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Mathematical Analysis of Bioheat Equation for the Study of Thermal Stress on Human Brain

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Abstract: This paper studies the analysis of temperature profiles in human brain and its overlying layers at various hypothermic and hyperthermic conditions. The head is modelled as hemisphere consisting of four layers viz. brain tissue, CSF (cerebrospinal fluid), skull and scalp. The variational finite element method is employed on bioheat equation with appropriate boundary conditions. To obtain the values of temperature profiles at different nodal points, the role of various parameters like density, thermal conductivity, metabolic heat generation, arterial blood flow and optical properties like absorption and scattering are taken into consideration. The temperature distribution to the brain tissue with respect to the change in the ambient temperature are calculated and interpreted graphically.

Keywords: Pennes bioheat equation, variational finite element method, metabolic heat generation, thermal conductivity.

1 Introduction

The animal models currently used in clinical studies have smaller brain like those of typical adults. Thus the temperature distribution measured in animal models may not apply directly to human brain. Consequently it is not clear that the results for temperature profiles achieved in animal models are applicable to human subjects. Although attempts were made by Olsen et al. [1] on monkey's brain and Clifton et al. [2] on rat's brain but these models did not considered the temperature dependent properties of the domain. For clinical purposes, it is important to monitor closely both the brain and body core temperature as suggested by most of the experimental studies viz. Wass et al. [3] and Stone et al. [4]. Wass suggested that there can be significant temperature variation within the regions of human head. In order to monitor these variations of temperature, several clinical studies were conducted by Ruman et al. [5] and Stone et al. [4] but placement of probed was still a methodological problem. This paper attempts to view the numerical temperature profiles in the different regions of human head. A theoretical model, based on bioheat equation [6] with appropriate boundary conditions, has been developed to examine the transient brain temperature at various hyperthermic and hypothermic conditions.

During hyperthermia and hypothermia, the temporal and spatial temperature gradient in human brain depend on heat convection due to blood flow, heat conduction through the tissue and metabolic heat generation, Perl [7, 8]. The size of human brain play an important role in the temperature gradient by effecting heat conduction. Dexter and Hinderman [9] showed that conduction has a moderate effect on brain temperature of infants compared to little effect on that of adults because of the size of brain. Nelson and Nunneley [10] used selective brain cooling (SBC) to give quantitative modelling results on brain temperature and limits on transcranial cooling in humans. Khanday and Saxena [11] used one dimensional Pennes bioheat equation for the estimation of heat regulation in human head. The present paper gives insight to study transient temperature at extreme environmental conditions using bioheat equation. Due to the fact that human head is almost of spherical shape it will be most appropriate to use the radial form of bioheat equation.

For surgical processes or during therapy for selective cooling and/or destroying of target tissue, it is useful to note the behaviour of extreme temperatures falling with in the burning and freezing range. Therefore it becomes imperative to study the temperature distribution and its effects in the neighbourhood of targeted tissue. Earlier

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Ong et al. [12] proposed a self-tuning adaptive thermal controller, based on Hebbian feedback covariance learning where the system is to be regulated continually to best suit its environment. The implication of this model was that optimal physiological behaviors arising from self tuning adaptive regulation in the thermal controller may be responsible for the departure from homeostasis in abnormal states. Ng et al. [13] evaluated the individual effect of critical parameters on RFA (radiofrequency ablation) using Taguchi experimental design. The study of cumulative effect of parameters on RFA for different tissue types was carried out and the implications regarding control system design of RFA for various tissues were discussed. In 2010, to simulate the steady state temperature distribution during contact with hot solid, Ng et al. [14] developed an axisymmetric model of the human skin and the simulation was carried out using the boundary element method (BEM). The comparison of the results obtained by BEM was carried with the results obtained by finite element method (FEM). Ng and Sudharsan [15] established computer simulation model in conjunction with medical thermography as an adjunct tool for early breast cancer. Ng and Chua [16,17] investigated through numerical modelling the prediction of skin burn injury (part-1 and part-2). In 2009, Ng [18] gave a review of thermography as promising non-invasive detection modality for breast tumour and demonstrated that the breast thermography has achieved an average sensitivity and specificity of 90%. It was also discussed that the performance and environmental requirements in the characterizing thermography as being used for the breast tumor screening under strict indoor controlled environmental conditions.

2 Mathematical formulation

The domain that is human head has been modelled as hemisphere with four layers as Brain: $(0 \le r \le l_0)$, CSF: $(l_0 < r \le l_1)$, Skull:$(l_1 < r \le l_2)$, Scalp: $(l_2 < r \le l_3)$ these layers are treated as sub-domains(or elements) with four interfaces as; Brain-CSF, CSF-Skull, Skull-Scalp and Scalp-Atmosphere. The discretization of human head is based on the dimensions, physical properties and physiological characteristics as well as arterial blood temperature of different elements as discussed by Keener et al. [19], Guyton [20].

The temperature distribution in biological system is given by the bio-heat equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\rho\rho_b c_b F(T_A-T)+S=\rho c\frac{\partial T}{\partial t} \qquad (1)$$

where r is the radial distance from the centre of the brain towards the surface tissues, T is the variable temperature, k is the thermal conductivity of the medium, T_A is the arterial temperature, S is the rate of metabolic heat generation, c_b is the specific heat of the blood, F is the blood mass flow rate, ρ and ρ_b are tissue and blood densities respectively.

At $r = l_4$, the mixed boundary condition based on Newton's law of cooling, is given by

$$-k\frac{\partial T}{\partial r}\bigg|_{r=l_4} = h(T-T_a)+LE \qquad (2)$$

where h is the heat transfer coefficient and T_a is the ambient temperature. L is the latent heat and E is the evaporation rate.

Since the human head is known for its ability to maintain core temperature constant as discussed by Nelson and Nunneley [10], Khanday et al. [11,21], Zhu and Diao [22] therefore, the boundary condition taken at the core of human brain is assumed to be;

$$T = T_0 = constant\,(37^0C) \qquad (3)$$

To estimate the temperature profiles at the nodal points or the interfaces of the above four regions we solve the Eqs. (1) to (3) by variational finite element method.

3 Solution of the model

The parameters used in Pennes' bioheat equation are position dependent and have fixed value in every spherical region. So it will be most appropriate to use finite element method (FEM). As compared to other numerical methods this method gives better approximation of solving partial differential equation in a continuous domains as assumed by Khanday et al. [11,23, 24] and Aijaz et al. [21]. Since we intend to make use of the variational finite element method, therefore in order to solve the model we invoke the following procedure.

The variational integral,

$$I = \int F(T,T',r)dr \qquad (4)$$

in optimum form is equivalent to the following Euler-Lagrange differential equation (5) as discussed by Myers [25].

$$\frac{\delta F}{\delta T}-\frac{d}{dr}\left(\frac{\delta F}{\delta T'}\right)=0, \qquad (5)$$

where $T' = \dfrac{\delta T}{\delta r}$.

Comparing Eq. (1) with Eq. (5), after solving and rearranging we can find the variational integral for differential Eq. (1) together with boundary condition given by Eq. (2) as;

$$I=\frac{1}{2}\int_{l_0}^{l_3}\left[k\left(\frac{\partial T}{\partial r}\right)^2-\rho\rho_b c_b F(T_A-T)^2-2ST+\rho c\frac{\partial T^2}{\partial t}\right]rdr$$
$$+\frac{1}{2}\delta_e h(T-T_a)^2+2LE$$

where $\delta_e = 0$ for $r \le l_2$ and $\delta_e = 1$ for $l_2 < r \le l_3$, Now using the variation on the above defined sub-domains we come across the following variational integrals for each sub-domain

$$I_j = \frac{1}{2}\int_{l_{j-1}}^{l_j}\left[k_j r\left(\frac{\partial T^j}{\partial r}\right)^2 - \rho_j\rho_b C_b F_j(T_{A_j}-T^j)^2 r - 2ST^j r + \rho_j c_j r\frac{\partial (T^j)^2}{\partial t}\right]dr + \frac{1}{2}\delta_e h(T^j - T_a)^2 + 2LE \tag{6}$$

where $j = 0,1,2,3$ corresponding to brain, CSF, skull and scalp.

Assembling these integrals the resultant integral can be written as

$$I = \sum_{j=0}^{3} I_{(j)} \tag{7}$$

To solve Eq. (6), the metabolic heat generation and perfusion terms are taken as zero both in CSF and skull and for the estimation of temperature profiles $T(r)$ at nodal points of sub-domains of the head, it is important to make assumption of solution in terms of shape functions. Since the domain size is small, therefore the use of higher order shape function does not make significant changes in the output as described by Khanday et al. [11,26,27]. Therefore we assume the solution by means of linear shape function as;

$$T^j(r) = A_j + B_j r$$

so that the layer wise shape functions for the above defined sub-domains are:

$$T^j(r) = \frac{T_{j+1}-T_j}{l_{j+1}-l_j}r + \frac{l_{j+1}T_j - l_j T_{j+1}}{l_{j+1}-l_j} \tag{8}$$

where $j = 0,1,2,3$ corresponding to brain, CSF, skull and scalp. The solved variational integrals for these layers are given as:

$$\begin{aligned} J_0 =& \frac{1}{4}k_0(T_1-T_0)^2 - \frac{\alpha_0}{2}\left\{\frac{l_1^2}{2}(T_{A_0}-T_0)^2 - \frac{2}{3}l_1^2(T_{A_0}-T_0)^2\times (T_1-T_0) + \frac{l_1^2}{4}(T_1-T_0)^2\right\} - S_0 l_1^2\left\{\frac{T_0}{2}+\frac{T_1-T_0}{3}\right\} \\ &+ \frac{\rho_0 c_0}{2}\left\{\dot{T}_0 T_0 l_1^2 + \frac{l_1^2}{2}(T_1-T_0)(\dot{T}_1-\dot{T}_0) + \frac{2}{3}l_1^2(\dot{T}_0T_1 + T_0\dot{T}_1 - 2T_0\dot{T}_0)\right\} \end{aligned} \tag{9}$$

$$\begin{aligned} J_1 =& \frac{k_1(T_2-T_1)^2L_1}{4L_1'} + \frac{\rho_1 c_1}{2L_1'}\left\{L_1(l_2T_1 - l_1T_2)(l_2\dot{T}_1 - l_1\dot{T}_2) + N_1\frac{T_2-T_1}{2}(\dot{T}_2-\dot{T}_1) + \frac{2}{3}M_1\{L_1(\dot{T}_1T_2 + \dot{T}_2T_1) \right. \\ &\left. - 2(l_1\dot{T}_1T_2 - l_2\dot{T}_2T_1)\}\right\} \end{aligned} \tag{10}$$

$$\begin{aligned} J_2 =& \frac{k_2(T_3-T_2)^2L_2}{4L_2'} + \frac{\rho_2 c_2}{2L_2'}\left\{L_2(l_3T_2 - l_2T_3)(l_3\dot{T}_2 - l_2\dot{T}_3) + N_2\frac{T_3-T_2}{2}\times(\dot{T}_3-\dot{T}_2) + \frac{2}{3}M_2\{L_2(\dot{T}_2T_3 + \dot{T}_3T_2) \right. \\ &\left. - 2(l_2\dot{T}_2T_3 - l_3\dot{T}_3T_2)\}\right\} \end{aligned} \tag{11}$$

$$\begin{aligned} J_3 =& \frac{k_3(T_4-T_3)^2L_3}{4L_3'} - \frac{\alpha_3}{2}\left\{L_3\frac{(l_4T_{A_3} - l_3T_{A_3} - l_4T_3 + l_3T_4)^2}{2L_3'} + N_3\frac{(T_4-T_3)^2}{4L_3'} - M_3\frac{(l_4T_{A_3} - l_3T_{A_3} - l_4T_3 + l_3T_4)(T_4-T_3)}{3L_3'}\right\} \\ &- S_3\left\{L_3\frac{(l_4T_3 - l_3T_4)}{2} + M_3\frac{T_4-T_3}{3}\right\} \\ &+ \frac{\rho_3 c_3}{2L_3'}\left\{L_3(l_4T_3 - l_3T_4)(l_4\dot{T}_3 - l_3\dot{T}_4) + N_3\frac{T_4-T_3}{2}(\dot{T}_4-\dot{T}_3) \right. \\ &\left. + \frac{2}{3}M_3\{L_3(\dot{T}_3T_4 + \dot{T}_4T_3) - 2(l_3\dot{T}_3T_4 - l_4\dot{T}_4T_3)\}\right\} \\ &+ hL_3\frac{(T_4-T_a)^2}{2} + LET_4 \end{aligned} \tag{12}$$

On the basis of these variational integral, the assembled integral given in Eq. (7) can be optimized by differentiating it with respect to the nodal temperatures T_1, T_2, T_3, and T_4. Hence equating the resulting equations to zero we have

$$\begin{aligned} A_1T_1 + A_2T_2 + A_3\dot{T}_1 + A_4\dot{T}_2 &= Q_1 \\ B_1T_1 + B_2T_2 + B_3T_3 + B_4\dot{T}_1 + B_5\dot{T}_2 + B_6\dot{T}_3 &= Q_2 \\ C_1T_2 + C_2T_3 + C_3T_4 + C_4\dot{T}_2 + C_5\dot{T}_3 + C_6\dot{T}_4 &= Q_3 \\ D_1T_3 + D_2T_4 + D_3\dot{T}_3 + D_4\dot{T}_4 &= Q_4 \end{aligned}$$

where dots($\dot{}$) denote the derivative of T with respect to t and the value of the coefficients are given in the Appendix. In matrix representation we have

$$AT + B\frac{dT}{dt} = C$$

where

$$A = \begin{bmatrix} A_1 & A_2 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ 0 & C_1 & C_2 & C_3 \\ 0 & 0 & D_1 & D_2 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

$$B = \begin{bmatrix} A_3 & A_4 & 0 & 0 \\ B_4 & B_5 & B_6 & 0 \\ 0 & C_4 & C_5 & C_6 \\ 0 & 0 & D_3 & D_4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}$$

Taking Laplace transform on both sides we get

$$\begin{aligned} AL(T) + BsL(T) &= \frac{C}{s} + BT^0 \\ \Rightarrow (A+Bs)L(T) &= \frac{C}{s} + BT^0 \end{aligned} \tag{13}$$

Where $T^0 = [T(l_0,0), T(l_1,0), T(l_2,0), T(l_3,0)]'$. From Eq. (13), value of $T(r,t)$ can be obtained by taking inverse Laplace transform on $L(T)$.

Table 1: Size of Sub-domains

S. No	Region	Thickness of layer(in cms)
1	Brain	8.4
2	CSF	0.2
3	Skull	0.4
4	Scalp	0.5

Table 2: Physiological and Numerical value of parameters

Parameter	Region	Value
Thermal	Brain(k_0)	$5.03\times10^{-3}W\ cm^{-10}C^{-1}$
conductivity	CSF(k_1)	$5.82\times10^{-3}W\ cm^{-10}C^{-1}$
	Skull(k_2)	$1.16\times10^{-2}W\ cm^{-10}C^{-1}$
	Scalp(k_3)	$3.40\times10^{-3}W\ cm^{-10}C^{-1}$
Mass density:		
	Blood(ρ_b)	$1.05\ g\ cm^{-3}$
	Brain(ρ_0)	$1.05\ g\ cm^{-3}$
	CSF(ρ_1)	$1.00\ g\ cm^{-3}$
	Skull(ρ_2)	$1.50\ g\ cm^{-3}$
	Scalp(ρ_3)	$1.03\ g\ cm^{-3}$
Specific heat:		
	Blood(c_b)	$3.8\ J\ g^{-10}C^{-1}$
	Brain(c_0)	$3.7\ J\ g^{-10}C^{-1}$
	CSF(c_1)	$2.3\ J\ g^{-10}C^{-1}$
	Skull(c_2)	$2.3\ J\ g^{-10}C^{-1}$
	Scalp(c_3)	$4\ J\ g^{-10}C^{-1}$
Flow rate:		
	Brain(F_0)	$2.67\times10^{-3}\ ml\ g^{-1}sec^{-1}$
	Scalp(F_3)	$7.5\times10^{-4}\ ml\ g^{-1}sec^{-1}$
Metabolic heat generation:		
	Brain(S_1)	$1.10\times10^{-2}\ W\ cm^{-3}$
	Scalp(S_3)	$1.72\times10^{-4}\ W\ cm^{-3}$
Mass tran. coeff.(h)	Scalp-Env.	$4.00\times10^{-4}\ W\ cm^{-20}C^{-1}$
Latent heat(L)	-	$2.42\ J\ g^{-1}$
Ambient Temp.(T_{A_3})		37^0C
Evap. rate:		
E_1	-	$0\ ml\ cm^{-2}sec^{-1}$
E_2	-	$6.94\ ml\ cm^{-2}sec^{-1}$
E_3	-	$1.389\ ml\ cm^{-2}sec^{-1}$

4 Numerical Computation

In this paper various parameters have been used to formulate the model. The numerical and physiological values of these parameters are given in Tables 1 and 2. These values have been taken from various research papers: Khanday et al. [11,21,24,26,27], Zhu and Diao [22], Dexter and Hinderman [9], Nelson and Nunneley [10], Diller and Hayes [28] and Zhou et al. [29]. The role of metabolic heat generation and perfusion terms in CSF and skull is less significant Khanday and Saxena [11], and Mir Aijaz and Khanday [21]. Therefore these two terms were neglected in Pennes bioheat equation for CSF and Skull. Further the temperature dependence of optical properties in the domain were incorporated due to there role in temperature distribution and development of thermal injury as discussed by Mir Aijaz and Khanday. For scalp the optical properties like absorption and scattering coefficients were taken positive for the temperature ranging in $20-45^0C$ and their behaviour was seen to show negative behaviour at other temperatures. The temperature distribution and temperature gradient have been calculated in the four regions of human head at extreme temperatures viz. $50^0C, 45^0C, 25^0C, 5^0C, -5^0C$ and the results obtained were interpreted by graphs 1-2. The two graphs were drawn at various evaporation rates and results were calculated for $t = 15$ min, 30 min, 45 min and 60 min. The numerical computation was carried out by MATLAB software, however the model was solved manually.

5 Discussion and Conclusion

The temperature distribution in human head is determined by the intricate biological and physical processes. Therefore for the prediction of temperature profiles in the overlying layers of human brain it becomes optimistic to make some appropriate assumptions to the various parameters viz. thermal conductivity (k), tissue density (ρ), metabolic heat generation S and perfusion etc. The recent study on the temperature distribution in human head at cold temperature distribution in human head at cold temperature was carried out by Khanday and Saxena [11]. They studied the unsteady state of Pennes' bioheat equation on human brain and overlying layers to approximate the variation of temperature at cold environmental conditions. In the present study, the radial form of Pennes' bioheat equation has been used which is more suitable for the domain having spherical shape like human head. To find the temperature distribution in human head at both hot as well as cold temperatures, the results are shown graphically in the Figs. 1 - 2. The study reveals that there is a small variation in the brain temperature with respect to the large variation in outer temperature. The brain is spherical organ with high metabolic rate. The over lying layers insulate it against the direct effects of the extreme environmental temperatures. It is blessing that the brain is insulted from environment by the CSF, skull, scalp and hair which prevents direct link of temperature between outer surface of head and brain temperature. Moreover the spherical shape of the human head makes low surface to volume ratio which minimizes the superficial heat exchange. Due to change in external temperature, the fall or rise of brain temperature is minimized by the individual by engaging himself in exercises during cold and sweating during hot environment. The extreme temperatures influence the functioning of cells and hence the whole system of body. Therefore, changes in temperature can have significant consequences for the behaviour of individual cells and the body as a whole. In this paper the authors tried to develop

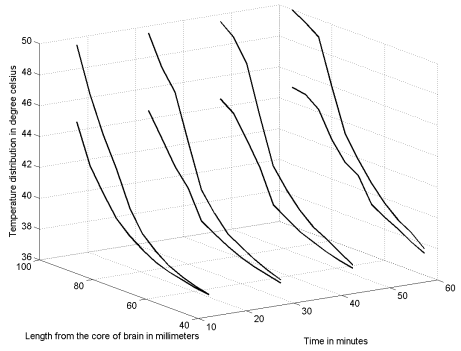

Fig. 1: Temperature distribution in human brain and its over laying layers at ambient temperatures $T_a = 45^0C$ and 50^0C

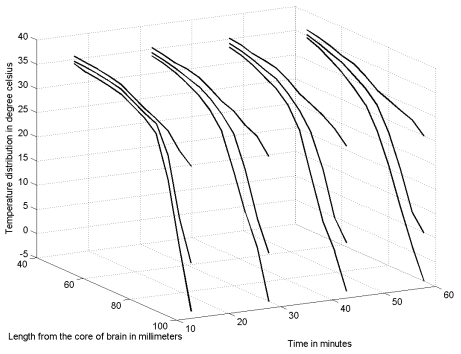

Fig. 2: Temperature distribution in human brain and its over laying layers at ambient temperatures $T_a = 25^0C$, 5^0C and -5^0C

a model to display the temperature profiles in human head and different medical diagnosis can be used to know the harmful impacts of extreme temperatures.

Further it becomes imperative to discuss that we have taken the Fourier model which is based on Classical Fourier's Law. We have not explored the possibility of non-Fourier effect on temperature distribution. Therefore this paper has scope to be generalised by taking both the parabolic nature (Fourier) and hyperbolic nature (non-Fourier) and exploring the optimum results. This model will be more realistic as in every thermodynamical transition, equilibrium state needs time to establish as discussed by Herwig and Peters [32].

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Appendix

$$L_i = l_i + l_{i+1}; \qquad L_i' = l_i - l_{i+1}; \qquad \alpha_i = \rho_i \rho_b c_b F_i$$

$$\gamma_i = \frac{\rho_i c_i}{2L_1'}; \quad R_i = \frac{k_i L_i}{2L_i'}; \quad M_i = l_i^2 + l_i l_{i+1} + l_{i+1}^2$$

$$N_i = l_i^3 + l_i^2 l_{i+1} + l_i l_{i+1}^2 + l_{i+1}^3; \quad \text{where} \quad i = 0,1,2,3$$

$$A_1 = \frac{k_0}{2}\frac{\alpha_1}{4} l_1^2 + R_1; \quad A_3 = \gamma_1\{\frac{L_1' l_i^2}{2} + l_2^2 L_1 + \frac{N_1}{2} + \frac{4l_2 M_1}{3}\}$$

$$A_2 = B_1 = -R_1; \quad A_4 = B_4 = \gamma_1\{-l_1 l_2 L_1 - \frac{N_2}{2} + \frac{2M_1 L_1}{3}\}$$

$$Q_1 = \frac{k_0 T_0}{2} - \frac{\alpha_1 l_1^2}{3}(T_{A_0} - T_0)^2 - \frac{\alpha_1 l_1^2 T_o}{4} + \frac{S_0 l_1^2}{3}$$

$$B_2 = R_2 + R_3; \qquad B_3 = C_1 = -R_3$$

$$B_5 = \gamma_1\{-l_1^2 L_1 + \frac{N_1}{2} - \frac{4l_1 M_1}{3}\} + \gamma_2\{l_3^2 L_2 + \frac{N_2}{2} - \frac{2M_1 l_3}{3}\}$$

$$B_6 = C_4 = \gamma_2\{-l_2 l_3 L_2 - \frac{N_2}{2} + \frac{M_2(l_2 + l_3)}{3}\}; \qquad Q_2 = 0$$

$$C_2 = R_2 + R_3 - \frac{\alpha_3}{L_3'}\{\frac{l_4^2 L_3}{2} + \frac{N_3}{4} + \frac{2l_4 M_3}{3}\}$$

$$C_3 = D_1 = R_3 + \frac{\alpha_3}{L_3'}\{\frac{l_3 l_4 L_3}{2} + \frac{N_3}{4} - \frac{l_3 M_3}{3} - \frac{l_4 M_3}{3}\}$$

$$C_4 = \gamma_2\{-l_2 l_3 L_2 - \frac{N_2}{2} + \frac{M_2 L_2}{2}\}$$

$$C_5 = \gamma_2\{l_2^2 L_2 + \frac{N_2}{2} - \frac{2l_2 M_2}{3}\} + \gamma_3\{l_4^2 L_3 + \frac{N_3}{2} - \frac{4l_4 M_4}{3}\}$$

$$C_6 = D_3 = \gamma_3\{-l_3 l_4 L_3 - \frac{N_3}{2} + \frac{4l_3 M_3}{3}\}$$

$$Q_3 = S_3\left(\frac{l_4 L_3}{2} - \frac{M_3}{3}\right) + \frac{\alpha_3 T_{A_3}}{2}\left(l_4 L_3 + \frac{2M_3}{3}\right)$$

$$D_2 = R_3 - \frac{\alpha_3}{2L_3'}\left(l_3^2 l_4 + \frac{N_3}{2} - \frac{4l_3 M_3}{3}\right) + hL_3$$

$$D_4 = \gamma_3\left(l_3^2 L_3 + \frac{N_3}{2} - \frac{4l_3 M_3}{3}\right)$$

$$Q_4 = S_3\left(-\frac{l_3 L_3}{2} + \frac{M_3}{3}\right) + \frac{\alpha_3 T_{A_3}}{2}\left(l_3 L_3 - \frac{2M_3}{3}\right) - hL_3 T_a - LE$$

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