloring equivalent to a symmetric one is also symmetric (see [6, Lemma 2.1]). Let $S_r(G)$ denote the set of all symmetric $r$-colorings of $G$.

Theorem 1.[5, Theorem 1] Let $G$ be a finite Abelian group. Then

$$|S_r(G)| = \sum_{X \leq G \subseteq Y} \frac{\mu(Y, X) |G/Y|}{|B(G/Y)|} \frac{r^{G/X} + |r^{G/X}|}{2},$$

$$|S_r(G)/\sim| = \sum_{X \leq G \subseteq Y} \frac{\mu(Y, X) |G/Y|}{|B(G/Y)|} \frac{1 + |r^{G/X}|}{2}.$$

Here, $X$ runs over subgroups of $G$, $Y$ over subgroups of $X$, $\mu(Y, X)$ is the Möbius function on the lattice of subgroups of $G$, and $B(G) = \{x \in G : x^2 = e\}$.
Given a finite partially ordered set, the Möbius function is defined as follows:

\[
\mu(a, b) = \begin{cases} 
1 & \text{if } a = b \\
-\sum_{a < z \leq b} \mu(z, b) & \text{if } a < b \\
0 & \text{otherwise}.
\end{cases}
\]

See [1, IV] for more information about the Möbius function.

In the case of \( \mathbb{Z}_n \) these formulas can be reduced to elementary ones.

Theorem 2.[5, Theorem 2] If \( n \) is odd then

\[
|S_r(\mathbb{Z}_n)|/\sim = r^{n-1},
\]

\[
|S_r(\mathbb{Z}_n)| = \sum_{d|n} \phi(d) \prod_{p|d} (1 - p) r^{d-1}.
\]

If \( n = 2^m l \) where \( l \geq 1 \) and \( m \) is odd, then

\[
|S_r(\mathbb{Z}_n)|/\sim = \frac{r^{2^{m+1}} + r^{m+1}}{2},
\]

\[
|S_r(\mathbb{Z}_n)| = \sum_{d|n} d \prod_{p|d} (1 - p) d^{d-1}.
\]

In the products \( p \) takes on values of prime divisors.

In this note by constructing the partially ordered set of optimal partitions we compute explicitly the number \( |S_r(D_p)| \) of symmetric \( r \)-colorings of \( D_p \) and the number \( |S_r(D_p)|/\sim \) of equivalence classes of symmetric \( r \)-colorings of the dihedral group \( D_p \), where \( p > 2 \) is prime. This generalises the result from [3]. Since \( D_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), every coloring of \( D_2 \) is symmetric, and so

\[
|S_r(D_2)| = r^4 \text{ and } |S_r(D_2)|/\sim = \frac{1}{4} r^4 + \frac{3}{4} r^2.
\]

2 Optimal partitions of \( D_p \)

In [6], Theorem 1 was generalized to an arbitrary finite group \( G \). The approach is based on constructing the partially ordered set of so called optimal partitions of \( G \).

Given a partition \( \pi \) of \( G \), the stabilizer and the center of \( \pi \) are defined by

\[
St(\pi) = \{ g \in G : \text{for every } x \in G, x g^{-1} \text{ belong to the same cell of } \pi \},
\]

\[
Z(\pi) = \{ g \in G : \text{for every } x \in G, x g^{-1} g \text{ belong to the same cell of } \pi \}.
\]

\( St(\pi) \) is a subgroup of \( G \) and \( Z(\pi) \) is an union of left cosets of \( G \) modulo \( St(\pi) \). Furthermore, if \( e \in Z(\pi) \), then \( Z(\pi) \) is also a union of right cosets of \( G \) modulo \( St(\pi) \) and for every \( a \in Z(\pi), (a) \subseteq Z(\pi) \). We say that a partition \( \pi \) of \( G \) is optimal if \( e \in Z(\pi) \) and for every partition \( \pi' \) of \( G \) with \( St(\pi') = St(\pi) \) and \( Z(\pi') = Z(\pi) \), one has \( \pi \leq \pi' \).

The latter means that every cell of \( \pi \) is contained in some cell of \( \pi' \), or equivalently, the equivalence corresponding to \( \pi \) is contained in that of \( \pi' \). The partially ordered set of optimal partitions of \( G \) can be naturally identified with the partially ordered set of pairs \( (A, B) \) of subsets of \( G \) such that \( A = St(\pi) \) and \( B = Z(\pi) \) for some partition \( \pi \) of \( G \) with \( e \in Z(\pi) \). For every partition \( \pi \), we write \( |\pi| \) to denote the number of cells of \( \pi \).

Theorem 3.[6, Theorem 2.11] Let \( P \) be the partially ordered set of optimal partitions of \( G \). Then

\[
|S_r(G)| = |G| \sum_{x \in P, y \leq x} \frac{\mu(y, x)}{|Z(y)|} |\pi|/|\pi|,
\]

\[
|S_r(G)/\sim | = \sum_{x \in P, y \leq x} \frac{\mu(y, x)}{|Z(y)|} |\pi|/|\pi|.
\]

The partially ordered set of optimal partitions \( \pi \) of \( G \) together with parameters \( |St(\pi)|, |Z(\pi)| \) and \( |\pi| \) can be constructed by starting with the finest optimal partition \( \{x, x^{-1} : x \in G \} \) and using the following fact:

Let \( \pi \) be an optimal partition of \( G \) and let \( A \subseteq G \). Let \( \pi_1 \) be the finest partition of \( G \) such that \( \pi \leq \pi_1 \) and \( A \subseteq St(\pi_1) \), and let \( \pi_2 \) be the finest partition of \( G \) such that \( \pi \leq \pi_2 \) and \( A \subseteq Z(\pi_2) \). Then the partitions \( \pi_1 \) and \( \pi_2 \) are also optimal.

In this section we construct the partially ordered set of optimal partitions of the dihedral group \( D_p \), where \( p > 2 \) is prime, and compute explicitly the number \( |S_r(D_p)| \) of symmetric \( r \)-colorings of \( D_p \) and the number \( |S_r(D_p)|/\sim | \) of equivalence classes of symmetric \( r \)-colorings.

The dihedral group \( D_p \) has the following lattice of subgroups:

Now we list all optimal partitions \( \pi \) of \( D_p, p > 2 \) together with parameters \( |St(\pi)|, |Z(\pi)| \) and \( |\pi| \).

The finest partition

\[
\pi : \{e\}, \{s\}, \{sa, ..., sa^{p-1}\}, \{a, a^{p-1}\},
\]

\[
St(\pi) = \{e\}, Z(\pi) = \{e\},
\]

\[
|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = p + 1 + \frac{p - 1}{2} = \frac{3p + 1}{2}.
\]
\( p \) partitions of the form

\[ \pi : \{e, \{a, a^{p-1}\}, \ldots, \{s, sa^{p-1}\}, \ldots \} \]

\[ \text{St}(\pi) = \{e, Z(\pi) = \{e, s\}, \}

\[ |\text{St}(\pi)| = 1, |Z(\pi)| = 2, |\pi| = \frac{p - 1}{2} + 2 = p + 1. \]

One partition

\[ \pi : \{e, a, \ldots, a^{p-1}\}, \{s, \{sa, sa^{p-1}\}, \ldots \}

\[ \text{St}(\pi) = \{e, Z(\pi) = \{e, a, \ldots, a^{p-1}\}, \}

\[ |\text{St}(\pi)| = 1, |Z(\pi)| = p + 1, |\pi| = \frac{p - 1}{2} + 2 = \frac{p + 3}{2}. \]

\( p \) partitions of the form

\[ \pi : \{e, a, \ldots, a^{p-1}\}, \{s, \{sa, sa^{p-1}\}, \ldots \} \]

\[ \text{St}(\pi) = \{e, Z(\pi) = \{e, a, \ldots, a^{p-1}, s\}, \}

\[ |\text{St}(\pi)| = 1, |Z(\pi)| = p + 1, |\pi| = \frac{p - 1}{2} + 2 = \frac{p + 3}{2}. \]

Finally, by Theorem 3, we obtain that

\[ |S_r(D_p)| = |D_p| \sum_{x \in P, y \leq x} \frac{\mu(y, x)}{|Z(y)|^2} \]

\[ = 2p \left( r^{\frac{3p+1}{2}} + pr^{p+1} \left( \frac{1}{2} - 1\right) + r^{p+1} \left( \frac{1}{p} - 1\right) + \right. \]

\[ + pr^{\frac{p+1}{2}} \left( \frac{1}{p+1} - \frac{1}{2} - \frac{1}{p} + 1\right) + \]

\[ + r^{\frac{p+1}{2}} \left( \frac{1}{p+1} - \frac{p}{2} + p - 1\right) + pr^{\frac{p+1}{2}} \left( \frac{1}{2} - \frac{1}{2}\right) + \]

\[ + r^2 \left( \frac{1}{2p} - \frac{1}{p+1} - \frac{p}{2} + \frac{p-1}{p} - p + 1\right) + \]

\[ + r \left( \frac{1}{2p} - \frac{p}{2} + p + 1 \right) \right) \]

\[ = 2p \left( r^{\frac{3p+1}{2}} - r^{p+1} - \frac{p-1}{p} r^{p+1} + (p-1) r^{\frac{p+1}{2}} + \right. \]

\[ + \frac{-p^2 + 2p - 1}{2p} r^2 \right) \]

\[ = 2p \left( r^{\frac{3p+1}{2}} + \frac{-p^2 - 2p + 2}{2p} r^{p+1} + (p-1) r^{\frac{p+1}{2}} - \left( \frac{p-1}{2p} \right)^2 \right) \]

\[ = 2p r^{\frac{3p+1}{2}} + \left( -p^2 - 2p + 2 \right) r^{p+1} + 2p(p-1) r^{\frac{p+1}{2}} - \left( \frac{p-1}{2} \right)^2 r^2, \]
\[|S_r(D_p)|/ \sim | = \sum_{x \in \mathbb{Z}} \frac{\mu(x,y)|S_r(y)|}{|Z(y)|} r^{|x|}\]

\[= r^{\frac{3p+1}{2}} + pr^{p+1}(\frac{1}{2} - 1) + r^{p+1}(\frac{1}{p} - 1) + \]

\[+ pr^{\frac{p+1}{2}}(\frac{1}{p+1} - \frac{1}{2} - \frac{1}{p} + 1) + \]

\[+ r^{\frac{p+1}{2}}(\frac{1}{p+1} - \frac{p}{2} + p - 1) + pr^{\frac{p+1}{2}}(\frac{2}{2} - \frac{1}{2}) + \]

\[+ \frac{r^2}{2} \left( \frac{2p}{p+1} - \frac{1}{p+1} - \frac{p}{2} + \frac{p-1}{p} - (p+1) \right) + \]

\[+ pr^\frac{p+1}{2} - \frac{p}{2} + p - 1 + r^{p+1} + (p-1)r^{\frac{p+1}{2}} + \]

\[+ \frac{p^2}{2} - \frac{p^2}{2} + 3p - 2 r^2 + \frac{1}{p-1} r = \]

\[= r^{\frac{3p+1}{2}} + \frac{-p^2 - 2p + 2}{2} r^{p+1} + (p-1) r^{\frac{p+1}{2}} + \]

\[+ \frac{p^2}{2} - \frac{p^2}{2} + 3p - 2 r^2 + \frac{1}{p-1} r.\]

Thus, we have showed that

**Theorem 4.** For every \( r \in \mathbb{N} \) and prime \( p > 2 \),

\[|S_r(D_p)| = 2pr^{\frac{3p+1}{2}} + (-p^2 - 2p + 2)r^{p+1} + \]

\[+ 2pr(p-1)r^{\frac{p+1}{2}} - (p-1)^2 r^2,\]

\[|S_r(D_p)/ \sim | = r^{\frac{3p+1}{2}} + \frac{p^2-2p+2}{2} r^{p+1} + (p-1) r^{\frac{p+1}{2}} + \]

\[+ \frac{p^2}{2} - \frac{p^2}{2} + 3p - 2 r^2 + \frac{1}{p-1} r.\]

Notice that the number of all \( r \)-colorings of \( D_p \) is \( r^{2p} \) and the number of equivalence classes of all \( r \)-colorings of \( D_p \) is

\[\frac{1}{|D_p|} \sum_{g \in D_p} |[D_p]_g| = \frac{1}{2} (r^{2p} + pr^p + (p-1)r^2).\]

### 3 Conclusion

We conclude with the following open question

**Question 1.** What is the number of equivalence classes of symmetric \( r \)-colorings of the dihedral group \( D_n \), where \( r, n \in \mathbb{N} \)?

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### References


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