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An Extended Theoretical and Numerical Study of two-Point Boundary Value Problems Using Collocation Method with Physical Applications

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Abstract: In this paper, we present a collocation method based on trigonometric quintic B-spline function is studied and applied for the numerical solutions of two-point boundary value problems arising in physical applications. The convergence of the trigonometric quintic B-spline collocation method is proved using diagonally dominant matrices. The numerical applications appear that our method is highly efficient and the maximum absolute error is convenient.
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1 Introduction

We consider the two boundary point value problem, given by

$$\alpha(\tau)Z''(\tau)+\beta(\tau)Z'(\tau)+\gamma(\tau)Z(\tau)=g(\tau,Z),\quad \tau\in[a,b], \tag{1}$$

under the boundary conditions

$$Z(a)=Z_0,\ Z(b)=Z_n, \tag{2}$$

where $\alpha(\tau),\beta(\tau)$ and $\gamma(\tau)$ are coefficients and $g(\tau,Z)$ is forcing function. Cubic splines are used in [1-4] and B-splines as cubic, quintic and septic are presented in [5-8]. The trigonometric cubic and quintic B-splines are investigated to show the numerical solutions of variety of the ordinary and the partial differential equations [9-15]. Second-order two-point boundary value problems subject to Dirichlet or Neumann boundary conditions are discussed by Polynomial and non-polynomial spline functions [16, 17], Sinc Galerkin and Sinc collocation methods [18] and parametric cubic spline method [19]. Many physical problems as heat transfer, plates deflection and many of other scientific applications are described mathematically by differential equations of second-order with different boundary conditions. This paper is orderly as follows: In section 2, we analyze the proposed collocation method and study the convergence of it. In section 3, we discuss some numerical models. In section 4, we introduce the conclusion of our results.

2 Analysis of Trigonometric Quintic B-spline Collocation Method

Let the domain of the problem is $[a,b]$ which is divided into n equal subintervals $[\tau_j,\ \tau_{j+1}],\ j=0,1,2,\ \dots,\ n-1$ by the nodal points $\tau_j=a+jh$ where $a=\tau_0<\tau_1<\dots<\tau_n=b$ and $h=\frac{b-a}{n}$. Then we use the trigonometric quintic B-spline collocation method to solve two-point value problem (1) numerically, where we find additional nodal points $\tau_{-5},\tau_{-4},\tau_{-3},\tau_{-2},\tau_{-1},\tau_{n+1},\ \tau_{n+2},\tau_{n+3},\tau_{n+4},\tau_{n+5}$ outside the domain of the problem, the trigonometric quintic B-spline function $TQB_j(\tau)$ is defined as follows:

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$$TQB_j(\tau)=\frac{1}{\rho}\begin{cases}\phi^5(\tau_{j-3}), & \tau\in[\tau_{j-3},\tau_{j-2}]\\ \phi^4(\tau_{j-3})\theta(\tau_{j-1})+\phi^3(\tau_{j-3})\theta(\tau_j)\phi(\tau_{j-2})+\phi^2(\tau_{j-3})\theta(\tau_{j+1})\phi^2(\tau_{j-2}) & \\ +\phi(\tau_{j-3})\theta(\tau_{j+2})\phi^3(\tau_{j-2})+\theta(\tau_{j+3})\phi^4(\tau_{j-2}), & \tau\in[\tau_{j-2},\tau_{j-1}]\\ \phi^3(\tau_{j-3})\theta^2(\tau_j)+\phi^2(\tau_{j-3})\theta(\tau_{j+1})\phi(\tau_{j-2})\theta(\tau_j) & \\ +\phi^2(\tau_{j-3})\theta^2(\tau_{j+1})\phi(\tau_{j-1})+\phi(\tau_{j-3})\theta(\tau_{j+2})\phi^2(\tau_{j-2})\theta(\tau_j) & \\ +\phi(\tau_{j-3})\theta(\tau_{j+2})\phi(\tau_{j-2})\theta(\tau_{j+1})\phi(\tau_{j-1})+\phi(\tau_{j-3})\theta^2(\tau_{j+2})\phi^2(\tau_{j-1}) & \\ +\theta(\tau_{j+3})\phi^3(\tau_{j-2})\theta(\tau_j)+\theta(\tau_{j+3})\phi^2(\tau_{j-2})\theta(\tau_{j+1})\phi(\tau_{j-1}) & \\ +\theta(\tau_{j+3})\phi(\tau_{j-2})\theta(\tau_{j+2})\phi^2(\tau_{j-1})+\theta^2(\tau_{j+3})\phi^3(\tau_{j-1}), & \tau\in[\tau_{j-1},\tau_j]\\ \phi^2(\tau_{j-3})\theta^3(\tau_{j+1})+\phi(\tau_{j-3})\theta(\tau_{j+2})\phi(\tau_{j-2})\theta^2(\tau_{j+1}) & \\ +\phi(\tau_{j-3})\theta^2(\tau_{j+2})\phi(\tau_{j-1})\theta(\tau_{j+1})+\phi(\tau_{j-3})\theta^3(\tau_{j+2})\phi(\tau_j) & \\ +\theta(\tau_{j+3})\phi^2(\tau_{j-2})\theta^2(\tau_{j+1})+\theta(\tau_{j+3})\phi(\tau_{j-2})\theta(\tau_{j+2})\phi(\tau_{j-1})\theta(\tau_{j+1}) & \\ +\theta(\tau_{j+3})\phi(\tau_{j-2})\theta^2(\tau_{j+2})\phi(\tau_j)+\theta^2(\tau_{j+3})\phi^2(\tau_{j-1})\theta(\tau_{j+1}) & \\ +\theta^2(\tau_{j+3})\phi(\tau_{j-1})\theta(\tau_{j+2})\phi(\tau_j)+\theta^3(\tau_{j+3})\phi^2(\tau_j), & \tau\in[\tau_j,\tau_{j+1}]\\ \phi(\tau_{j-3})\theta^4(\tau_{j+2})+\theta(\tau_{j+3})\phi(\tau_{j-2})\theta^3(\tau_{j+2})+\theta^2(\tau_{j+3})\phi(\tau_{j-1})\theta^2(\tau_{j+2}) & \\ +\theta^3(\tau_{j+3})\phi(\tau_j)\theta(\tau_{j+2})+\theta^4(\tau_{j+3})\phi(\tau_{j+1}), & \tau\in[\tau_{j+1},\tau_{j+2}]\\ \theta^5(\tau_{j+3}), & \tau\in[\tau_{j+2},\tau_{j+3}]\\ 0, & otherwise\end{cases}$$

$$(i=-2,-1,0,\cdots,N+2), \tag{3}$$

where,

$$\rho=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)\sin\left(\frac{5h}{2}\right),\ \phi(\tau_j)=\sin\left(\frac{\tau-\tau_j}{2}\right),\ \theta(\tau_j)=\sin\left(\frac{\tau_j-\tau}{2}\right).$$

Let $z(\tau)$ be the approximate solution to the exact solution $Z(\tau)$ of the problems (1) and (2) as expansion of the $TQB_j(\tau)$

$$z(\tau)=\sum_{j=-2}^{n+2}\mu_j TQB_j(\tau), \tag{4}$$

where the constants μ_j's are to be determined from the collocation points and the boundary conditions. The values of the trigonometric quintic B-spline functions and its first and second derivatives and at the nodal points are shown in Table 1.

Table 1: Values of $TQB_j(\tau), TQB'_j(\tau)$ and $TQB''_j(\tau)$ at the nodal points.

	τ_{j-3}	τ_{j-2}	τ_{j-1}	τ_j	τ_{j+1}	τ_{j+2}	τ_{j+3}
$TQB_j(\tau)$	0	δ_1	δ_2	δ_3	δ_2	δ_1	0
$TQB'_j(\tau)$	0	$-\delta_4$	$-\delta_5$	0	δ_5	δ_4	0
$TQB''_j(\tau)$	0	δ_6	δ_7	δ_8	δ_7	δ_6	0

where

$$\delta_1=\frac{\sin^5\left(\frac{h}{2}\right)}{\rho},\ \delta_2=\frac{\sin^4\left(\frac{h}{2}\right)\sin(h)(8\cos(h)+5)}{\rho},\ \delta_3=\frac{2\sin^5\left(\frac{h}{2}\right)(6\cos(2h)+16\cos(h)+11)}{\rho},$$

$$\delta_4=\frac{\frac{5}{4}\sin^3\left(\frac{h}{2}\right)\sin(h)}{\rho},\delta_5=\frac{5\sin^4\left(\frac{h}{2}\right)\cos^2\left(\frac{h}{2}\right)(4\cos(h)+1)}{\rho},\delta_6=\frac{\frac{5}{8}\sin^3\left(\frac{h}{2}\right)(5\cos(h)+3)}{\rho}.$$

$$\delta_7=\frac{\frac{5}{4}\sin^3\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)\left(4\cos(2h)+\cos(h)+3\right)}{\rho},\delta_8=\frac{-\frac{5}{4}\sin^3\left(\frac{h}{2}\right)\left(\cos(3h)+6\cos(2h)+10\cos(h)+7\right)}{\rho}.$$

Using Eqs. (3) and (4), the values of z_j and their its first and second derivatives at the nodal points are

$$\left.\begin{aligned}z_j&=\delta_1\,\mu_{j-2}+\delta_2\,\mu_{j-1}+\delta_3\,\mu_j+\delta_2\,\mu_{j+1}+\delta_1\,\mu_{j+2}\\ z'_j&=-\delta_4\,\mu_{j-2}-\delta_5\,\mu_{j-1}+\delta_5\,\mu_{j+1}+\delta_4\,\mu_{j+2}\\ z''_j&=\delta_6\,\mu_{j-2}+\delta_7\,\mu_{j-1}+\delta_8\,\mu_j+\delta_7\,\mu_{j+1}+\delta_6\,\mu_{j+2}\end{aligned}\right\},$$

$$j=0,1,\ldots,n. \qquad (5)$$

Substituting from Eq. (5) in Eqs. (1) and (2) we find,

$$\begin{aligned}&\left(\delta_6\,\alpha(\tau_j)-\delta_4\,\beta(\tau_j)+\delta_1\,\gamma(\tau_j)\right)\mu_{j-2}+\left(\delta_7\,\alpha(\tau_j)-\delta_5\,\beta(\tau_j)+\delta_2\,\gamma(\tau_j)\right)\mu_{j-1}\\&+\left(\delta_8\,\alpha(\tau_j)+\delta_3\,\gamma(\tau_j)\right)\mu_j+\left(\delta_7\,\alpha(\tau_j)+\delta_5\,\beta(\tau_j)+\delta_2\,\gamma(\tau_j)\right)\mu_{j+1}\\&+\left(\delta_6\,\alpha(\tau_j)+\delta_4\,\beta(\tau_j)+\delta_1\,\gamma(\tau_j)\right)\mu_{j-2}=g_j,\ j=0,1,\ldots,n,\end{aligned}$$

(6)

$$\begin{aligned}&\delta_1\,\mu_{-2}+\delta_2\,\mu_{-1}+\delta_3\,\mu_0+\delta_2\,\mu_1+\delta_1\,\mu_2=Z_0\\&\delta_1\,\mu_{n-2}+\delta_2\,\mu_{n-1}+\delta_3\,\mu_n+\delta_2\,\mu_{n+1}+\delta_1\,\mu_{n+2}=Z_n,\end{aligned}$$

(7)

system of Eqs. (6), as follows:

$$\begin{aligned}&-\delta_4\,\mu_{-2}-\delta_5\,\mu_{-1}+\delta_5\,\mu_1+\delta_4\,\mu_2=Z'_0\\&-\delta_4\,\mu_{n-2}-\delta_5\,\mu_{n-1}+\delta_5\,\mu_{n+1}+\delta_4\,\mu_{n+2}=Z'_n.\end{aligned}$$

(8)

Eliminate $\mu_{-2},\mu_{-1},\mu_{n+1}$ and μ_{n+2} in Eqs. (7) and (8), then Eq. (6) can be converted to matrix form:

$$A\mu=g(\mu), \qquad (9)$$

where A is an $(n+1)\times(n+1)$ matrix, μ is an $(n+1)$ dimensional column vector with components μ_j and the right hand side $g(\mu)$ is an $(n+1)$ dimensional column vector.

For appropriately enough small h, the matrix A is diagonally dominant if

$$|\delta_8\alpha+\delta_3\gamma|-\{|\delta_6\alpha-\delta_4\beta+\delta_1\gamma|+|\delta_7\alpha-\delta_5\beta+\delta_2\gamma|+|\delta_7\alpha+\delta_5\beta+\delta_2\gamma|+|\delta_6\alpha+\delta_4\beta+\delta_1\gamma|\}$$

$$=\begin{cases}(-\delta_8\alpha-\delta_3\gamma)-\left\{\begin{array}{l}(\delta_6\alpha-\delta_4\beta+\delta_1\gamma)+(\delta_7\alpha-\delta_5\beta+\delta_2\gamma)+(\delta_7\alpha+\delta_5\beta+\delta_2\gamma)\\+(\delta_6\alpha+\delta_4\beta+\delta_1\gamma)\end{array}\right\}\\[2mm] =\left(\dfrac{40\csc(h)\csc(2h)\sin^6\left(\frac{h}{4}\right)}{\left(2\cos\left(\frac{h}{2}\right)-1\right)\left(-2\cos(h)+2\cos\left(\frac{h}{2}\right)-1\right)}\right)\alpha-\left(\dfrac{\sec(h)\sec\left(\frac{h}{2}\right)\left(\sec\left(\frac{h}{2}\right)-3\right)}{2\left(2\cos\left(\frac{h}{2}\right)-1\right)\left(-2\cos(h)+2\cos\left(\frac{h}{2}\right)-1\right)}\right)\gamma\\[2mm] \approx-\gamma>0, \qquad (\alpha>0),\\[2mm] (\delta_8\alpha+\delta_3\gamma)-\left\{\begin{array}{l}(-\delta_6\alpha+\delta_4\beta-\delta_1\gamma)+(-\delta_7\alpha+\delta_5\beta-\delta_2\gamma)+(-\delta_7\alpha-\delta_5\beta-\delta_2\gamma)\\+(-\delta_6\alpha-\delta_4\beta-\delta_1\gamma)\end{array}\right\}\\[2mm] =\left(\dfrac{40\csc(h)\csc(2h)\sin^6\left(\frac{h}{4}\right)}{\left(2\cos\left(\frac{h}{2}\right)-1\right)\left(2\cos(h)-2\cos\left(\frac{h}{2}\right)+1\right)}\right)\alpha+\left(\dfrac{\sec(h)\sec\left(\frac{h}{2}\right)\left(3-\sec\left(\frac{h}{2}\right)\right)}{2\left(2\cos\left(\frac{h}{2}\right)-1\right)\left(2\cos(h)-2\cos\left(\frac{h}{2}\right)+1\right)}\right)\gamma\\[2mm] \approx\gamma>0, \qquad (\alpha<0),\end{cases}$$

and

$$\alpha(\tau_j)\gamma(\tau_j) < 0, \quad j = 0,1,\ldots,n. \tag{11}$$

Furthermore,

$$\|A^{-1}\|_\infty \le \frac{1}{\min\limits_{0\le j\le n}|\gamma(\tau_j)|} \equiv M. \tag{12}$$

where $\|.\|_\infty$ indicates the maximum norm in $[a,b]$.

Since D is a linear second order differential operator at the nodal points, we write

$$D(\bar{z}(\tau_j)) = g(\tau_j, \bar{z}(\tau_j)) + \varepsilon(\tau_j),\ j = 0,1,\ldots,n, \tag{13}$$

where $\bar{z}(\tau)$ be the trigonometric quintic B-spline of interpolation to the unique solution of the problems (1) and (2), $\varepsilon(\tau)$ mean that error function.

Thus, Eq. (6) becomes

$$A\bar{\mu} = g(\bar{\mu}) + \varepsilon, \tag{14}$$

where A is an $(n+1)\times(n+1)$ matrix, $\bar{\mu}$ is an $(n+1)$ dimensional vector with components $\bar{\mu}_j$

and the right hand side $g(\bar{\mu})$ and ε are an $(n+1)$ dimensional vector.

Let the Lipschitz condition on the forcing function is of the form:

$$|g(\tau, z_1) - g(\tau, z_2)| \le L|z_1 - z_2|, \text{ for all } \tau \in [a,b], \tag{15}$$

where the constant L is independent of τ.

From Eqs. (9) and (14), we obtain

$$Ae = \varepsilon + g(\mu) - g(\bar{\mu}), \tag{16}$$

where $e_j = \mu_j - \bar{\mu}_j = (e_0, e_1, \ldots, e_n)^T$ and applying the Lipschitz condition (15) on the forcing function then

$$|g(\tau_j, z(\tau_j)) - g(\tau_j, \bar{z}(\tau_j))| = L_j|z(\tau_j) - \bar{z}(\tau_j)|$$

$$= \begin{cases} 0, & j = 0,1, \\ L_j\left(\frac{1}{120}e_{j-2} + \frac{13}{60}e_{j-1} + \frac{33}{60}e_j + \frac{13}{60}e_{j+1} + \frac{1}{120}e_{j+2}\right), & 2 \le j \le n-2, \\ 0, & j = n-1, n. \end{cases} \tag{17}$$

Substituting Eq. (17) in Eq. (16), we obtain

$$Ae = \varepsilon + \widetilde{L}Ne, \tag{18}$$

Where $\widetilde{L} = Diag\{L_0, L_1, \ldots, L_n\}$,

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{120} & \frac{13}{60} & \frac{33}{60} & \frac{13}{60} & \frac{1}{120} & 0 & 0 & 0 \\ 0 & \frac{1}{120} & \frac{13}{60} & \frac{33}{60} & \frac{13}{60} & \frac{1}{120} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{120} & \frac{13}{60} & \frac{33}{60} & \frac{13}{60} & \frac{1}{120} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So that $e = A^{-1}\varepsilon + A^{-1}\widetilde{L}Ne$, when A^{-1} exists and bounded as in Eq. (12), then

$$\|e\|_\infty \le M\|\varepsilon\|_\infty + ML\|e\|_\infty, \tag{19}$$

where M is abound on $\|A^{-1}\|_\infty$ and $\|N\|_\infty = 1$ then

$$1 - ML > 0, \tag{20}$$

and we have

$$\|e\|_\infty \le \frac{\|\varepsilon\|_\infty}{\min\limits_{0\le j\le n}\left|\gamma\left(\tau_j\right)\right| - L}, \qquad (21)$$

when the forcing function is a function of τ only, the Lipschitz constant $L = 0$ and Eq. (21) becomes

$$\|e\|_\infty \le \frac{\|\varepsilon\|_\infty}{\min\limits_{0\le j\le n}\left|\gamma\left(\tau_j\right)\right|}. \qquad (22)$$

Then the following theorem shows the convergence of the trigonometric quintic B-spline.

Theorem 2.1. Let us have the second-order two-point boundary value problems (1) and (2) where the coefficients $\alpha(\tau), \beta(\tau), \gamma(\tau)$ and the forcing function $g(\tau, Z)$ are satisfied the conditions (11), (15) and (20), then the trigonometric quintic B-spline $\bar{z}(\tau)$ interpolating to $z(\tau)$ converges to $Z(\tau)$ in the maximum error over $[a, b]$ with error norm as (21).

3 Numerical Examples

We investigate some examples of the second-order two-point boundary value problems. We take the interval $[0, 1]$ and divide it into different values of n. At each value of n, we calculate the absolute error of the difference between the exact solution and the numerical solution.

Example 1. We investigate a linear boundary value problem [19]

$$Z''(\tau) - 100Z(\tau) = 0,$$

$$Z(0) = Z(1) = 1, \qquad (23)$$

and the exact solution is given by $Z(\tau) = Cosh(10\,\tau - 5)/Cosh(5)$. Thus, we can calculate the numerical solution and compare it with the exact solution to obtain the maximum absolute errors as shown in Tables 2 and 3.

Table 2: The maximum absolute errors for Example 1.

n	Trigonometric quintic B-spline errors	Cubic B-spline errors [5]	Quintic B-spline errors [5]
5	1.30×10^{-3}	1.00×10^{-1}	7.88×10^{-3}
10	1.57×10^{-4}	1.69×10^{-2}	2.91×10^{-4}
15	4.39×10^{-5}	7.30×10^{-3}	4.87×10^{-5}
20	1.49×10^{-5}	3.93×10^{-3}	2.15×10^{-5}

Table 3: The maximum absolute errors for Example 1.

n	Trigo.nometric quintic B-spline Errors	Fourth order parametric cubic spline errors [19]
8	3.25×10^{-4}	1.74×10^{-3}
16	3.48×10^{-5}	1.12×10^{-4}
32	2.38×10^{-6}	7.29×10^{-6}

Example 2. We discuss a linear boundary value problem [18]

$$Z''(\tau) - 4Z(\tau) - 4Cosh(1) = 0,$$

$$Z(0) = Z(1) = 0, \qquad (24)$$

and the exact solution is given by $Z(\tau) = Cosh(2\,\tau - 1) - Cosh(1)$. Thus, we can do a comparison between the numerical results and the exact solution to obtain the maximum absolute errors as shown in Tables 4 and 5.

Table 4: The maximum absolute errors for Example 2.

n	Trigonometric quintic B-spline errors	Cubic B-spline errors [5]	Quintic B-spline errors [5]
3	2.45×10^{-4}	1.53×10^{-2}	1.01×10^{-4}
5	4.57×10^{-5}	5.23×10^{-3}	1.34×10^{-5}
7	1.27×10^{-5}	2.63×10^{-3}	3.44×10^{-6}
9	4.85×10^{-6}	1.58×10^{-3}	1.22×10^{-6}

Table 5: The maximum absolute errors for Example 2.

n	Trigonometric quintic B-spline errors	Sinc-collocation errors [18]	Sinc–Galerkin errors [18]
50	5.5706×10^{-9}	1.1605×10^{-9}	1.5022×10^{-9}
75	1.1057×10^{-9}	1.4987×10^{-11}	2.6068×10^{-11}
100	3.5076×10^{-10}	2.5024×10^{-13}	3.7025×10^{-13}

Example 3. Finally, we examine the nonlinear boundary value problem [18]

$$Z''(\tau) - 0.5(Z(\tau) + \tau + 1)^3 = 0,$$

$$Z(0) = Z(1) = 0, \qquad (25)$$

and the exact solution is given by $Z(\tau) = \frac{2}{2-\tau} - \tau - 1$. Thus, we can calculate the numerical solution and compare it with the exact solution to obtain the maximum absolute errors as shown in Tables 6 and 7.

Table 6: The maximum absolute errors for Example 3.

n	Trigonometric quintic B-spline errors	Cubic B-spline errors [5]	Quintic B-spline errors [5]
4	7.40×10^{-5}	5.04×10^{-3}	9.91×10^{-5}
6	2.02×10^{-5}	2.13×10^{-3}	1.56×10^{-5}
8	6.88×10^{-6}	1.18×10^{-3}	5.24×10^{-6}

Table 7: The maximum absolute errors for Example 3.

n	Trigonometric quintic B-spline errors	Sinc-collocation errors [18]	Sinc–Galerkin errors [18]
50	5.6981×10^{-9}	3.1411×10^{-10}	7.4032×10^{-10}
75	1.1369×10^{-9}	9.5502×10^{-13}	4.2103×10^{-12}
100	3.6148×10^{-10}	2.0625×10^{-14}	5.1452×10^{-14}

4 Conclusions

In this paper, we investigated the trigonometric quintic B-spline collocation method to find numerical solutions of two-point boundary value problems associated with deflection of plates as in examples (1- 3). From the numerical results, the proposed collocation method gives reliable solutions of the presented problems if compared with the existing results at different values of n. The numerical examples are compared with the analytic solutions by finding the maximum absolute errors and are compared with numerical results in [5], [18] and [19] as shown in the Tables (2- 7), where we can see that the numerical accuracy in our method is better than results obtained in [19] and similar to results obtained in [5, 18].

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