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New method to evaluate divergent series via the Wigner function

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Abstract: It is shown how a physical function, namely the Wigner function, that in principle may be measured, can be used to evaluate divergent series.

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1 Divergent series

The process of measuring the field wave function is usually realized through the reconstruction of quasiprobability distribution functions (QDF) [1–5]. One of the most prominent QDF is the Wigner function [6]. They allow us, among other things, the visualization of nonclassical states of light (see for instance [7]). Besides their applications in quantum mechanics, they may also be used in classical optics [8]

Here we will give a physical way to evaluate a class of divergent series by using the Wigner function for the position operator (and its powers). Abel quoted on these kind of series:

”Divergent series are on the whole devil's work, and it is a shame that one dares to found any proof on them. One can get out of them what one wants if one uses them, and it is they which have made so much unhappiness and so many paradoxes” [9].

We can evaluate the alternating series $S = 1 - 1 + 1 - 1 + 1 + ...$ by considering the alternating geometric series $1 - x + x^2 - x^3 + x^4 - ....$ provided $|x| < 1$, we know that it converges to $1/(1 + x)$. By allowing x to get close to 1, the series will tend to a value close to 0.5. Therefore, we can say that the (divergent) sum S evaluates $1/2$ in the Abel sense [9].

Another way of defining convergence is by using Césaro's sums. The sum S diverges not because the partial sums grow uncontrollably, but rather because the partial sums oscillate. If we could find a way of averaging the sums in order to smooth them out, maybe this series will converge. Using Abel and Césaro's limits it has been shown also that $1 - 2 + 3 - 4 + 5 + \ldots$ has discrete sum $-1/4$ [10], values that agree with evaluations of the Riemann zeta function [10].

In this contribution we propose a new method to evaluate divergent series by using a physical function, namely the Wigner function [6], which may be written in the form, [11]

$$W(\alpha) = \frac{1}{\pi} Tr\{(-1)^{\hat{n}} D^{\dagger}(\alpha) \rho D(\alpha)\} \tag{1.1}$$

where ρ is the system's density matrix, the number operator $\hat{n} = a^{\dagger}a$ with a the annihilation operator and $D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^* a)$ the Glauber displacement operator, where $\alpha = (q + ip)/\sqrt{2}$. Another common form for the Wigner function is

$$W_A(\alpha) = \frac{1}{2\pi} \int du e^{iup} \langle q + u/2 | A | q - u/2 \rangle \tag{1.2}$$

where we define it for an arbitrary operator A. If we consider a function of the position operator, we find

$$W_{f(\hat{q})}(\alpha) = \frac{1}{2\pi} \int du e^{iup} f(q - u/2) \delta(u) = \frac{f(q)}{2\pi}. \tag{1.3}$$

It is worth to note that the Wigner function is in general a physical function and that has been measured in experiments. In particular by using the expansion in terms of Laguerre polynomials for Hamiltonians in the interaction of light and trapped ions [12], Leibfried *et al.* measured the Wigner function for the first excited state of the vibrational motion of an ion [13].

For an arbitrary operator, for instance, position to an arbitrary power $\hat{q}^k$ the Wigner function may also be given in a series representation [14]

$$\begin{aligned} \frac{q^k}{2\pi} &= \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle n | D^{\dagger}(\alpha) \hat{q}^k D(\alpha) | n \rangle \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle n | (\hat{q} + q)^k | n \rangle. \end{aligned} \tag{1.4}$$

where $|n\rangle$ is a number state. Note that equation (1.4) has already the alternating form of the series considered above.

By doing $k = 0$, we obtain

$$\frac{1}{2} = \sum_{n=0}^{\infty} (-1)^n, \tag{1.5}$$

doing $k = 2$ gives [15]

$$\frac{q^2}{2} = \sum_{n=0}^{\infty}(-1)^n(q^2 + \langle n|\hat{q}^2|n\rangle), \tag{1.6}$$

which by using that $\langle n|\hat{q}^2|n\rangle = n + 1/2$ allows us to evaluate the sum

$$\sum_{n=0}^{\infty}(-1)^n n = -\frac{1}{4}, \tag{1.7}$$

where we have used (1.5) to evaluate (1.7). Eq. (1.6) then shows that there is a recursion relation between the higher order sums in terms of the lower order ones. We can find a general expression that will have this recursion in it. In order to do this we write $\hat{q}$ in terms of annihilation and creation operators,

$$\hat{q} = \frac{a + a^\dagger}{\sqrt{2}}, \qquad a|n\rangle = \sqrt{n}|n-1\rangle,$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{1.8}$$

and insert it in (1.4), this gives

$$\frac{q^k}{2} = \sum_{n=0}^{\infty}(-1)^n \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} \frac{q^{k-s}}{2^{s/2}} \langle n|(a + a^\dagger)^s|n\rangle. \tag{1.9}$$

by equating the coefficients of powers of q at right and left of the equal sign, we finally obtain ($s > 0$)

$$0 = \sum_{n=0}^{\infty}(-1)^n \langle n|(a + a^\dagger)^{2s}|n\rangle. \tag{1.10}$$

This equation is the main result of the manuscript, as it will allow the evaluation of the divergent series considered earlier.

Because of the average with the numbers states, we can remove all terms that do not contain an equal number of $a^\dagger$'s and a's, as they are the only terms that will contribute to the sum of diagonal matrix elements. Therefore by considering those elements we may neglect all other terms of the sum. We have [16]

$$\left(a^\dagger + a\right)^m \Rightarrow \begin{cases} \begin{pmatrix} m \\ m/2 \end{pmatrix} : (a^\dagger a)^{m/2} :_W, & m \text{ even} \\ \\ 0 & m \text{ odd} \end{cases} \tag{1.11}$$

where $: (a^\dagger a)^s :_W$ denotes the Weyl (symmetric) ordering of the operator $\hat{n}^s$. It may be transformed into normal ordering using [16]

$$: (a^\dagger a)^m :_W = \sum_{l=0}^{m} \frac{l!}{2^l} \begin{pmatrix} m \\ l \end{pmatrix}^2 a^{\dagger m-l} a^{m-l}. \tag{1.12}$$

Inserting (1.12) into (1.10) and making use of the expression

$$a^{\dagger m}a^m|n\rangle = \frac{n!}{(n-m)!}|n\rangle, \tag{1.13}$$

we obtain

$$\sum_{n=0}^{\infty}\sum_{l=0}^{s}\frac{(-1)^n}{2^l}\left(\begin{array}{c} s \\ l \end{array}\right)\left(\begin{array}{c} n \\ s-l \end{array}\right) = 0. \tag{1.14}$$

For $s = 1$ we obtain (1.5) and for $s = 2$ we obtain (1.7), while for $s > 2$ we obtain the sums

$$\sum_{n=0}^{\infty}(-1)^n n^{(s-1)} \tag{1.15}$$

in terms of the *lower* sums, i.e. as a recursion formula.

In conclusion, we have shown a physical form, to evaluate some divergent sums, by means of a function that in principle may be measured, namely, the Wigner function.

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