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New High-Precision Strapdown Navigation Attitude Algorithm under Angular-Rate Input Condition

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Abstract: Coning motion is a standard test input to evaluate the performance of the strapdown attitude algorithms. Angular-rate coning algorithms error consists of two parts: drift error and approximation error. Traditional angular-rate coning algorithms usually improve the algorithm performance by increasing the sampling number in one update period. However the increase of the sampling number can only reduce the drift error, it has few effects on reducing approximation error. And the approximation error compensation is neglected in traditional angular-rate coning algorithms. In this paper the calculation result shows that the approximation error is comparable with drift error for the most general case, which means the approximation error can not be neglected in high-precision strapdown navigation systems. A new angular-rate coning algorithm with an additional second-order noncommutativity error compensation term is developed. Without increasing sampling number, the new angular-rate coning algorithm can reduce the approximation error greatly. Theoretical analyses and digital simulations indicate that the new algorithm has advantages over the traditional coning algorithms for the general case.

Keywords: strapdown navigation, coning algorithm, rotation vector, approximation error

1 Introduction

In strapdown navigation systems, the rotation of a body is measured and integrated to form an attitude matrix or attitude quaternion which describes the attitude (head, roll and pitch angles) of the body. But from the theory of finite rotations we know that when the axis of rotation changes directions, the attitude can not be determined by direct integration of the body angular rate , otherwise noncommutativity error will be caused. This is because the attitude of a rotating body not only depends on the magnitude, but also depends on the order of the rotations [1]. To eliminate the noncommutativity error caused by body rotation, rotation vector concept is developed. By rotation vector we can describe the rotation of a body accurately. The first-order solution to the rotation vector differential equation is [2]:

$$\Phi \approx \int_t^{t+H} \omega dt + \frac{1}{2}\int_t^{t+H} (\Phi \times \omega) dt = \Delta\theta + \delta\Phi, \quad (1)$$

where H is the update period, and Φ is the rotation vector defining the body attitude at time t+H relative to the body attitude at time t. The first term of the Eq.(1) is the integration of body angular rate vector. The second term $\delta\Phi$ is the firtst-order noncommutativity error compensation term by the rotation vector. Eq.(1) is a theoretical equation. However the practical digital rotation vector algorithms derived from Eq.(1) can take various forms.

A classical coning motion is defined by quaternion as [3]:

$$Q(t) = [cos\frac{\alpha}{2}, 0, sin\frac{\alpha}{2}cos\Omega t, sin\frac{\alpha}{2}sin\Omega t]. \quad (2)$$

The body angular rate ω in a coning environment described by Eq.(2) is:

$$\omega = 2Q^{-1}(t) \otimes \dot{Q}(t) = \begin{bmatrix} -2\Omega \sin^2\frac{\alpha}{2} \\ -\Omega \sin\alpha \sin\Omega t \\ \Omega \sin\alpha \cos\Omega t \end{bmatrix}. \quad (3)$$

Ref.[4] proved that when the aircraft is in a coning environment, the noncommutativity compensation term of the Eq.(1) has a maximum value. Hence coning motion is

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usually used as a standard input to test the attitude integration algorithms based on rotation vector. In 1996, Ignagni proved that attitude algorithms work satisfactorily in a coning environment would satisfy most other environments requirements[5]. Moreover, "Coning motion is also a nonnegligible effect for fast, highly maneuverable precision-pointing spacecraft and alignment calibration for maneuvering spacecraft because state propagation errors can bias the calibration estimates"[6]. Hence the researches on coning algorithm have great practical significance. In 1983 Miller proposed the classical three-sample coning algorithm[3]. The algorithm proposed by Miller uses a gyro with incremental angle output (e.g. Ring laser gyro). Based on Miller's algorithm, other improved coning algorithms using the gyro incremental angle output or angular rate output are developed [7,8,9,10].The algorithm error of all those coning algorithms consists of two parts: drift error and approximation error. Drift error is caused by the residual constant error on coning axis (x) in the derivation of the coning algorithms. Approximation error is caused by the approximations in the derivation of the coning algorithms. Hence the approximation error, like the drift error, is a theoretical error which can not be reduced by the performance improvement of the navigation computer.

2 Error analysis of traditional angular-rate coning algorithms

2.1 Derivation process of traditional coning algorithms

For modern-day strapdown navigation systems, gyros with angular rate output (e.g. fiber optic gyro) are widely used. And the corresponding angular rate coning algorithms have been developed. The famous algorithm is the two-interval angular- rate coning algorithm[10]. The derivation is as follows.

For a classical coning motion described by Eq.(2), the corresponding truth value of updating quaternion is:

$$q(H)=\begin{bmatrix} 1-2\sin^2\frac{\alpha}{2}\sin^2\frac{\Omega H}{2} \\ -\sin^2\frac{\alpha}{2}\sin\Omega H \\ -\sin\alpha\sin(\frac{\Omega H}{2})\sin\Omega(t+\frac{H}{2}) \\ \sin\alpha\sin(\frac{\Omega H}{2})\cos\Omega(t+\frac{H}{2}) \end{bmatrix}. \quad (4)$$

The estimation value for updating quaternion $q(H)$ corresponding to the rotation vector Φ is:

$$\hat{q}(H)=\begin{bmatrix} \cos(|\Phi|/2) \\ (\Phi_x/|\Phi|)\sin(|\Phi|/2) \\ (\Phi_y/|\Phi|)\sin(|\Phi|/2) \\ (\Phi_z/|\Phi|)\sin(|\Phi|/2) \end{bmatrix}=\begin{bmatrix} C \\ \Phi_x S \\ \Phi_y S \\ \Phi_z S \end{bmatrix}, \quad (5)$$

where Φ is the magnitude of rotation vector: $|\Phi|=\sqrt{\Phi^2{}_x+\Phi^2{}_y+\Phi^2{}_z}$. The error quaternion is:

$$\tilde{q}(H)=q(H)\otimes \hat{q}^{-1}(H)=\begin{bmatrix} q_0C-S(-q_1\Phi_x-q_2\Phi_y-q_3\Phi_z) \\ q_1C-S(q_0\Phi_x-q_3\Phi_y+q_2\Phi_z) \\ q_2C-S(q_3\Phi_x+q_0\Phi_y-q_1\Phi_z) \\ q_3C-S(-q_2\Phi_x+q_1\Phi_y+q_0\Phi_z) \end{bmatrix}, \quad (6)$$

where q_2, q_3, Φ_y, Φ_z are all periodic, so $\tilde{q}_2$, $\tilde{q}_3$ are also periodic. $\tilde{q}_2$, $\tilde{q}_3$ contribute a reciprocating error which can be canceled in the long run. But $\tilde{q}_1$ has nonperiodic term: q_1C-S$q_0\Phi_x$. Nonperiodic error will cause drift error during the quaternion update. So $\tilde{q}_1$ must be restrained.

To simplify the analysis, in traditional coning algorithms some variables in Eq.(6) was approximated as [3,10]:

$$C\approx 1, S\approx 1/2, q_0\approx 1. \quad (7)$$

Then $\tilde{q}_1$ can be simplified to the following form [3]:

$$\tilde{q}_1(H)\approx q_1-1/2\Phi_x. \quad (8)$$

From the first-order rotation vector equation (Eq.(1)), a conclusion can be made that the two-interval first-order angular-rate coning algorithm form is [10]:

$$\hat{\Phi}=\Delta\theta+k_1(\omega_1\times\omega_3)H^2+k_2(\omega_2\times\omega_3)H^2, \quad (9)$$

where $\Delta\theta$ is the incremental angle vector over an update interval (t_{m-1}, t_m), and H= t_m-t_{m-1}. Note that for a two-interval algorithm under angular-rate condition, the gyro has three outputs over (t_{m-1}, t_m) : ω_1(t=t_{m-1}), ω_2 (t=$t_{m-1/2}$), ω_3 (t=t_m). It can be calculated by the digital integral of the angular rate ω_i from the gyro outputs

$$\Delta\hat{\theta}=(\omega_1+4\omega_2+\omega_3)\frac{H}{6}. \quad (10)$$

We substitute Eq.(3) into Eq.(9). Based on the minimum error criteria, the optimal coefficient k_i are achieved: k_1=1/180, k_2=7/45. So the two-interval angular-rate coning algorithm is gotten:

$$\hat{\Phi}=(\omega_1+4\omega_2+\omega_3)\frac{H}{6}+\frac{H^2}{180}(\omega_1\times\omega_3)+\frac{7H^2}{45}(\omega_2\times\omega_3). \quad (11)$$

It should be noted that in some published papers (e.g. Ref.[10]), H was replaced by the subminor interval h. For a two-interval coning algorithm, $H=2h$. So the Eq.47 in Ref.[10] is:

$$\hat{\Phi}=(\omega_1+4\omega_2+\omega_3)\frac{H}{6}+\frac{1}{45}(\omega_1\times\omega_3)h^2+\frac{28}{45}(\omega_2\times\omega_3)h^2. \quad (12)$$

It is obvious that Eq.(11) is as same as Eq.(12) actually.The drift error (per unit time) of the coning algorithm defined by Eq.(11) and Eq.(12) was given in Ref.[10]:

$$\Phi_\varepsilon=\sin^2\frac{\alpha}{2}[\frac{(\Omega H)^7}{20160}]/H\approx\frac{\Omega^7H^6}{80640}\sin^2\alpha. \quad (13)$$

But approximation error is not considered in Eq.(13). And the complete algorithm error analysis (including both drift error and approximation error) will be given in the following discussion (Eq.(21)).

To further improve the algorithm accuracy, other traditional algorithms usually increase the sampling number of one attitude update period to reduce the drift error. For example, when we use four gyro samples (three-interval) in one attitude update period, the three-interval angular-rate coning algorithm will be gotten[10]:

$$\hat{\Phi} = (\omega_1 + 3\omega_2 + 3\omega_3 + \omega_4)\frac{H}{8} + \frac{2619}{2240}(\omega_1 \times \omega_2)h^2 + \frac{27}{56}(\omega_1 \times \omega_3)h^2 + \frac{87}{2240}(\omega_1 \times \omega_4)h^2. \quad (14)$$

For a three-interval coning algorithm, there is:$H = 3h$. The corresponding residual drift error (per unit time) is:

$$\Phi_\varepsilon = \frac{\Omega^9 H^8}{8899200}\sin^2\alpha. \quad (15)$$

Note that in practice the output data rate of a digital gyro is usually fixed. So the increase of the sampling number will cause a longer attitude update period. This problem will be analyzed in the following section 4 (below Table 1).

2.2 Error analysis

As stated in section 2.1 (Eq.(7)), there are some approximations in the derivation of the traditional angular rate coning algorithms. For those high-precision strapdown navigation systems, the approximation error can not be neglected.

To reduce the approximation error, Taylor series is used:

$$\begin{cases} C = \cos\frac{|\Phi|}{2} = 1 - \frac{|\Phi|^2}{8} + \ldots \approx 1 - \frac{|\Phi|^2}{8}, \\ S = \frac{\sin\frac{|\Phi|}{2}}{|\Phi|} = \frac{\frac{|\Phi|}{2} - \frac{1}{3!}\left(\frac{|\Phi|}{2}\right)^3}{|\Phi|} + \ldots \approx \frac{1}{2} - \frac{|\Phi|^2}{48}, \\ q_0 = 1 - 2\sin^2\frac{\alpha}{2}\sin^2\frac{\Omega H}{2}. \end{cases} \quad (16)$$

Substituting Eq.(16) into Eq.(6) gives:

$$\begin{aligned} \tilde{q}_1 &= q_1 C - S(q_0\hat{\Phi}_x - q_3\Phi_y + q_2\Phi_z) \\ &= -\sin^2\frac{\alpha}{2}\sin\Omega H\left(1 - \frac{|\hat{\Phi}|^2}{8}\right) - \left(\frac{1}{2} - \frac{1}{48}|\hat{\Phi}|^2\right)\left(1 - 2\sin^2\frac{\alpha}{2}\sin^2\frac{\Omega H}{2}\right)\hat{\Phi}_x \\ &= -\sin^2\frac{\alpha}{2}\sin\Omega H - \frac{\hat{\Phi}_x}{2} + \frac{|\hat{\Phi}|^2}{8}\sin^2\frac{\alpha}{2}\sin\Omega H + \frac{|\hat{\Phi}|^2}{48}\hat{\Phi}_x + \sin^2\frac{\alpha}{2}\sin^2\frac{\Omega H}{2}\hat{\Phi}_x. \end{aligned} \quad (17)$$

From Eq.(11) we can get:

$$|\hat{\Phi}| = \sqrt{\hat{\Phi}_x^2 + \hat{\Phi}_y^2 + \hat{\Phi}_z^2} \approx 4\sin\frac{\alpha}{2}\sin\frac{\Omega H}{2}. \quad (18)$$

Substituting Eq.(18) into Eq.(17) gives:

$$\tilde{q}_1 = \left(-\sin^2\frac{\alpha}{2}\sin\Omega H - \frac{\hat{\Phi}_x}{2}\right) + \left(2\sin^4\frac{\alpha}{2}\sin^2\frac{\Omega H}{2}\sin\Omega H + \frac{4}{3}\sin^2\frac{\alpha}{2}\sin^2\frac{\Omega H}{2}\hat{\Phi}_x\right). \quad (19)$$

We can substitute Eq.(11) into Eq.(19), and use Taylor series to expand "ΩH" term:

$$\tilde{q}_1 = \sin^2\frac{\alpha}{2}\left[\frac{1}{40320}(\Omega H)^7 + \ldots\right] + \sin^4\frac{\alpha}{2}\left[\frac{1}{30}(\Omega H)^5 - \frac{23}{5760}(\Omega H)^7 + \cdots\right]. \quad (20)$$

As is known from the Ref.[9], the coning algorithm error equals twice of the quantization error. So the two-interval coning algorithm error (per unit time) is:

$$\Phi_\varepsilon = \overbrace{\left\{\left(\sin^2\frac{\alpha}{2}\right)\left[\frac{(\Omega H)^7}{20160} + \ldots\right]/H\right\}}^{-drifterror-} + \overbrace{\left\{\left(\sin^4\frac{\alpha}{2}\right)\left[\frac{(\Omega H)^5}{15} - \frac{23(\Omega H)^7}{2880} + \ldots\right]/H\right\}}^{-approximation\ error-}. \quad (21)$$

As is seen from Eq.(21), error of traditional two-interval angular- rate coning algorithm consists of two parts. On the right-hand side the term in the first brace is the drift error (same to Eq.(13)). The term in the second brace is the approximation error. The existing coning algorithms (Ref.[7]-[10] usually reduce the algorithm error by increasing the sampling number. But the increase of sampling number can only reduce the drift error. It has few improvements on approximation error. But in fact approximation error can not be neglected for the general case. For example, when $\alpha = 1^\circ$, $\Omega = 2\pi$ *rad/s*, H=0.1*s*, the value of : "$\sin^4(\alpha/2)(\Omega^5 H^4)/15$" is 3.79×10^{-10} *rad/s*, which is comparable with the value of "$\sin^2(\alpha/2)(\Omega^7 H^6)/20160$" ($1.46 \times 10^{-9}$ *rad/s*). Therefore high-precision coning algorithms should also compensate the approximation error. That means an additional approximation error compensation term should be added to the traditional two-interval angular-rate coning algorithm. That is:

$$\sin^4\frac{\alpha}{2}\left[\frac{1}{15}(\Omega H)^5 - \frac{23}{2880}(\Omega H)^7 + \ldots\right]. \quad (22)$$

3 New second-order two-interval angular-rate coning algorithm

To get higher precision algorithm, second-order rotation vector equation is employed:

$$\Phi = \Delta\theta + \frac{1}{2}\int_{t_{m-1}}^{t_m} \Phi \times \omega dt + \frac{1}{12}\int_{t_{m-1}}^{t_m} \Phi \times (\Phi \times \omega)dt. \quad (23)$$

As is stated in Eq.(1), from the traditional first-order coning algorithm we can get:

$$\Phi \approx \Delta\theta + \delta\Phi, \delta\Phi = \frac{1}{2}\int_{t_{m-1}}^{t_m} \Delta\theta \times \omega dt. \quad (24)$$

Substituting Eq.(24) into Eq.(23) gives:

$$\begin{aligned}\Phi &= \Delta\theta + \tfrac{1}{2}\int_{t_{m-1}}^{t_m}(\Delta\theta\times\omega)dt + (\tfrac{1}{4}\int_{t_{m-1}}^{t_m}(\int_{t_{m-1}}^{t_m}\Delta\theta\times\omega dt) \\ &\quad \times\omega dt + \tfrac{1}{12}\int_{t_{m-1}}^{t_m}\Delta\theta\times(\Delta\theta\times\omega)dt) \\ &= \Delta\theta + \delta\Phi + \delta\delta\Phi,\end{aligned} \tag{25}$$

where $\delta\Phi$ is the first-order noncommutativity error compensation term, its digital algorithm is the traditional angular-rate coning algorithm (Eq.11, Eq.14). $\delta\delta\Phi$ is the second-order noncommutativity error compensation term. The digital algorithm of $\delta\delta\Phi$ is discussed as follows.

Suppose that the body's angular rate over an update interval (t_{m-1},t_m) is:

$$\omega = a + 2b(t-t_{m-1}) + 3c(t-t_{m-1})^2, t\in(t_{m-1},t_m). \tag{26}$$

So:

$$\begin{cases} a=\omega_1, \\ bH = \frac{1}{2}(-3\omega_1+4\omega_2-\omega_3), \\ cH^2 = \frac{2}{3}(\omega_1-2\omega_2+\omega_3), \end{cases} \tag{27}$$

where $\omega_1,\omega_2,\omega_3$ are the ideal gyro outputs at $t_{m-1},t_{m-1/2},t_m$. Substituting Eq.(26) into the $\delta\delta\Phi$ term of Eq.(25) gives:

$$\begin{aligned}&\delta\delta\Phi \\ &= \tfrac{1}{4}\int_{t_{m-1}}^{t_m}(\int_{t_{m-1}}^{t}\Delta\theta\times\omega dt)\times\omega dt + \tfrac{1}{12}\int_{t_{m-1}}^{t_m}\Delta\theta\times(\Delta\theta\times\omega)dt \\ &= -\tfrac{1}{60}b\times(a\times b)H^5 - \tfrac{1}{36}c\times(a\times b)H^6 + \tfrac{1}{120}a\times(a\times c)H^5 \\ &-\tfrac{1}{72}b\times(a\times c)H^6 - \tfrac{5}{168}c\times(a\times c)H^7 + \tfrac{1}{180}a\times(b\times c)H^6 \\ &-\tfrac{1}{420}b\times(b\times c)H^7 - \tfrac{1}{120}c\times(b\times c)H^8.\end{aligned} \tag{28}$$

Substituting Eq.(27) into the first term of Eq.(28) gives:

$$\begin{aligned}&-\tfrac{1}{60}bh\times(a\times bh)H^3 = \tfrac{1}{20}\omega_1\times(\omega_1\times\omega_2)H^3 - \tfrac{1}{80}\omega_1 \\ &\times(\omega_1\times\omega_3)H^3 - \tfrac{1}{15}\omega_2\times(\omega_1\times\omega_2)H^3 + \tfrac{1}{60}\omega_2\times(\omega_1 \\ &\times\omega_3)H^3 + \tfrac{1}{60}\omega_3\times(\omega_1\times\omega_2)H^3 - \tfrac{1}{240}\omega_3\times(\omega_1\times\omega_3)H^3.\end{aligned} \tag{29}$$

All other terms in Eq.(28) can also be processed into a form like Eq.(29). Therefore the second-order coning algorithm should consist of the sum of all possible second-order cross products from the angular rate gyro outputs over the update period. That is:

$$\delta\delta\hat{\Phi} = \sum_{i=1}^{N+1}\sum_{j=2}^{N+1}K_{ij}\omega_i\times(\omega_1\times\omega_j)H^3,\ N=2, \tag{30}$$

where N is the subminor interval number. From Eq.(3), we know that ideal gyro outputs over an update period are:

$$\omega_i = \begin{bmatrix} -2\Omega\sin^2(\frac{\alpha}{2}) \\ -\Omega\sin\alpha\sin\Omega(t+\frac{i-1}{N}H) \\ \Omega\sin\alpha\cos\Omega(t+\frac{i-1}{N}H)\end{bmatrix}, i=1,2\ldots N+1. \tag{31}$$

Substituting Eq.(31) into Eq.(30) gives:

$$\begin{aligned}&\omega_i\times(\omega_1\times\omega_j){H^3}_x \\ &= 4(\Omega H)^3\sin^2\alpha\sin^2\tfrac{\alpha}{2}\sin(\tfrac{j-1}{2N}\Omega H)\sin(\tfrac{j+1-2i}{2N}\Omega H).\end{aligned} \tag{32}$$

It can be easily seen that the value of Eq.(32) depends on the value of $|j-1|$ and $|j+1-2i|$, there are three different combinations all together:

$$\begin{aligned}&\omega_1\times(\omega_1\times\omega_2)_xH^3 = 16(\sin^4\tfrac{\alpha}{2})(\Omega H)^3\sin(\tfrac{\Omega H}{4})\sin(\tfrac{\Omega H}{4}), \\ &\omega_3\times(\omega_1\times\omega_2)_xH^3 = -16(\sin^4\tfrac{\alpha}{2})(\Omega H)^3\sin(\tfrac{\Omega H}{4})\sin(\tfrac{3\Omega H}{4}), \\ &\omega_1\times(\omega_1\times\omega_3)_xH^3 = 16(\sin^4\tfrac{\alpha}{2})(\Omega H)^3\sin(\tfrac{\Omega H}{2})\sin(\tfrac{\Omega H}{2}).\end{aligned} \tag{33}$$

Note that "$\omega_1\times(\omega_1\times\omega_3)_x$" term in Eq.(33) can be expressed by other two terms in Eq.(33):

$$\begin{aligned}&\sin(\tfrac{\Omega H}{2})\sin(\tfrac{\Omega H}{2}) \\ &= \sin(\tfrac{\Omega H}{4})\sin(\tfrac{\Omega H}{4}) + \sin(\tfrac{\Omega H}{4})\sin(\tfrac{3\Omega H}{4}).\end{aligned} \tag{34}$$

Therefore "$\omega_1\times(\omega_1\times\omega_3)_x$" term can be neglected. The second-order compensation coning algorithm should be:

$$\delta\delta\hat{\Phi} = k_{112}\omega_1\times(\omega_1\times\omega_2)H^3 + k_{312}\omega_3\times(\omega_1\times\omega_2)H^3. \tag{35}$$

We substitute Eq.(33) into Eq.(35), and use Taylor series to expand "(ΩH)" term:

$$\begin{aligned}\delta\delta\hat{\Phi} &= \sin^4\tfrac{\alpha}{2}[(k_{112}-3k_{312})(\Omega H)^5 + (-\tfrac{1}{48}k_{112} \\ &\quad + \tfrac{5}{16}k_{312})(\Omega H)^7 + \ldots].\end{aligned} \tag{36}$$

As is stated in section 2.2 (Eq.(22)), the second-order noncommutativity error compensation term of new coning algorithm should be equal to the approximation error of the traditional angular-rate coning algorithm. That is:

$$\begin{cases}(\Omega H)^5: k_{112}-3k_{312} = \frac{1}{15}, \\ (\Omega H)^7: -\frac{1}{48}k_{112} + \frac{5}{16}k_{312} = -\frac{23}{2880}.\end{cases} \tag{37}$$

The solution is:k_{112} = -1/80,k_{312} = -19/720. Then the second-order coning algorithm is achieved:

$$\delta\delta\hat{\Phi} = -\frac{1}{80}\omega_1\times(\omega_1\times\omega_2)H^3 - \frac{19}{720}\omega_3\times(\omega_1\times\omega_2)H^3. \tag{38}$$

Then the new second-order two-interval angular-rate coning algorithm is:

$$\begin{aligned}\hat{\Phi} &= \tfrac{(\omega_1+4\omega_2+\omega_3)H}{6} + \tfrac{H^2}{180}(\omega_1\times\omega_3) + \tfrac{7H^2}{45}(\omega_2\times\omega_3) \\ &\quad - \tfrac{H^3}{80}\omega_1\times(\omega_1\times\omega_2) - \tfrac{19H^3}{720}\omega_3\times(\omega_1\times\omega_2).\end{aligned} \tag{39}$$

Obviously the drift error of new algorithm is as same as traditional two-interval angular-rate coning algorithm(Eq.11), but from the Eq.(37) we can see that approximation error has been reduced greatly to the order of $(\Omega H)^9$:

$$\Phi_{\varepsilon/tru} = (\sin^4\frac{\alpha}{2})\,O(\Omega H)^9. \tag{40}$$

4 Error comparison

Algorithm 1: Traditional two-interval angular-rate coning algorithm (Eq.(11))

Algorithm 1 is based on the first-order noncommutativity error compensation model, the algorithm error (per unit time) e_1 consists of drift error e_{1d} and approximation error e_{1T}. Both them have been calculated in Eq.(21):

$$e_1 : \begin{cases} e_{1d} = \Phi_\varepsilon = \sin^2\frac{\alpha}{2}[\frac{(\Omega H)^7}{20160}]/H \approx \frac{\Omega^7 H^7}{80640}\sin^2\alpha/H, \\ e_{1T} = \sin^4\frac{\alpha}{2}[\frac{1}{15}(\Omega H)^5 - \frac{23}{2880}(\Omega H)^7 \ldots]/H. \end{cases} \quad (41)$$

Algorithm 2: Traditional three-interval angular-rate coning algorithm (Eq.(14))

Algorithm 2 is based on the first-order noncommutativity error compensation model too. With the increase of sampling number, the drift error e_{2d} of algorithm 2 has been reduced greatly (Eq.(15)). Similar to Eq.(16)-Eq.(22), the exact value of approximation error e_{2T} can be calculated. For simplicity, it can approximately be considered as e_{1T}:

$$e_2 : \begin{cases} e_{2d} = \frac{\sin^2\alpha\Omega^9 H^9}{8899200}/H, \\ e_{2T} = \sin^4\frac{\alpha}{2}[\frac{1}{15}(\Omega H)^5 + \ldots]/H. \end{cases} \quad (42)$$

Algorithm 3: New second-order two-interval angular-rate coning algorithm (Eq.(39))

Algorithm 3 is based on the second-order noncommutativity error compensation model. The approximation error e_{3T} is given in Eq.(40). From Eq.(40) we can see that the approximation error e_{3T} has been reduced greatly. But the drift error e_{3d} is unchanged (as same as algorithm 1). Hence it has higher precision than Algorithm 1 and 2.

$$e_3 : \begin{cases} e_{3d} = (\sin^2\frac{\alpha}{2})(\frac{\Omega^7 H^7}{21060})/H, \\ e_{3T} \approx \sin^4\frac{\alpha}{2}[O(\Omega H)^9 + \ldots]/H. \end{cases} \quad (43)$$

As a summarization of Eq.(41)-Eq.(43), the error analyses (per unit time) of three coning algorithms are listed in Table 1. From Eq.(41)-Eq.(43) and Table 1 we can see that the approximation error (per unit time) of traditional two-interval and three-interval coning algorithms (1, 2) are the same: the order of $\Omega^5 H^4$, which means the approximation error is proportional to the update period H. In practical strapdown inertial systems, the output data rate of a digital gyro is usually fixed by manufacturer (generally is between 10-200Hz), so the update period H is proportional to the sampling number. The approximation error of the three-interval algorithm is larger than that of the two-interval algorithm actually.

Table 1: Error analyses of three coning algorithms (per unit time)

Algo.	Φ	X-axis algorithm error (rad/s)
1	Eq.(11)	$e_1 = (\sin^2\alpha)(\frac{\Omega^7 H^6}{80640}) + (\sin^4\frac{\alpha}{2})(\frac{\Omega^5 H^4}{15})$
2	Eq.(14)	$e_2 = (\sin^4\frac{\alpha}{2})\frac{\Omega^5 H^4}{15} + (\sin^2\alpha)\frac{\Omega^9 H^8}{8899200}$
3	Eq.(39)	$e_3 = (\sin^2\alpha)(\frac{\Omega^7 H^6}{80640}) + (\sin^4\frac{\alpha}{2})O(\Omega^9 H^8)$

Fox example in an inertial system with a gyro of 20Hz output rate, the shortest update period H of the two-interval coning algorithm is 0.1s, the shortest update period H of the three-interval coning algorithm is 0.15s. When $\alpha = 1^\circ$, $\Omega = 2\pi\ rad/s$, there is:

$$e_1 \approx 1.84\times 10^{-9} rad/s, e_2 \approx 2.05\times 10^{-9} rad/s, \\ e_3 \approx 1.46\times 10^{-9} rad/s. \quad (44)$$

Nowadays in some high-precision navigation systems (e.g. long-range bomber), the bias stability of the used high quality gyro can be less than $0.005^\circ/hr \approx 2.4\text{e}-008 rad/\text{s}$. So in these high-precision systems, the approximation error of coning attitude algorithm can not be neglected compared with the sensor error. The coning attitude algorithm used in these systems still needs further improvement. It can be seen from Eq.(44) that the traditional two-interval coning algorithm (algorithm 1) error is about 7.7% of $0.005^\circ/hr$ and the traditional 3-interval coning algorithm (algorithm 2) error is about 8.5% of $0.005^\circ/hr$. But the developed two-interval algorithm (algorithm 3) error is only about 6.1% of the same sensor error.The algorithm accuracy is improved by more than 20%. So the developed coning algorithm has the certainly practical value.

5 Digital Simulations

Validation of the new second-order strapdown attitude integration algorithm is achieved in two steps: 1) To verify the error analysis of the coning algorithms given in Eq.(41)-(43) is correct; 2) To verify the advantage of the new second-order angular-rate coning algorithm with different angular rate and gyro output rate.

5.1 Verification of the correctness of error analysis given in Eq. (41)-Eq.(43)

For this 60s duration test, a classical coning motion described by Eq.(2) is used as a test input to verify the correctness of error analysis given in Eq.(41)-Eq.(43). The ideal gyro outputs in a classical coning environment are given in Eq.(31). Three coning algorithms:1, 2, and 3 are defined in section 4.

Rotation vectors calculated by algorithm 1, 2, and 3 separately, are compared with the truth value of rotation vector generated by conversion from the attitude quaternion $q(H)$ (Eq.(4)). The $q(H)$ to Φ conversion formula refers to Ref.[11](Sec. 3.2.4.5). Suppose that three coning algorithms use a same gyro with a 20Hz output data rate. So the shortest update period H of two-interval coning algorithms (1,3) is 0.1s, the update period H of algorithm 2 (three-interval) is 0.15s. Table 2 lists the error comparison (per unit time) of three coning algorithms. The unit of the algorithm error mean (per unit time) is *rad/s*.

Table 2: Error Mean comparisons (per unit time)

$\alpha = 1^\circ$	Algorithm 1 $H = 0.1s$	Algorithm 2 $H = 0.15s$	Algorithm 3 $H = 0.1s$
$\Omega = \pi/2$	4.58e-013	1.86e-012	8.91e-014
$\Omega = \pi$	2.31e-011	5.86e-011	1.14e-011
$\Omega = 2\pi$	1.82e-009	1.85e-009	1.46e-009
$\Omega = 3\pi$	2.75e-008	1.63e-008	2.49e-0084

As is seen from Table 2, the simulation results are similar to the analytical predictions given in Eq.(41)-Eq.(43) and Table 1. For example, when $\alpha = 1^\circ$, $\Omega = 2\pi$ *rad/s*, the error mean of algorithm 1, 2, and 3 is similar to the theoretical analysis in Eq.(44). These results provide confidence in the validity of the accuracy analysis for the new algorithm in Eq.(41)-Eq.(43) and Table 1.

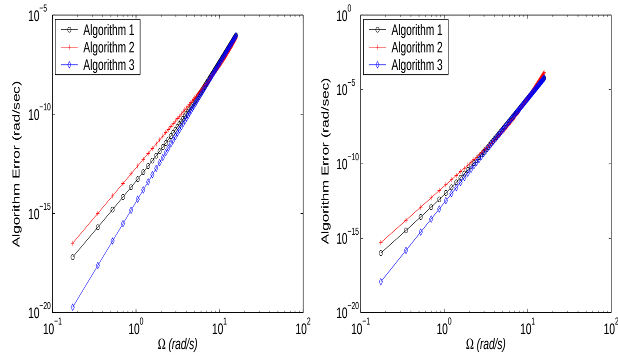
And from the column 3 of Table 2, we can see that for the most case the traditional three-interval angular-rate coning algorithm performance is worse than two-interval algorithms actually. The reason has already been given in section 4 (below the Table 1). So a new three-interval angular-rate coning algorithm with a second-order noncommutativity error compensation term should be developed in a same way to reduce the approximation error and improve the three-interval coning algorithm accuracy. For example, from Eq.(30) we can get:

$$\delta\delta\hat{\Phi} = \sum_{i=1}^{N+1}\sum_{j=2}^{N+1} K_{ij}\omega_i \times (\omega_1 \times \omega_j)H^3, \; N = 3. \qquad (45)$$

Similar to the derivation of Eq.(31)-Eq.(38), the optimal value of coefficient K can be calculated and the new second-order three-interval angular-rate coning algorithm will be developed.

5.2 Verification of the advantages for the new second-order angular-rate strapdown attitude integration algorithm

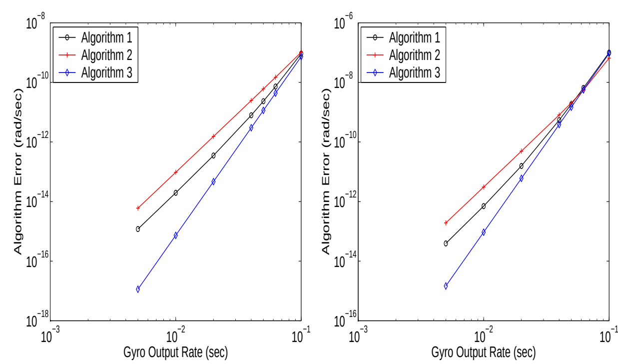
For this 60s duration test, the (per unit time) errors of three coning algorithms (1, 2, 3) with different angular oscillations frequency Ω are compared. The coning half-angle α is 1°. The simulations results are shown in Fig.1. Fig.1 is a log-log plot.



(a)h=0.05s (b)h=0.1s

Fig. 1: Algorithm Error (per unit time) comparisions with different angular oscillations frequency Ω

From Fig.1 we can see that when the angular oscillations frequency Ω increases, the error of each coning algorithm (1, 2, and 3) increases too. But if the angular oscillations frequency Ω is the same, the error of the algorithm 3 (new two-interval coning algorithm) is far smaller than that of the traditional algorithm 1 (traditional two-interval coning algorithm), no matter the gyro output rate h (i.e. subminor interval in one update period) is. And the error of algorithm 3 is smaller than that of algorithm 2 (traditional 3-interval coning algorithm) too when $h = 0.05s$ & $\Omega < 2.5\pi$ (Fig.1.*a*), and when $h = 0.1s$ & $\Omega < 1.2\pi$ (Fig.1.*b*). These results are close to the theoretical predictions given in Eq.(41-Eq.(43).



(a)$\Omega = \pi$ *rad/s* (b)$\Omega = 2\pi$ *rad/s*

Fig. 2: Algorithm Error (per unit time) comparisions with different gyro output rate h

Fig.2 is also a log-log plot. In the figure the error (per unit time) comparison of the three algorithms with different gyro output rates (i.e. subminor interval h) is shown. The coning half-angle α is 1°. The simulation time is 60s. From Fig.2 we can see that the new algorithm 3 performance is better than that of the traditional algorithm 1 in the whole gyro output rate range (no matter the angular oscillations frequency Ω is). And the algorithm 3 performance is also better than algorithm 2 in most cases (Fig.2*a*: when $\Omega = \pi$, the suitable range includes the entire general gyro output rate (10-200Hz); Fig.2*b*: when $\Omega = 2\pi$, the suitable range of gyro output rate is greater than 16Hz). These results are close to the theoretical predictions given in Eq.(41)-Eq.(43) and provide confidence in the validity of the error analysis for the new algorithm in section 4.

6 Conclusion

A second-order strapdown angular-rate attitude integration algorithm is developed for strapdown inertial navigation systems. The key contributions of the study are:

1) For the first time, the approximation error of the angular-rate coning algorithm is calculated in this paper. The calculation result shows that for the two (or more)-interval angular-rate coning algorithms, approximation error is comparable with drift error. So it can not be neglected in high-precision strapdown navigation systems.

2) A new second-order angular-rate coning algorithm with an additional second-order noncommutativity error compensation term has been developed. Comparing with the traditional coning algorithms, the new algorithm can effectively reduce the algorithm error without increasing the sampling number. Simulations have been presented to illustrate the advantages of the developed second-order angular-rate attitude integration algorithm. The new attitude integration algorithm can be applied to the high-precision strapdown inertial navigation systems, especially the navigation systems in an angular vibration environment. Beyond that, the developed second-order angular-rate coning attitude algorithm can also be applied to highly maneuverable precision-pointing spacecraft, alignment calibration for maneuvering spacecraft, and rotation inertial navigation system because Ref.[12]-[16] have demonstrated that in these cases coning motion is a nonnegligible factor.

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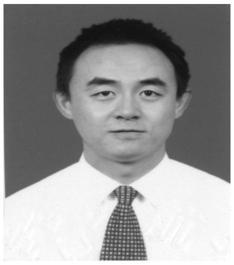

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