An axiomatization of the Hirsch-index without adopting monotonicity

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Abstract: The Hirsch-index is an index for measuring and comparing the output of researchers. Under the condition of monotonicity, Woeginger [7] provides a characterization of the Hirsch-index by three axioms in 2008. Replacing monotonicity by expansion consistency, we characterize the Hirsch-index by only two of Woeginger’s axioms. Besides, we also introduce an axiom contraction consistency. It is a dual viewpoint of expansion consistency. Based on contraction consistency, an additional characterization of the Hirsch-index is reported.

Keywords: Hirsch-index, monotonicity, consistency.

1. Introduction

In 2005, Jorge Hirsch proposed the Hirsch-index to quantify both the scientific productivity and the scientific impact of a scientist. The Hirsch-index is based on the scientist’s most cited papers and on the number of citations that they have received in other people’s publications. It reflects both the number of publications and the number of citations per publication.

Under the condition of monotonicity, two characterizations of the Hirsch-index may be found in [5, 7]. [7] provides a characterization of the Hirsch-index if indexes are assumed to be integer-valued. When indexes are allowed to be real-valued, [5] offers a characterization of the Hirsch index.1

Monotonicity requires that more citations or papers do not lower the index. Woeginger ([7], p.227) stresses that his axioms should be interpreted within the context of monotonicity. It may be difficult to question monotonicity as an appropriated property of an index, but it is worth providing a characterization of the Hirsch-index without adopting monotonicity. The aim of this note is to do so.

Consistency2 is a crucial property in axiomatic theory. It says that the alternative chosen for each admissible problem should always be “in agreement” with the alternative chosen for each of the “reduced” problems that result when some agents have received their components of the alternative and left, and the situation is reassessed at that point. This fundamental property has been investigated in various classes of problems such as apportionment problems, bankruptcy problems, bargaining problems, cost allocation problems, fair assignment problems, matching problems, resource allocation problems and taxation problems. The reader is referred to [6] for a survey of this literature.

This note establishes an axiomatization of the Hirsch-index by means of a self-consistency property. We introduce a property—expansion consistency in Section 3. Rough speaking, consistency concerns two problems, an original problem and its “reduced” problem in literature. Expansion consistency is slightly different from consistency. It concerns two vectors, an original vector and its “extended” vector. It requires that if the extended vector results from the original vector by adding some “irrelevant” articles, then their indexes should be consistent. Based on expansion consistency, we present a

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1 Based on the Hirsch-index, [4] presents a characterization of the ranking and [1] introduces two alternate indices that can be used to estimate of the impact of Journals published in Arabic Language as well as scientist’ cumulative research contributions.

2 The axiom was originally introduced by [2] under the name of bilateral equilibrium. For discussion of this axiom, see [6].
characterization of the Hirsch-index without adopting monotonicity.

In Section 5 we take into account the dual viewpoint of expansion consistency, contraction consistency. It is also a condition of self-consistency. Contraction consistency concerns two vectors, an original vector and its “reduced” vector. It requires that if the reduced vector results from the original vector by deleting some “irrelevant” articles, then their indexes should be consistent. As a by-product, we provide a characterization of the Hirsch-index based on contraction consistency. In Section 6 we introduce an axiom weak contraction consistency. It is logically weaker than contraction consistency. Weak contraction consistency instead of contraction consistency, we obtain the coincident result as that in Section 5. Finally, an axiom quality-quantity rationality is introduced in Section 7. Based on this property, we present three characterizations of the Hirsch-index without adopting monotonicity.

2. Preliminaries

We follow the notation and terminology of [7]. A researcher with \( n \geq 0 \) publications is formally described by a vector \( x = (x_1, x_2, \ldots, x_n) \) with non-negative integer components \( x_1 \geq x_2 \geq \cdots \geq x_n \), the \( k \)th component \( x_k \) of this vector states the total number of citations to this researcher’s \( k \)th most important publication. If \( n = 0 \), the researcher has no publications and the vector is empty. Let \( X \) denote the set of all such vectors. We say that a vector \( x = (x_1, x_2, \ldots, x_n) \) is dominated by a vector \( y = (y_1, y_2, \ldots, y_m) \), if \( n \leq m \) holds and if \( x_k \leq y_k \) for \( 1 \leq k \leq n \); we will write \( x \preceq y \) to denote this situation.

Definition 1. A scientific impact index (or index, for short) is a function \( f \) from the set \( X \) into the set \( \mathbb{N} \) of non-negative integers that satisfies the following condition:

\[-\text{If } x = (0, 0, \cdots, 0) \text{ or if } x \text{ is the empty vector, then } f(x) = 0.\]

A Hirsch-index of at least \( k \) means that there are \( k \) distinct publications that all have at least \( k \) citations. The following definition provides a formal mathematical description of the Hirsch-index.

Definition 2. The Hirsch-index (or \( h \)-index) is the scientific impact index \( h: X \rightarrow \mathbb{N} \) that assigns to vector \( x = (x_1, x_2, \cdots, x_n) \) the value

\[ h(x) = \max \{ k : x_k \geq k \}. \]

Woeginger’s theorem is based on the following four properties Monotonicity, A1, B and D. The first property Monotonicity requires that more citations or articles do not lower the index. [7] postulates Monotonicity in the definition of an index.

Monotonicity (MON): If \( x \preceq y \), then \( f(x) \leq f(y) \).

The property A1 concerns the addition of a single publication to a publication list. It requires that if the publication is only average with respect to the current index, then it should not raise the index.

A1: If the \((n+1)\)-dimensional vector \( y \) results from the \( n \)-dimensional vector \( x \) by adding a new article with \( f(x) \) citations, then \( f(y) \leq f(x) \).

The property B concerns the addition of new citations to old publications. It requires that minor changes in the citation record should not lead to major changes in the index.

B: If the \( n \)-dimensional vector \( y \) results from the \( n \)-dimensional vector \( x \) by increasing the number of citations of a single article, then \( f(y) \leq f(x) + 1 \).

The final property D concerns the case where both the number of publication and the number of citations go up. It requires that adding a strong new publication and consistently improving the citations to one’s old publications should also raise the index.

D: If the \((n+1)\)-dimensional vector \( y \) results from the \( n \)-dimensional vector \( x \) by first adding an article with \( f(x) \) citations and afterwards increasing the number of citations of every article by at least one, then \( f(y) > f(x) \).


Theorem 1. (Theorem 4.1, [7]) A scientific impact index \( f : X \rightarrow \mathbb{N} \) satisfies the four properties MON, A1, B and D, if and only if it is the \( h \)-index.

To conclude this section, we illustrate that MON is needed in Woeginger’s characterization.

Proposition 17. The \( h \)-index is not the only one to satisfy A1, B, and D.

Proof. The proof is by way of an example of an index that satisfies the three axioms but differs from the \( h \)-index: Let \( \sigma^1 \) be the scientific impact index defined by

\[ \sigma^1(x) = \begin{cases} 0 , & \text{if } x = (1, 0, 0, \cdots, 0) \\ h(x) , & \text{otherwise}. \end{cases} \]

Clearly \( \sigma^1 \) differs from the \( h \)-index. It is immediate to verify that \( \sigma^1 \) as constructed satisfies A1, B and D, but it violates MON.

\[^3\] See Definition 2.1 in [7].

\[^4\] \( x = (1, 0, 0, \cdots, 0) \) means that \( n \geq 2 \), \( x_1 = 1 \) and \( x_i = 0 \forall 2 \leq i \leq n. \)
3. Main Result

In this section we present a characterization for the Hirsch-index by the following property.

**Expansion Consistency (ECON):** If the \((n + k)\)-dimensional extended vector \(y\) results from the \(n\)-dimensional vector \(x\) by adding \(k\) articles with the number of citations of every article being at most \(f(x)\), then \(f(y) = f(x)\), where \(k \geq 1\).

ECON is a condition of self-consistency. It concerns the addition of publications to a publication list. It requires that if the number of citations of every added publication is not above the current index, then the index should not change. That is, if the \((n + k)\)-dimensional extended vector \(y\) results from the \(n\)-dimensional vector \(x\) by adding \(k\) “irrelevant” articles, then their indexes should be consistent. In fact, if an index satisfies ECON for \(k = 1\), by a repeated application of ECON for \(k = 1\) would yield ECON for all \(k > 1\).

Clearly, if an index satisfies ECON then it satisfies A1. The converse statement is not true. The counterexample is as follows:

- The minimum-index is the scientific impact index \(f_{\text{min}} : X \to \mathbb{N}\) that assigns to vector \(x = (x_1, x_2, \cdots, x_n)\) the value \(f_{\text{min}}(x) = x_n\). Then the minimum-index satisfies A1, but it violates ECON.

**Remark 1** It is easy to see that A1 and MON together imply ECON. That is, if an index satisfies A1 and MON then it also satisfies ECON.

The main result is as follows:

**Theorem 2.** A scientific impact index \(f : X \to \mathbb{N}\) satisfies B, D and ECON if and only if it is the \(h\)-index.

**Proof.** Clearly the \(h\)-index satisfies B, D, and ECON. Suppose \(f\) is an index satisfying B, D, and ECON. Our argument proceeds in four steps. The Steps (1) and (2) are the same as that in [7]. For completeness, we copy them.

**Step (1):** We argue that any vector \(x\) with at most \(k\) non-zero components has \(f(x) \leq k\). This follows by induction, starting from Definition 1 and then repeatedly applying property B.

**Step (2):** We consider for every \(k \geq 0\) the vector \(u^{[k]}\) that consists of exactly \(k\) components of value exactly \(k\). We prove by induction on \(k \geq 0\) that \(f(u^{[k]}) = k\). The statement for \(k = 0\) follows from Definition 1. In the inductive step, we derive from the inductive assumption and from property D that \(f(u^{[k+1]}) > f(u^{[k]}) = k\), whereas the statement in Step (1) yields \(f(u^{[k+1]}) \leq k + 1\). This yields the desired \(f(u^{[k+1]}) = k + 1\).

**Step (3):** Let \(x\) be a \(k\)-dimensional vector every component of which is at least \(k\). It is easy to see that the \(k\)-dimensional vector \(x\) results from the \((k - 1)\)-dimensional vector \(u^{[k-1]}\) as in the statement of axiom D. Hence by D, \(f(u^{[k-1]}) < f(x)\). Combining this with Steps (1) and (2), \(k - 1 = f(u^{[k-1]}) < f(x) \leq k\). Hence \(f(x) = k\).

**Step (4):** We establish \(f(x) = h(x) \forall x\). Let \(x\) be an \(n\)-dimensional vector in \(X\), and let \(k = h(x)\). Let \(y = (x_1, x_2, \cdots, x_k)\) denote the vector that consists of the first \(k\) components of \(x\). Since these components all are at least \(k\), by Step (3), \(f(y) = k\). Since vector \(x\) results from vector \(y\) by adding components of values at most \(k\), by ECON, \(f(x) = f(y) = k\). Therefore \(f(x) = h(x)\).

Finally, the independence of properties listed in Theorem 2 can be established by adopting the following indexes, the maximum-index, the zero-index and the \(w\)-index in [7].

- The maximum-index is the scientific impact index \(f_{\text{max}} : X \to \mathbb{N}\) that assigns to vector \(x = (x_1, x_2, \cdots, x_n)\) the value \(f_{\text{max}}(x) = x_1\). Then the maximum-index satisfies D and ECON, but it violates B.
- The zero-index assigns to every vector \(x\) the value 0. Then the zero-index satisfies B and ECON, but it violates D.
- The \(w\)-index is the scientific impact index \(w : X \to \mathbb{N}\) that assigns to vector \(x = (x_1, x_2, \cdots, x_n)\) the value \(w(x) = \max\{k : x_m \geq k - m + 1 \forall m \leq k\}\). Then the \(w\)-index satisfies B and D, but it violates ECON.

4. Comparison

In this section we show that our result (Theorem 2) directly implies Woeginger’s result (Theorem 1). We firstly investigate the logical relations between A1 and ECON. The logical implications are summarized in the following:

1. \(\text{ECON} \Rightarrow \text{A1}; \text{ECON} \not\Rightarrow \text{A1};\)
   Clearly, ECON implies A1. The converse statement is not true. The counterexample is as follows:
   - The minimum-index \(f_{\text{min}}\) satisfies A1, but it violates ECON.

2. \(\text{ECON} \Leftrightarrow \text{A1\&MON}; \text{ECON} \not\Leftrightarrow \text{A1\&MON};\)
   In Remark 1 we see that A1 and MON together imply ECON. The converse statement is not true. The counterexample is as follows:
   - Let \(\sigma^2\) be the scientific impact index defined by \(\forall x\)
     \[\sigma^2(x) = \begin{cases} 1, & \text{if } x_1 = 1 \\ 0, & \text{otherwise}. \end{cases}\]
   Then \(\sigma^2\) satisfies ECON, but it violates MON.
By the statement of Point 2, ECON ⇐ A1&MON, we conclude that Theorem 2 (our result) directly implies Theorem 1 (Woeginger’s result).

5. Contraction Consistency

In this section we take into account the dual viewpoint of ECON, Contraction Consistency. By investigating the logical relations among A1, ECON and Contraction Consistency, we derive another characterization of the h-index by four properties, MON, Contraction Consistency, B and D.

Contraction Consistency (CCON): If the n-dimensional reduced vector x results from the (n + k)-dimensional vector y by deleting k articles with the number of citations of every article being at most (f(y) − 1), then f(x) = f(y), where k ≥ 1.

CCON is a condition of self-consistency. It concerns the deletion of publications from a publication list. It requires that if the number of citations of every deleted publication is under the current index, then the index should not change. That is, if the n-dimensional reduced vector x results from the (n + k)-dimensional vector y by deleting k “irrelevant” articles, then their indexes should be consistent.5 In fact, if an index satisfies CCON for k = 1, by a repeated application of CCON for k = 1 would yield CCON for all k > 1.

The following proposition states that CCON is a more restrictive variant of A1.

Proposition 2 If an index f satisfies CCON then it satisfies A1.

Proof. Let f be an index satisfying CCON. Consider two vectors x and y as in the statement of A1, and suppose for the sake of contradiction that f(y) > f(x) ≥ 0. Two cases can be distinguished:

Case (1): f(x) > x1
Let y′ be the vector that results from y by removing all articles with value strictly less than f(y). Since f(x) > x1 and f(y) ≥ f(x), these imply y′ = ∅. Axiom CCON yields f(y) = f(∅) = 0. This is a contradiction.

Case (2): f(x) ≤ x1
Let y′ be the vector that results from y by removing one article of value f(x). This means y′ = x. Axiom CCON yields f(y) = f(y′) = f(x). This is a contradiction.

The logical implications are summarized in the following:

1. About A1 and CCON:
   (a) CCON ⇒ A1; CCON ≠ A1:
   In Proposition 2 we see that CCON implies A1. The converse statement is not true. The counterexample is as follows:

   5 Clearly the h-index satisfies CCON.

   −Let σ3 be the scientific impact index defined by ∀ x
   σ3(x) = \{0 \text{, if } n = 1 \text{ or } x_i = 0 \forall 2 \leq i \leq n, \text{ otherwise.}

   Then σ3 satisfies A1, but it violates CCON.

   (b) CCON ≠ A1&MON; CCON ≠ A1&MON:
   If an index f satisfies A1 and MON then it may not satisfy CCON. Conversely, if an index f satisfies CCON then it may not satisfy A1 and MON. The counterexamples are as follows:
   ¬σ3 satisfies A1 and MON, but it violates CCON.
   ¬σ1 satisfies CCON, but it violates MON.6

2. About CCON and ECON:
   (a) CCON ≠ ECON; CCON ≠ ECON:
   If an index f satisfies CCON then it may not satisfy ECON. Conversely, if an index f satisfies ECON then it may not satisfy CCON. The counterexamples are as follows:
   ¬fmin satisfies CCON, but it violates ECON.
   ¬σ3 satisfies ECON, but it violates CCON.

   (b) CCON& MON ⇒ ECON; ECON& MON ≠ CCON:
   If an index f satisfies CCON and MON then it satisfies ECON. This statement is true by combining “ECON ⊆ A1&MON” (the Point 2 in Section 4) with “CCON ⇒ A1” (1-(a)). On the other hand, if an index f satisfies ECON and MON then it may not satisfy CCON. The counterexample is as follows:
   ¬σ3 satisfies ECON and MON, but it violates CCON.

By 2-(b), CCON& MON ⇒ ECON, Theorem 2 directly implies the following result:7

Theorem 3 A scientific impact index f : X → N satisfies the four properties MON, CCON, B and D, if and only if it is the h-index.

6 We can conclude that the h-index is not the only one to satisfy CCON, B, and D.

7 Theorem 3 could be also derived by applying Theorem 1 and Proposition 2.
denote \( m_{y,f} = \max P_{y,f} \). The reduced vector \( y_f \) with respect to \( f \) is defined by
\[
y_f = \begin{cases} 
(y_1, y_2, \cdots, y_{m_{y,f}}), & \text{if } P_{y,f} \neq \emptyset \\
\text{empty vector}, & \text{otherwise}.
\end{cases}
\]
Note that \( P_{y,f} \) is the set of all “relevant” articles in \( y \) with respect to \( f \). The reduced vector \( y_f \) results from the \( n \)-dimensional vector \( y \) by deleting all “irrelevant” articles. Hence if all articles are irrelevant in \( y \), then the reduced vector \( y_f \) should be the empty vector.

**Weak Contraction Consistency (WCCON):** For each \( n \)-dimensional vector \( y \in X \), \( f(y) = f(y_f) \).

WCCON requires that if the reduced vector \( y_f \) results from the \( n \)-dimensional vector \( y \) by deleting all irrelevant articles, then their indexes should be consistent.

It is easy to see that both CCON and WCCON concern two vectors, an original vector \( y \) and a reduced vector \( x \) of the vector \( y \) by deleting “irrelevant articles”. More precisely, WCCON specifies the reduced vector \( x \) which is reduced by deleting “all irrelevant articles” in \( y \). Hence, if an index \( f \) satisfies CCON then it satisfies WCCON. The converse statement is not true. The counterexample is as follows:

Let \( \sigma^4 \) be the scientific impact index defined by \( \forall x \)
\[
\sigma^4(x) = \begin{cases} 
x_2, & \text{if } n = 2 \\
x_1, & \text{otherwise}.
\end{cases}
\]
Then \( \sigma^4 \) satisfies WCCON, but it violates CCON.

**Remark 2** Under the condition of MON, WCCON is an alternative form of CCON. We have known that if an index satisfies CCON then it also satisfies WCCON. Conversely, by MON, an application of WCCON would yield CCON.

By Remark 2, Theorem 3 directly implies the following result:

**Theorem 4** A scientific impact index \( f : X \rightarrow \mathbb{N} \) satisfies the four properties MON, WCCON, B and D, if and only if it is the \( h \)-index.

### 7. Quality-Quantity Rationality (QQR)

The quality-quantity rationality (QQR) reflects both the number of publications (quantity) and the number of citations (quality). If both the quality and the quantity are greater than and equal to \( k \), then the index should reflect this situation. A such axiom is here formulated as follows.

\textbf{Quality-Quantity Rationality (QQR):} If the \( n \)-dimensional vector \( x \) with \( x_k \geq k \), then \( f(x) \geq k \), where \( k \leq n \).

Based on QQR, we present three characterizations of the Hirsch-index without adopting MON.

**Theorem 5.**

1. A scientific impact index \( f : X \rightarrow \mathbb{N} \) satisfies B, QQR and ECON if and only if it is the \( h \)-index.
2. A scientific impact index \( f : X \rightarrow \mathbb{N} \) satisfies B, QQR and CCON if and only if it is the \( h \)-index.
3. A scientific impact index \( f : X \rightarrow \mathbb{N} \) satisfies B, QQR and WCCON if and only if it is the \( h \)-index.

**Proof.** Clearly the \( h \)-index satisfies all axioms. To verify Statement (1), suppose \( f \) is an index satisfying B, QQR, and ECON. Our argument proceeds in three steps.

**Step (1’):** Step (1’) is the same as Step (1) in [7]. We argue that any vector \( x \) with at most \( k \) non-zero components has \( f(x) \leq k \). We omit it.

**Step (2’):** Let \( x \) be a \( k \)-dimensional vector every component of which is at least \( k \). By applying QQR, \( f(x) \geq k \). Combining Step (1’) with (2’), we derive that \( f(x) = k \).

**Step (3’):** Step (3’) is the same as Step (4) in [7]. We omit it. This completes the proof of Statement (1).

Since WCCON is logically weaker than CCON, it remains to verify Statement (3). Suppose \( f \) is an index satisfying B, QQR, and WCCON. By Step (2’), we have known that if \( x \) is a \( k \)-dimensional vector every component of which is at least \( k \), then \( f(x) = k \). Next, we establish \( f(x) = h(x) \) \( \forall x \).

Let \( x \) be an \( n \)-dimensional vector in \( X \), and let \( k = h(x) \). By QQR, \( f(x) \geq k \). If \( f(x) = k \), we are done. Suppose \( f(x) > k \). Two cases can be distinguished:

**Case (1):** \( f(x) > x_1 \) i
If \( f(x) > x_1 \), then \( x_f \) is the empty vector. By WCCON, \( f(x) = f(x_f) = 0 \). The desired contradiction has been obtained.

**Case (2):** \( f(x) \leq x_1 \)
Let \( y = (x_1, x_2, \cdots, x_1) \) denote the vector that consists of the first \( i \) components of \( x \), where \( 1 \leq i \leq k \). For each \( y_i \), since these components all are at least \( i \), by Step (2’), \( f(y_i') = i \) for \( 1 \leq i \leq k \). Besides, since \( x_1 \geq f(x) > k \), it is easy to see that there exists \( i \) such that \( y_i = x_f \).

Hence, by WCCON, \( f(x) = f(y_i') = i \). Thus, \( k < f(x) = f(y_i') = i \leq k \). The desired contradiction has been obtained. Therefore \( f(x) = h(x) \).

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