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A Measure of Departure from Partial Marginal Homogeneity for Square Contingency Tables

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Abstract: For square contingency tables, Tomizawa and Makii [1] considered the measure represents the degree of departure from marginal homogeneity model. The present paper proposes a measure for the partial marginal homogeneity model which indicates that there is a homogeneous structure for at least one of pairs of row and column marginal probabilities. Examples are given.

Keywords: Marginal homogeneity; Measure; Partial marginal homogeneity; Square contingency table

1 Introduction

Consider an $r \times r$ square contingency table with the same row and column classifications. Let p_{ij} denote the cell probability that an observation will fall in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$).

The marginal homogeneity (MH) model is defined by

$$p_{i\cdot} = p_{\cdot i} \qquad (i = 1, \ldots, r),$$

where $p_{i\cdot} = \sum_{t=1}^{r} p_{it}$ and $p_{\cdot i} = \sum_{s=1}^{r} p_{si}$ (Stuart, [2]). This model indicates that the row marginal distribution is identical to the column marginal distribution.

Assume that $p_{i\cdot} + p_{\cdot i} > 0$ for $i = 1, \ldots, r$. Let $\pi_i = (p_{i\cdot} + p_{\cdot i})/2$, $p_{1(i)} = p_{i\cdot}/(p_{i\cdot} + p_{\cdot i})$ and $p_{2(i)} = p_{\cdot i}/(p_{i\cdot} + p_{\cdot i})$. Tomizawa and Makii [1] considered the measure to represent the degree of departure from the MH model as follows:

$$\Psi^{(\lambda)} = \sum_{i=1}^{r} \pi_i \left[1 - \frac{\lambda 2^{\lambda}}{2^{\lambda} - 1} H_i^{(\lambda)} \right] \quad \text{for} \quad \lambda > -1,$$

where

$$H_i^{(\lambda)} = \frac{1}{\lambda} \left[1 - \left(p_{1(i)}\right)^{\lambda+1} - \left(p_{2(i)}\right)^{\lambda+1} \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$, and λ is a real value chosen by the user. Note that $H_i^{(\lambda)}$ is the Patil and Taillie's [3] diversity index of degree λ which includes the Shannon entropy as a special case. We point out that $\Psi^{(\lambda)}$ is expressed as the weighted arithmetic mean of the diversity index $H_i^{(\lambda)}$.

The data in Table 1a are taken from Hashimoto ([4], p.151), and the data in Table 1b is taken from Bishop, Fienberg and Holland ([5], p.100). These data describe the cross-classifications of father's and his son's occupational status in Japan and in Denmark, respectively. The smaller category number means the higher status in each table. For example, if there is a structure of MH in Table 1b, the probability that a Danish father's occupational status is i is equal to the probability that his son's occupational status is i for all i ($i = 1, \ldots, 5$). If there is not a structure of MH in each table, we are next interested in whether there is a weaker marginal homogeneous structure (e.g., marginal means equality) than

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Table 1: Cross-classifications of father's and his son's occupational status (a) in Japan (Hashimoto, [4], p.151) and (b) in Denmark (Bishop et al., [5], p.100)

(a) in Japan

Father's status	Son's status (1)	(2)	(3)	(4)	(5)	Total
(1)	29	43	25	31	4	132
(2)	23	159	89	38	14	323
(3)	11	69	184	34	10	308
(4)	42	147	148	184	17	538
(5)	42	176	377	114	298	1007
Total	147	594	823	401	343	2308

(b) in Denmark

Father's status	Son's status (1)	(2)	(3)	(4)	(5)	Total
(1)	18	17	16	4	2	57
(2)	24	105	109	59	21	318
(3)	23	84	289	217	95	708
(4)	8	49	175	348	198	778
(5)	6	8	69	201	246	530
Total	79	263	658	829	562	2391

MH structure. The weakest marginal homogeneous structure may be complete marginal inhomogeneity, i.e., $p_{i\cdot} \neq p_{\cdot i}$ for any i. However, many readers may not be interested in such structure. Thus we shall consider a structure which indicates marginal homogeneity in some pairs of row and column categories instead of complete marginal inhomogeneity.

For square contingency tables, Bowker [6] introduced the symmetry model which states a cell probability equals to the cell probability corresponding to the symmetric cell with respect to the main diagonal of table. Thus the symmetry model indicates homogeneous for all pairs of symmetric cell probabilities. Saigusa, Tahata and Tomizawa [7] considered the partial symmetry model which indicates homogeneous for one or more pairs of symmetric cell probabilities.

We consider new marginal homogeneity model which has weaker restrictions than those of the MH model as follows:

$$p_{i\cdot} = p_{\cdot i} \qquad \text{for at least one } i \ (i = 1, \ldots, r).$$

We shall refer to this model as the partial marginal homogeneity (PMH) model. This model indicates that homogeneous for one or more pairs of marginal probabilities. Note that the PMH model is implied by the MH model. For example, on the data in Table 1b, holding the PMH model reveals that there are one or more occupational status categories such that the probability that Danish father's status category is i is equal to the probability that his son's status category is also i.

We are now interested in proposing the measure represents the degree of departure from the PMH model. The measure may enable us to compare the degrees of departure between two different tables.

In the present paper, Section 2 proposes a new measure which represents the degree of departure from the PMH model. Section 3 gives the approximate confidence interval of the proposed measure. Section 4 gives some examples of data analysis using the measure. Section 5 illustrates the property of the measure with artificial examples. Section 6 gives the concluding remarks.

2 Measure

Assume that $p_{i\cdot} + p_{\cdot i} > 0$ for $i = 1, \ldots, r$. Consider the measure to represent the degree of departure from the PMH model defined by

$$\Phi^{(\lambda)} = \prod_{i=1}^{r} \left[1 - \frac{\lambda 2^{\lambda}}{2^{\lambda} - 1} H_i^{(\lambda)} \right]^{\pi_i} \quad \text{for} \quad \lambda > -1,$$

where λ is a real value chosen by the user. Note that the value of $\Phi^{(\lambda)}$ at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$, namely,

$$\Phi^{(0)} = \prod_{i=1}^{r} \left[1 - \frac{1}{\log 2} H_i^{(0)} \right]^{\pi_i},$$

where $H_i^{(0)} = -p_{1(i)} \log p_{1(i)} - p_{2(i)} \log p_{2(i)}$. This measure $\Phi^{(\lambda)}$ must lie between 0 and 1. For any $\lambda\ (> -1)$, (i) $\Phi^{(\lambda)} = 0$ if and only if there is a structure of the PMH model, namely $p_{i\cdot} = p_{\cdot i}$ for at least one i, (ii) $\Phi^{(\lambda)} = 1$ if and only if there is a structure of complete marginal inhomogeneity in the sense that $p_{i\cdot} = 0$ (then $p_{\cdot i} > 0$) or $p_{\cdot i} = 0$ (then $p_{i\cdot} > 0$) for all i $(i = 1,\ldots,r)$. We shall show an example of structure of complete marginal inhomogeneity in Section 5.

The measure $\Phi^{(\lambda)}$ is expressed as the weighted geometric mean of the diversity index whereas $\Psi^{(\lambda)}$ is the weighted arithmetic mean. It is easily seen that the value of $\Phi^{(\lambda)}$ is less than or equal to the value of $\Psi^{(\lambda)}$. It would be natural relationship because (i) the necessary and sufficient condition that $\Phi^{(\lambda)}$ takes the minimum value zero is weaker than one that $\Psi^{(\lambda)}$ also takes the minimum value zero, (ii) the necessary and sufficient conditions that $\Phi^{(\lambda)}$ takes the maximum value one is identical with one that $\Psi^{(\lambda)}$ also takes the maximum value one. When $\{H_i^{(\lambda)}\}$ take the same value as one another, the value of $\Phi^{(\lambda)}$ is equal to the value of $\Psi^{(\lambda)}$. Namely, when $H_i^{(\lambda)} = c$ for every i,

$$\Phi^{(\lambda)} = \Psi^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} c.$$

$H_i^{(\lambda)}$ would be considered as a submeasure which reflects the degree of departure from marginal homogeneity in a category i. Thus $\Phi^{(\lambda)}$ is identical to $\Psi^{(\lambda)}$ if the degrees of departure are the same in all categories, in the sense that $p_{1(i)} = q$ (then $p_{2(i)} = 1-q$) or $p_{1(i)} = 1-q$ (then $p_{2(i)} = q$) for $i = 1,\ldots,r; 0 \le q \le 1$.

The measure $\Phi^{(\lambda)}$ is appropriate for the nominal contingency tables, because the value of $\Phi^{(\lambda)}$ is invariant under arbitrary same permutations of the row and column categories, namely, the value of $\Phi^{(\lambda)}$ does not depend on the order of the categories.

3 Approximate confidence interval of measure

Let n_{ij} denote the observed frequency in the ith row and jth column of the table $(i = 1,\ldots,r; j = 1,\ldots,r)$, and let n denote the total number of observations, i.e. $n = \sum\sum n_{ij}$. Assuming that a multinomial distribution applies to the $r \times r$ table, we shall consider the approximate standard error and the large-sample confidence interval of $\Phi^{(\lambda)}$. The sample version of $\Phi^{(\lambda)}$, denoted by $\hat{\Phi}^{(\lambda)}$, is given by $\Phi^{(\lambda)}$ with (p_{ij}) replaced by $(\hat{p}_{ij})$, where $\hat{p}_{ij} = n_{ij}/n$. Using the delta method (Agresti, [8], p.583), $\sqrt{n}(\hat{\Phi}^{(\lambda)} - \Phi^{(\lambda)})$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance σ^2, where

$$\sigma^2 = \sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij} (\gamma_{ij}^{(\lambda)})^2 - \left(\sum_{i=1}^{r}\sum_{j=1}^{r} p_{ij} \gamma_{ij}^{(\lambda)} \right)^2,$$

with

$$\gamma_{ij}^{(\lambda)} = \begin{cases} \dfrac{\Phi^{(\lambda)}}{2} \left\{ \log\left(\omega_i^{(\lambda)} \omega_j^{(\lambda)}\right) + \dfrac{2^\lambda(\lambda+1)}{2^\lambda - 1} \left(\dfrac{\left(p_{i\cdot}^\lambda - p_{\cdot i}^\lambda\right) p_{\cdot i}}{(p_{i\cdot} + p_{\cdot i})^{\lambda+1} \omega_i^{(\lambda)}} - \dfrac{\left(p_{j\cdot}^\lambda - p_{\cdot j}^\lambda\right) p_{j\cdot}}{(p_{j\cdot} + p_{\cdot j})^{\lambda+1} \omega_j^{(\lambda)}} \right) \right\} & (\lambda > -1\,;\, \lambda \neq 0), \\ \dfrac{\Phi^{(\lambda)}}{2} \left\{ \log\left(\omega_i^{(\lambda)} \omega_j^{(\lambda)}\right) + \dfrac{1}{\log 2} \left(\dfrac{p_{2(i)}}{\omega_i^{(\lambda)}} \log\left(\dfrac{p_{i\cdot}}{p_{\cdot i}}\right) - \dfrac{p_{1(j)}}{\omega_j^{(\lambda)}} \log\left(\dfrac{p_{j\cdot}}{p_{\cdot j}}\right) \right) \right\} & (\lambda = 0), \end{cases}$$

$$\omega_i^{(\lambda)} = \begin{cases} 1 - \dfrac{\lambda 2^\lambda}{2^\lambda - 1} H_i^{(\lambda)} & (\lambda > -1\,;\, \lambda \neq 0), \\ 1 - \dfrac{1}{\log 2} H_i^{(0)} & (\lambda = 0). \end{cases}$$

We point out that the asymptotic normal distribution of $\sqrt{n}(\hat{\Phi}^{(\lambda)} - \Phi^{(\lambda)})$ is applicable only when $0 < \Phi^{(\lambda)} < 1$. Let $\hat{\sigma}^2$ denote σ^2 with (p_{ij}) replaced by $(\hat{p}_{ij})$. Then $\hat{\sigma}/\sqrt{n}$ is the estimated approximate standard error of $\hat{\Phi}^{(\lambda)}$, and the approximate $100(1-\alpha)\%$ confidence interval of $\Phi^{(\lambda)}$ is $\hat{\Phi}^{(\lambda)} \pm z_{\alpha/2} \hat{\sigma}/\sqrt{n}$, where $z_{\alpha/2}$ is the percentage point from standard normal distribution corresponding to a two-tail probability equal to α.

Table 2: The estimates of $\Psi^{(\lambda)}$, estimated approximate standard errors of $\hat{\Psi}^{(\lambda)}$, and approximate 95% confidence intervals of $\Psi^{(\lambda)}$, applied to each of Tables 1a and 1b

(a) Table 1a

λ	Estimated measure $\hat{\Psi}^{(\lambda)}$	Standard error	Confidence interval
-0.5	0.0656	0.0042	$(0.0574, 0.0739)$
0	0.1073	0.0066	$(0.0943, 0.1203)$
0.5	0.1316	0.0080	$(0.1160, 0.1472)$
1	0.1434	0.0086	$(0.1266, 0.1602)$
1.5	0.1464	0.0087	$(0.1293, 0.1635)$
2	0.1434	0.0086	$(0.1266, 0.1602)$
2.5	0.1365	0.0082	$(0.1203, 0.1526)$
3	0.1271	0.0078	$(0.1118, 0.1424)$

(b) Table 1b

λ	Estimated measure $\hat{\Psi}^{(\lambda)}$	Standard error	Confidence interval
-0.5	0.0012	0.0006	$(0.0001, 0.0023)$
0	0.0020	0.0009	$(0.0001, 0.0038)$
0.5	0.0025	0.0012	$(0.0002, 0.0048)$
1	0.0028	0.0013	$(0.0002, 0.0053)$
1.5	0.0028	0.0013	$(0.0002, 0.0054)$
2	0.0028	0.0013	$(0.0002, 0.0053)$
2.5	0.0026	0.0012	$(0.0002, 0.0050)$
3	0.0024	0.0011	$(0.0002, 0.0046)$

Table 3: The estimates of $\Phi^{(\lambda)}$, estimated approximate standard errors of $\hat{\Phi}^{(\lambda)}$, and approximate 95% confidence intervals of $\Phi^{(\lambda)}$, applied to each of Tables 1a and 1b

(a) Table 1a

λ	Estimated measure $\hat{\Phi}^{(\lambda)}$	Standard error	Confidence interval
-0.5	0.0398	0.0064	$(0.0273, 0.0523)$
0	0.0657	0.0105	$(0.0451, 0.0863)$
0.5	0.0812	0.0129	$(0.0558, 0.1065)$
1	0.0888	0.0141	$(0.0611, 0.1165)$
1.5	0.0908	0.0145	$(0.0625, 0.1191)$
2	0.0888	0.0141	$(0.0611, 0.1165)$
2.5	0.0842	0.0134	$(0.0579, 0.1105)$
3	0.0779	0.0125	$(0.0535, 0.1023)$

(b) Table 1b

λ	Estimated measure $\hat{\Phi}^{(\lambda)}$	Standard error	Confidence interval
-0.5	0.0006	0.0004	$(-0.0002, 0.0015)$
0	0.0011	0.0007	$(-0.0003, 0.0025)$
0.5	0.0014	0.0009	$(-0.0003, 0.0031)$
1	0.0015	0.0010	$(-0.0004, 0.0034)$
1.5	0.0015	0.0010	$(-0.0004, 0.0035)$
2	0.0015	0.0010	$(-0.0004, 0.0034)$
2.5	0.0014	0.0009	$(-0.0004, 0.0032)$
3	0.0013	0.0008	$(-0.0003, 0.0029)$

4 Examples

Consider the data in Table 1 again. We are interested in whether there is a structure of the MH model in Tables 1a and 1b. Table 2 gives the estimated values of measure $\Psi^{(\lambda)}$ and the approximate 95% confidence intervals of the measure applied to each of Tables 1a and 1b. From Table 2, for any λ (>-1), none of the confidence intervals of $\Psi^{(\lambda)}$ applied to each table includes zero. Thus it is inferred that there is not a structure of MH in each table.

We are next interested in whether there is a weaker marginal homogeneous structure, PMH, in Tables 1a and 1b. Table 3 gives the estimated values of measure $\Phi^{(\lambda)}$ and the approximate confidence intervals applied to each table. From Table 3, for any λ (>-1), the confidence interval of $\Phi^{(\lambda)}$ applied to the data in Table 1a does not include zero. Thus there may be not even the structure of PMH in Table 1a. On the other hand, for any λ (>-1), the confidence interval applied to the data in Table 1b includes zero. So there may be a structure of PMH in Table 1b. Therefore it is inferred that there is a homogeneous structure in the one or more pairs of Danish father's and his son's occupational status (1) to (5).

Furthermore, we would like to compare the degrees of departure from PMH for Tables 1a and 1b using $\Phi^{(\lambda)}$. Comparing the confidence intervals of $\Phi^{(\lambda)}$ for Tables 1a and 1b, for any fixed λ (>-1), they are not overlapped. Thus it is inferred that the degree of departure from PMH for Table 1a is larger than that for Table 1b. Namely, a structure of Danish father's and his son's occupational status would have less degree of departure from PMH than that of Japanese father's and his son's.

We note that, for any fixed λ (>-1), the estimated value of $\Phi^{(\lambda)}$ is less than that of $\Psi^{(\lambda)}$.

5 Artificial examples

Consider the 4×4 cell probability tables given in Table 4. In Table 4a, there are structures of the MH and PMH models. In Table 4b, there is a structure of the PMH model whereas there is not a structure of the MH model. In Table 4f, there is not a structure of each of the MH and PMH models, and the degrees of departure from the MH and PMH models are the largest. See Tomizawa and Makii [1] and Section 2 in the present paper for the definitions of the largest departure from the MH and PMH models. In Tables 4b-4d, either of row and column marginal probabilities is zero in categories 2-4 and the ratio of row and column marginal probabilities in category 1, i.e. $p_{1\cdot}/p_{\cdot 1}$, is greater than one and increases in the order of Tables 4b, 4c and 4d. Thus it may be natural to consider that the degree of departure from the PMH model increases in the order of Tables 4b, 4c and 4d. In Table 4e, the degrees of departure from marginal homogeneity in every categories are the same, in the sense that $p_{1(i)}=1/4$ or $p_{1(i)}=3/4$ for $i=1,\ldots,4$. Table 5 shows the values of $\Psi^{(\lambda)}$ and $\Phi^{(\lambda)}$ applied

Table 4: Artificial cell probability tables

(a)

	(1)	(2)	(3)	(4)	Total
(1)	0.02	0.01	0.08	0.04	0.15
(2)	0.05	0.03	0.06	0.06	0.20
(3)	0.06	0.09	0.12	0.08	0.35
(4)	0.02	0.07	0.09	0.12	0.30
Total	0.15	0.20	0.35	0.30	1

(b)

	(1)	(2)	(3)	(4)	Total
(1)	0.35	0.05	0.20	0	0.60
(2)	0	0	0	0	0
(3)	0	0	0	0	0
(4)	0.25	0.05	0.10	0	0.40
Total	0.60	0.10	0.30	0	1

(c)

	(1)	(2)	(3)	(4)	Total
(1)	0.35	0.05	0.30	0	0.70
(2)	0	0	0	0	0
(3)	0	0	0	0	0
(4)	0.15	0.05	0.10	0	0.30
Total	0.50	0.10	0.40	0	1

(d)

	(1)	(2)	(3)	(4)	Total
(1)	0.35	0.05	0.40	0	0.80
(2)	0	0	0	0	0
(3)	0	0	0	0	0
(4)	0.05	0.05	0.10	0	0.20
Total	0.40	0.10	0.50	0	1

(e)

	(1)	(2)	(3)	(4)	Total
(1)	0.02	0.09	0.12	0.04	0.27
(2)	0.02	0.03	0.03	0.02	0.10
(3)	0.02	0.01	0.08	0.04	0.15
(4)	0.03	0.17	0.22	0.06	0.48
Total	0.09	0.30	0.45	0.16	1

(f)

	(1)	(2)	(3)	(4)	Total
(1)	0	0.20	0	0.45	0.65
(2)	0	0	0	0	0
(3)	0	0.05	0	0.30	0.35
(4)	0	0	0	0	0
Total	0	0.25	0	0.75	1

Table 5: Values of $\Psi^{(\lambda)}$ and $\Phi^{(\lambda)}$ for Table 4

(a) Values of $\Psi^{(\lambda)}$

Applied tables	λ 0	0.5	1.5
Table 4a	0	0	0
Table 4b	0.400	0.400	0.400
Table 4c	0.412	0.415	0.417
Table 4d	0.449	0.461	0.468
Table 4e	0.189	0.230	0.255
Table 4f	1	1	1

(b) Values of $\Phi^{(\lambda)}$

Applied tables	λ 0	0.5	1.5
Table 4a	0	0	0
Table 4b	0	0	0
Table 4c	0.096	0.110	0.118
Table 4d	0.223	0.253	0.271
Table 4e	0.189	0.230	0.255
Table 4f	1	1	1

to each of Tables 4a-4f. From Tables 4a, 4b, 4f and 5, we see that $\Phi^{(\lambda)}$ takes zero for the cell probability table there is a structure of the PMH model, and takes one for the table there is the largest departure from the PMH model, whereas $\Psi^{(\lambda)}$ does not take zero for the table there is a structure of the PMH model. We can see from Tables 4b-4d and 5b, for fixed λ, the value of $\Phi^{(\lambda)}$ increases in the order of Tables 4b, 4c and 4d. Therefore the measure $\Phi^{(\lambda)}$ would be appropriate for measuring the degree of departure from the PMH model. We also see from Tables 4e and 5, for fixed λ the value of $\Phi^{(\lambda)}$ equals to the value of $\Psi^{(\lambda)}$. We note that, for fixed λ, the value of $\Phi^{(\lambda)}$ is less than or equal to the value of $\Psi^{(\lambda)}$.

6 Concluding remarks

For an $r \times r$ square contingency table, we have newly considered the PMH model which has weaker restriction than that of the MH model. We also have proposed the measure to represent the degree of departure from the PMH model. The PMH model indicates homogeneity for at least one of pairs of row and column marginal probabilities rather than all pairs. The proposed measure $\Phi^{(\lambda)}$ would be appropriate for quantifying how far the marginal distributions are distant from those there is a PMH structure. The measure $\Phi^{(\lambda)}$ would also be useful for comparing the degrees of departure from the PMH model between two different tables.

In general, the goodness-of-fit test is employed for testing a model. It may be difficult to provide a goodness-of-fit test for the PMH model. Therefore the proposed measure may be useful as one of the methods to see whether the PMH model holds or not. We would like to compare between judgments by goodness-of-fit test and confidence interval of the measure in future work.

Note that by using $\Phi^{(\lambda)}$ we cannot identify which pairs of row and column marginal probabilities are identical when there would be a structure of the PMH model in the table.

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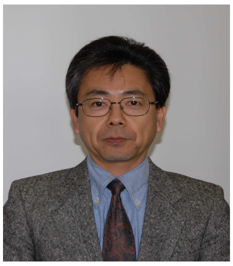

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