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The Number of Symmetric Colorings of the Dihedral Group D_p

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Abstract: We compute the number of symmetric r-colorings and the number of equivalence classes of symmetric r-colorings of the dihedral group D_p, where p is prime.

Keywords: Dihedral group, symmetric coloring, optimal partition, Möbius function, lattice of subgroups

1 Introduction

The symmetries on a group G are the mappings $G \ni x \mapsto gx^{-1}g \in G$, where $g \in G$. This is an old notion, which can be found in the book [4]. It has also interesting relations to Ramsey theory and to enumerative combinatorics [2], [7].

Let G be a finite group and let $r \in \mathbb{N}$. An *r-coloring* of G is any mapping $\chi : G \to \{1,\dots,r\}$. The group G naturally acts on the colorings. For every coloring χ and $g \in G$, the coloring χg is defined by

$$\chi g(x) = \chi(xg^{-1}).$$

Let $[\chi]$ and $St(\chi)$ denote the orbit and the stabilizer of a coloring χ, that is,

$$[\chi] = \{\chi g : g \in G\} \text{ and } St(\chi) = \{g \in G : \chi g = \chi\}.$$

As in the general case of an action, we have that

$$|[\chi]| = |G : St(\chi)| \text{ and } St(\chi g) = g^{-1}St(\chi)g.$$

Let $\sim$ denote the equivalence on the colorings corresponding to the partition into orbits, that is, $\chi \sim \varphi$ if and only if there exists $g \in G$ such that $\chi(xg^{-1}) = \varphi(x)$ for all $x \in G$.

Obviously, the number of all r-colorings of G is $r^{|G|}$. Applying Burnside's Lemma [1, I, §3] shows that the number of equivalence classes of r-colorings of G is equal to

$$\frac{1}{|G|}\sum_{g\in G} r^{|G:\langle g\rangle|},$$

where $\langle g\rangle$ is the subgroup generated by g.

A coloring χ of G is *symmetric* if there exists $g \in G$ such that

$$\chi(gx^{-1}g) = \chi(x)$$

for all $x \in G$. That is, if it is invariant under some symmetry. A coloring equivalent to a symmetric one is also symmetric (see [6, Lemma 2.1]). Let $S_r(G)$ denote the set of all symmetric r-colorings of G.

Theorem 1.*[5, Theorem 1] Let G be a finite Abelian group. Then*

$$|S_r(G)| = \sum_{X\le G}\sum_{Y\le X}\frac{\mu(Y,X)|G/Y|}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}},$$

$$|S_r(G)/\sim| = \sum_{X\le G}\sum_{Y\le X}\frac{\mu(Y,X)}{|B(G/Y)|} r^{\frac{|G/X|+|B(G/X)|}{2}}.$$

Here, X runs over subgroups of G, Y over subgroups of X, $\mu(Y,X)$ is the Möbius function on the lattice of subgroups of G, and $B(G) = \{x \in G : x^2 = e\}$.

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Given a finite partially ordered set, the Möbius function is defined as follows:

$$\mu(a,b) = \begin{cases} 1 & \text{if } a = b \\ -\sum_{a<z\leq b}\mu(z,b) & \text{if } a < b \\ 0 & \text{otherwise .} \end{cases}$$

See [1, IV] for more information about the Möbius function.

In the case of $\mathbb{Z}_n$ these formulas can be reduced to elementary ones.

Theorem 2.[5, *Theorem 2]* *If n is odd then*

$$|S_r(\mathbb{Z}_n)/\sim| = r^{\frac{n+1}{2}},$$
$$|S_r(\mathbb{Z}_n)| = \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}.$$

If $n = 2^l m$, *where* $l \geq 1$ *and m is odd, then*

$$|S_r(\mathbb{Z}_n)/\sim| = \frac{r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}}{2},$$
$$|S_r(\mathbb{Z}_n)| = \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1}.$$

In the products p takes on values of prime divisors.

In this note by constructing the partially ordered set of optimal partitions we compute explicitly the number $|S_r(D_p)|$ of symmetric r-colorings of D_p and the number $|S_r(D_p)/\sim|$ of equivalence classes of symmetric r-colorings of the dihedral group D_p, where $p > 2$ is prime. This generalises the result from [3]. Since $D_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, every coloring of D_2 is symmetric, and so

$$|S_r(D_2)| = r^4 \text{ and } |S_r(D_2)/\sim| = \frac{1}{4}r^4 + \frac{3}{4}r^2.$$

2 Optimal partitions of D_p

In [6], Theorem 1 was generalized to an arbitrary finite group G. The approach is based on constructing the partially ordered set of so called optimal partitions of G.

Given a partition π of G, the *stabilizer* and the *center* of π are defined by

$St(\pi) = \{g \in G :$ for every $x \in G, x$ and xg^{-1} belong to the same cell of $\pi\}$,

$Z(\pi) = \{g \in G :$ for every $x \in G, x$ and $gx^{-1}g$ belong to the same cell of $\pi\}$.

$St(\pi)$ is a subgroup of G and $Z(\pi)$ is a union of left cosets of G modulo $St(\pi)$. Furthermore, if $e \in Z(\pi)$, then $Z(\pi)$ is also a union of right cosets of G modulo $St(\pi)$ and for every $a \in Z(\pi)$, $\langle a\rangle \subseteq Z(\pi)$. We say that a partition π of G is *optimal* if $e \in Z(\pi)$ and for every partition π' of G with $St(\pi') = St(\pi)$ and $Z(\pi') = Z(\pi)$, one has $\pi \leq \pi'$. The latter means that every cell of π is contained in some cell of π', or equivalently, the equivalence corresponding to π is contained in that of π'. The partially ordered set of optimal partitions of G can be naturally identified with the partially ordered set of pairs (A,B) of subsets of G such that $A = St(\pi)$ and $B = Z(\pi)$ for some partition π of G with $e \in Z(\pi)$. For every partition π, we write $|\pi|$ to denote the number of cells of π.

Theorem 3.[6, *Theorem 2.11]* *Let P be the partially ordered set of optimal partitions of G. Then*

$$|S_r(G)| = |G| \sum_{x\in P}\sum_{y\leq x} \frac{\mu(y,x)}{|Z(y)|} r^{|x|},$$

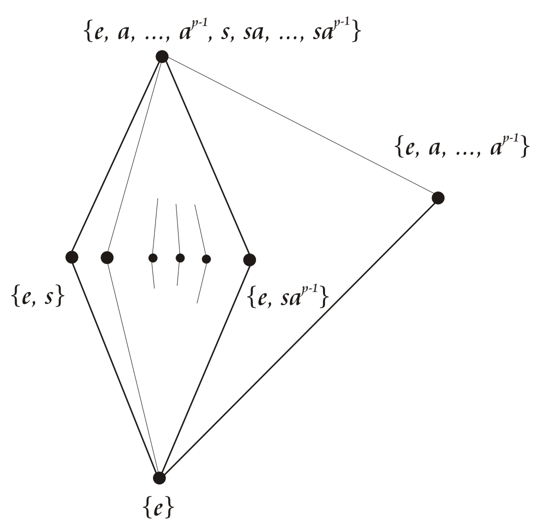
$$|S_r(G)/\sim| = \sum_{x\in P}\sum_{y\leq x} \frac{\mu(y,x)|St(y)|}{|Z(y)|} r^{|x|}.$$

The partially ordered set of optimal partitions π of G together with parameters $|St(\pi)|$, $|Z(\pi)|$ and $|\pi|$ can be constructed by starting with the finest optimal partition $\{\{x,x^{-1}\} : x \in G\}$ and using the following fact:

Let π be an optimal partition of G and let $A \subseteq G$. Let π_1 be the finest partition of G such that $\pi \leq \pi_1$ and $A \subseteq St(\pi_1)$, and let π_2 be the finest partition of G such that $\pi \leq \pi_2$ and $A \subseteq Z(\pi_2)$. Then the partitions π_1 and π_2 are also optimal.

In this section we construct the partially ordered set of optimal partitions of the dihedral group D_p, where $p > 2$ is prime, and compute explicitly the number $|S_r(D_p)|$ of symmetric r-colorings of D_p and the number $|S_r(D_p)/\sim|$ of equivalence classes of symmetric r-colorings.

The dihedral group D_p has the following lattice of subgroups:



Now we list all optimal partitions π of $D_p, p > 2$ together with parameters $|St(\pi)|$, $|Z(\pi)|$ and $|\pi|$.

The finest partition

$\pi : \{e\},\{s\},\{sa\},\ldots,\{sa^{p-1}\},\{a,a^{p-1}\},\ldots$

$St(\pi) = \{e\}, Z(\pi) = \{e\},$

$$|St(\pi)| = 1, |Z(\pi)| = 1, |\pi| = p+1+\frac{p-1}{2} = \frac{3p+1}{2}.$$

p partitions of the form

$$\pi : \{e\}, \{a, a^{p-1}\}, \ldots, \{s\}, \{sa, sa^{p-1}\}, \ldots$$
$$St(\pi) = \{e\}, Z(\pi) = \{e, s\},$$
$$|St(\pi)| = 1, |Z(\pi)| = 2, |\pi| = \frac{p-1}{2} \cdot 2 + 2 = p + 1.$$

One partition

$$\pi : \{e, a, \ldots, a^{p-1}\}, \{s\}, \{sa\}, \ldots, \{sa^{p-1}\}$$
$$St(\pi) = \{e\}, Z(\pi) = \{e, a, \ldots, a^{p-1}\},$$
$$|St(\pi)| = 1, |Z(\pi)| = p, |\pi| = p + 1.$$

One partition

$$\pi : \{e\}, \{a, a^{p-1}\}, \ldots, \{s, sa, \ldots, sa^{p-1}\}$$
$$St(\pi) = \{e\}, Z(\pi) = \{e, s, sa, \ldots, sa^{p-1}\},$$
$$|St(\pi)| = 1, |Z(\pi)| = p + 1, |\pi| = \frac{p-1}{2} + 2 = \frac{p+3}{2}.$$

p partitions of the form

$$\pi : \{e, a, \ldots, a^{p-1}\}, \{s\}, \{sa, sa^{p-1}\}, \ldots$$
$$St(\pi) = \{e\}, Z(\pi) = \{e, a, \ldots, a^{p-1}, s\},$$
$$|St(\pi)| = 1, |Z(\pi)| = p + 1, |\pi| = \frac{p-1}{2} + 2 = \frac{p+3}{2}.$$

p partitions of the form

$$\pi : \{e, s\}, \{a, a^{p-1}, sa, sa^{p-1}\}, \ldots$$
$$St(\pi) = \{e, s\}, Z(\pi) = \{e, s\},$$
$$|St(\pi)| = 2, |Z(\pi)| = 2, |\pi| = \frac{p-1}{2} + 1 = \frac{p+1}{2}.$$

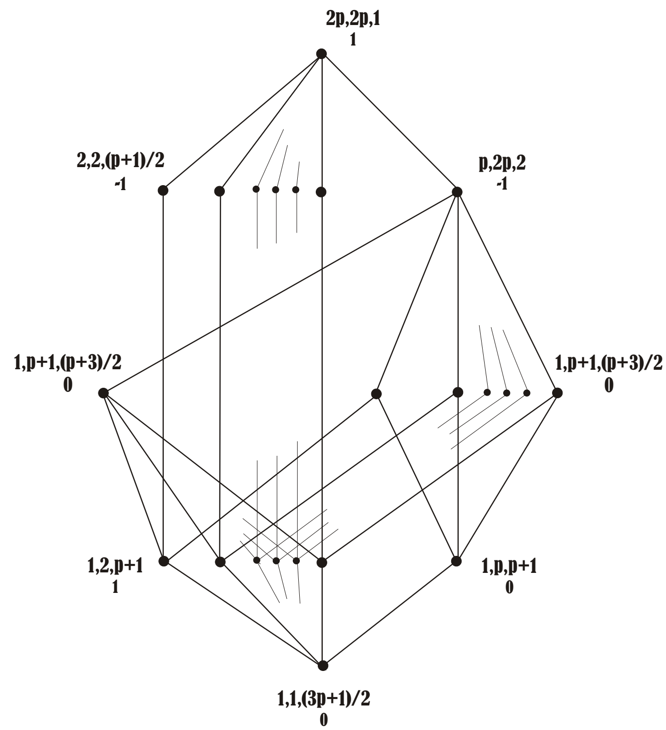
One partition

$$\pi : \{e, a, \ldots, a^{p-1}\}, \{s, sa, \ldots, sa^{p-1}\}$$
$$St(\pi) = \{e, a, \ldots, a^{p-1}\}, Z(\pi) = D_p,$$
$$|St(\pi)| = p, |Z(\pi)| = 2p, |\pi| = 2.$$

And the coarsest partition

$$\pi : \{D_p\}$$
$$St(\pi) = D_p, Z(\pi) = D_p,$$
$$|St(\pi)| = 2p, |Z(\pi)| = 2p, |\pi| = 1.$$

Next, we draw the partially ordered set of optimal partitions π together with parameters $|St(\pi)|$, $|Z(\pi)|$ and $|\pi|$. The picture below shows also the values of the Möbius function of the form $\mu(a, 1)$.



Finally, by Theorem 3, we obtain that

$$\begin{aligned}
|S_r(D_p)| &= |D_p| \sum_{x \in P} \sum_{y \leq x} \frac{\mu(y,x)}{|Z(y)|} r^{|x|} \\
&= 2p(r^{\frac{3p+1}{2}} + pr^{p+1}(\frac{1}{2} - 1) + r^{p+1}(\frac{1}{p} - 1) + \\
&+ pr^{\frac{p+3}{2}}(\frac{1}{p+1} - \frac{1}{2} - \frac{1}{p} + 1) + \\
&+ r^{\frac{p+3}{2}}(\frac{1}{p+1} - \frac{p}{2} + p - 1) + pr^{\frac{p+1}{2}}(\frac{1}{2} - \frac{1}{2}) + \\
&+ r^2(\frac{1}{2p} - \frac{1}{p+1} - \frac{p}{p+1} + \frac{p}{2} + \frac{p-1}{p} - p + 1) + \\
&+ r(\frac{1}{2p} - \frac{1}{2p} - \frac{p}{2} + \frac{p}{2})) = \\
&= 2p(r^{\frac{3p+1}{2}} - \frac{p}{2} r^{p+1} - \frac{p-1}{p} r^{p+1} + (p-1) r^{\frac{p+3}{2}} + \\
&+ \frac{-p^2 + 2p - 1}{2p} r^2) = \\
&= 2p(r^{\frac{3p+1}{2}} + \frac{-p^2 - 2p + 2}{2p} r^{p+1} + (p-1) r^{\frac{p+3}{2}} - \\
&- \frac{(p-1)^2}{2p} r^2) = \\
&= 2pr^{\frac{3p+1}{2}} + (-p^2 - 2p + 2) r^{p+1} + 2p(p-1) r^{\frac{p+3}{2}} - \\
&- (p-1)^2 r^2,
\end{aligned}$$

$$\begin{aligned}
|S_r(D_p)/\sim| &= \sum_{x\in P}\sum_{y\le x}\frac{\mu(y,x)|St(y)|}{|Z(y)|}r^{|x|}\\
&= r^{\frac{3p+1}{2}} + pr^{p+1}(\frac{1}{2}-1) + r^{p+1}(\frac{1}{p}-1)+\\
&+ pr^{\frac{p+3}{2}}(\frac{1}{p+1}-\frac{1}{2}-\frac{1}{p}+1)+\\
&+ r^{\frac{p+3}{2}}(\frac{1}{p+1}-\frac{p}{2}+p-1) + pr^{\frac{p+1}{2}}(\frac{2}{2}-\frac{1}{2})+\\
&+ r^2(\frac{p}{2p}-\frac{1}{p+1}-\frac{p}{p+1}+\frac{p}{2}+\frac{p-1}{p}-p+1)+\\
&+ r(\frac{2p}{2p}-\frac{p}{2p}-\frac{2p}{2}+\frac{p}{2}) =\\
&= r^{\frac{3p+1}{2}} - \frac{p}{2}r^{p+1} - \frac{p-1}{p}r^{p+1} + (p-1)r^{\frac{p+3}{2}}+\\
&+ \frac{p}{2}r^{\frac{p+1}{2}} + \frac{-p^2+3p-2}{2p}r^2 + \frac{1-p}{2}r =\\
&= r^{\frac{3p+1}{2}} + \frac{-p^2-2p+2}{2p}r^{p+1} + (p-1)r^{\frac{p+3}{2}}+\\
&+ \frac{p}{2}r^{\frac{p+1}{2}} + \frac{-p^2+3p-2}{2p}r^2 + \frac{1-p}{2}r.
\end{aligned}$$

Thus, we have showed that

Theorem 4. *For every $r \in \mathbb{N}$ and prime $p > 2$,*

$$|S_r(D_p)| = 2pr^{\frac{3p+1}{2}} + (-p^2-2p+2)r^{p+1}+ +2p(p-1)r^{\frac{p+3}{2}} - (p-1)^2r^2,$$

$$|S_r(D_p)/\sim| = r^{\frac{3p+1}{2}} + \frac{-p^2-2p+2}{2p}r^{p+1} + (p-1)r^{\frac{p+3}{2}}+ +\frac{p}{2}r^{\frac{p+1}{2}} + \frac{-p^2+3p-2}{2p}r^2 + \frac{1-p}{2}r.$$

Notice that the number of all r-colorings of D_p is r^{2p} and the number of equivalence classes of all r-colorings of D_p is

$$\frac{1}{|D_p|}\sum_{g\in D_p} r^{|D_p/\langle g\rangle|} = \frac{1}{2p}(r^{2p} + pr^p + (p-1)r^2).$$

3 Conclusion

We conclude with the following open question

Question 1. What is the number of equivalence classes of symmetric r-colorings of the dihedral group D_n, where $r, n \in \mathbb{N}$?

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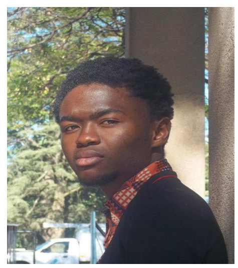

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