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A generalization of the companion of Ostrowski-like inequality and applications

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Abstract: By introducing a parameter, we establish in this paper a generalization of the companion of Ostrowski-like type integral inequality for mappings whose second derivatives belong to L^∞-spaces. Our result in special cases not only recapture known results, but also give a smaller estimator than that of the known results. Some applications to composite quadrature rules in numerical integration and to probability density functions are also given.

Keywords: Ostrowski-like type inequality; twice differentiable mappings; composite quadrature rule; probability density function.

1. Introduction

An extensive literature deals with inequalities between an integral $\frac{1}{b-a}\int_a^b f(t)dt$ and its various approximations, such as trapezoidal approximation, midpoint approximation, Simpson approximation and so on. In 1938, Ostrowski established the following interesting integral inequality (see [1]) for differentiable mappings with bounded derivatives:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a,b]$ we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In [2], Dragomir and Sofo developed the Ostrowski-like type integral inequality for twice differentiable mapping.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative f'' is in $L^\infty([a,b])$. Then for all $x \in [a,b]$ we have the following inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{2}\left(x - \frac{a+b}{2}\right)f'(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{48} + \frac{1}{3}\frac{|x-\frac{a+b}{2}|^3}{(b-a)^3}\right](b-a)^2\|f''\|_\infty. \tag{2}$$

In (2), the midpoint $x = \frac{a+b}{2}$ gives the best estimator, i.e.,

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{48}(b-a)^2\|f''\|_\infty. \tag{3}$$

In fact, we can choose $f(t) = (t-a)^2$ in (3) to prove that the constant $\frac{1}{48}$ in inequality (3) is sharp. In [3], by introducing a parameter, Xue et al. derive a new Ostrowski's integral inequality with a parameter and generalized the result of [2].

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Theorem 3. *Let $f:[a,b]\to R$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume that the second derivative f'' is in $L^\infty([a,b])$. Then we have the inequality*

$$\begin{aligned}&\left|\frac{1}{2}\left[(1-h)f(x)+(1+h)\frac{f(a)+f(b)}{2}\right]\right.\\&-\frac{1}{2}(1-h)\left(x-\frac{a+b}{2}\right)f'(x)\\&\left.-h\frac{b-a}{8}[f'(b)-f'(a)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\le\|f''\|_\infty\left[\frac{1}{3}\left|x-\frac{a+b}{2}\right|^3\right.\\&\left.+\frac{(b-a)^3}{48}(1-3h+6h^2-2h^3)\right]\end{aligned}\tag{4}$$

for all $a+h((b-a)/2)\le x\le b-h((b-a)/2)$ and $h\in[0,1]$. If we choose $x=\frac{a+b}{2}$ and $h=1-\frac{\sqrt{2}}{2}$, the right hand of (4) obtains its minimum, i.e., the following inequality is optimal:

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{\sqrt{2}}{2}f\left(\frac{a+b}{2}\right)+\left(2-\frac{\sqrt{2}}{2}\right)\frac{f(a)+f(b)}{2}\right]\right.\\&\left.-\left(1-\frac{\sqrt{2}}{2}\right)\frac{b-a}{8}[f'(b)-f'(a)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\le\frac{2-\sqrt{2}}{48}(b-a)^2\|f''\|_\infty.\end{aligned}\tag{5}$$

On the other hand, motivated by [17], Dragomir [14] proved some companions of Ostrowski's inequality, as follows:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then the following inequalities*

$$\begin{aligned}&\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\begin{cases}\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & f'\in L^\infty[a,b],\\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p, & \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1\ \textit{and} & f'\in L^p[a,b],\\ \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_1, & f'\in L^1[a,b]\end{cases}\end{aligned}\tag{6}$$

hold, for all $x\in[a,\frac{a+b}{2}]$

Recently, in [5] Alomari introduced a companion of Dragomir's generalization of Ostrowsk's inequality for absolutely continuous mappings whose first derivatives are in $L^\infty([a,b])$.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on (a,b) whose derivative is bounded on $[a,b]$. Then the inequality*

$$\begin{aligned}&\left|\left[(1-\lambda)\frac{f(x)+f(a+b-x)}{2}+\lambda\frac{f(a)+f(b)}{2}\right]\right.\\&\left.-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\left[\frac{1}{8}(2\lambda^2+(1-\lambda)^2)+2\frac{\left(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right)^2}{(b-a)^2}\right]\\&\times(b-a)\|f'\|_\infty\end{aligned}\tag{7}$$

holds, for all $\lambda\in[0,1]$ and $x\in[a+\lambda\frac{b-a}{2},\frac{a+b}{2}]$.

Recently, Liu [21] derived a companion of Ostrowski like inequality for mappings whose second derivatives belong to L^∞-spaces as

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\right.\\&-\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}\\&\left.-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\left[\frac{(\frac{a+3b}{4}-x)(x-a)^2}{3(b-a)^3}+\frac{1}{3}\frac{(\frac{a+b}{2}-x)^3}{(b-a)^3}\right]\\&\times(b-a)^2\|f''\|_\infty\end{aligned}\tag{8}$$

for all $x\in[a,\frac{a+b}{2}]$.

For other related results, the reader may see the papers [4,6–11,13,15,16,18–20,22–29,31,32] and the references therein.

Motivated by previous works [5,12,14,2,21], by introducing a parameter, we investigate in this paper a generalization of the above mentioned companion of Ostrowski-like type integral inequality (8). Our result in special cases not only recapture (8) and the inequality in [3, Corollary 3], but also give a smaller estimator than that of both (3) and (5) (see Remark 2 below). Some applications to composite quadrature rules in numerical integration and to probability density functions are also given.

2. A generalization of the companion of Ostrowski-like type integral inequality

The following generalization of the companion of Ostrowski-like type integral inequality holds:

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a,b]$ and assume* *that the second derivative f'' is in $L^\infty([a,b])$. Then the following inequality*

$$\left|\frac{1}{2}\left[(1-\lambda)\frac{f(x)+f(a+b-x)}{2}+(1+\lambda)\frac{f(a)+f(b)}{2}\right]\right.$$
$$-\frac{1}{2}\left[(1-\lambda)\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}\right.$$
$$\left.\left.+\lambda\frac{b-a}{2}\frac{f'(b)-f'(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\left[\frac{1}{48}\lambda^2(3-\lambda)+\frac{1}{3}\frac{(\frac{a+b}{2}-x)^3}{(b-a)^3}\right.$$
$$\left.+\frac{1}{3}\frac{\left[\frac{(1+\lambda)a+(3-\lambda)b}{4}-x\right]\left[x-\left(a+\lambda\frac{b-a}{2}\right)\right]^2}{(b-a)^3}\right]$$
$$\times(b-a)^2\|f''\|_\infty \tag{9}$$

holds, for all $\lambda\in[0,1]$ and $x\in\left[a+\lambda\frac{b-a}{2},\frac{a+b}{2}\right]$.

The first constant $\frac{1}{3}$ in the right hand side of (9) is sharp in the sense that it can not be replaced by a smaller one provided that $x\neq\frac{(1+\lambda)a+(3-\lambda)b}{4}$ and $x\neq a+\lambda\frac{b-a}{2}$.

Proof. Define the kernel $K(t):[a,b]\to\mathbb{R}$ by

$$K(t):=\begin{cases} t-\left(a+\lambda\frac{b-a}{2}\right), & t\in[a,x],\\ t-\frac{a+b}{2}, & t\in(x,a+b-x],\\ t-\left(b-\lambda\frac{b-a}{2}\right), & t\in(a+b-x,b]\end{cases} \tag{10}$$

for all $\lambda\in[0,1]$ and $x\in\left[a+\lambda\frac{b-a}{2},\frac{a+b}{2}\right]$.

Recall that, integrating by parts, the following identity holds: (see [5])

$$\frac{1}{b-a}\int_a^b K(t)g'(t)dt$$
$$=\left[(1-\lambda)\frac{g(x)+g(a+b-x)}{2}+\lambda\frac{g(a)+g(b)}{2}\right]$$
$$-\frac{1}{b-a}\int_a^b g(t)dt. \tag{11}$$

Now in (11) choose $g(x)=(x-\frac{a+b}{2})f'(x)$ to get

$$\frac{1}{b-a}\int_a^b K(t)\left[f'(t)+\left(t-\frac{a+b}{2}\right)f''(t)\right]dt$$
$$=(1-\lambda)\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}$$
$$+\lambda\frac{b-a}{2}\frac{f'(b)-f'(a)}{2}$$
$$-\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt. \tag{12}$$

Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt$$
$$=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt. \tag{13}$$

Also, by using (11), we get

$$\frac{1}{b-a}\int_a^b K(t)\left[f'(t)+\left(t-\frac{a+b}{2}\right)f''(t)\right]dt$$
$$=\frac{1}{b-a}\int_a^b K(t)f'(t)dt$$
$$+\frac{1}{b-a}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt$$
$$=\left[(1-\lambda)\frac{f(x)+f(a+b-x)}{2}+\lambda\frac{f(a)+f(b)}{2}\right]$$
$$-\frac{1}{b-a}\int_a^b f(t)dt$$
$$+\frac{1}{b-a}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt. \tag{14}$$

It follows from (12)–(14) that

$$\frac{1}{2(b-a)}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt$$
$$=\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[(1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right.$$
$$\left.+(1+\lambda)\frac{f(a)+f(b)}{2}\right]$$
$$+\frac{1}{2}\left[(1-\lambda)\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}\right.$$
$$\left.+\lambda\frac{b-a}{2}\frac{f'(b)-f'(a)}{2}\right]. \tag{15}$$

Now using (15) we obtain the following inequality:

$$\left|\frac{1}{2}\left[(1-\lambda)\frac{f(x)+f(a+b-x)}{2}+(1+\lambda)\frac{f(a)+f(b)}{2}\right]\right.$$
$$-\frac{1}{2}\left[(1-\lambda)\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}\right.$$
$$\left.\left.+\lambda\frac{b-a}{2}\frac{f'(b)-f'(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{\|f''\|_\infty}{2(b-a)}\int_a^b|K(t)|\left|t-\frac{a+b}{2}\right|dt. \tag{16}$$

Since $\lambda\in[0,1]$ and $x\in\left[a+\lambda\frac{b-a}{2},\frac{a+b}{2}\right]$, we have

$$I:=\int_a^b|K(t)|\left|t-\frac{a+b}{2}\right|dt$$

$$=2\int_a^x \left|t-\left(a+\lambda\frac{b-a}{2}\right)\right|\left|t-\frac{a+b}{2}\right| dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt$$

$$=2\int_a^{a+\lambda\frac{b-a}{2}}\left[\left(a+\lambda\frac{b-a}{2}\right)-t\right]\left(\frac{a+b}{2}-t\right)dt + 2\int_{a+\lambda\frac{b-a}{2}}^x\left[t-\left(a+\lambda\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2}-t\right)dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt$$

$$=\frac{1}{24}\lambda^2(3-\lambda)(b-a)^3+\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3 + \frac{[(a+3b-4x)-\lambda(b-a)]\left[(x-a)-\frac{\lambda}{2}(b-a)\right]^2}{6},$$

and referring to (16), we obtain the result (9) of Theorem 6.

The sharpness of the constant $\frac{1}{3}$ can be proved in a special case for $x=\frac{a+b}{2}$ and $\lambda=0$ (see the line behind (3)).

Remark.
(a) If we choose $\lambda=0$ in (9), we recapture (8).
(b) If we take $\lambda=1$ (then $x=\frac{a+b}{2}$) in (9), we obtain the inequality in [3, Corollary 3].

Remark. Consider

$$F(x,\lambda)=\frac{1}{48}\lambda^2(3-\lambda)+\frac{1}{3}\frac{\left(\frac{a+b}{2}-x\right)^3}{(b-a)^3} + \frac{\left[\frac{(1+\lambda)a+(3-\lambda)b}{4}-x\right]\left[x-\left(a+\lambda\frac{b-a}{2}\right)\right]^2}{3(b-a)^3}$$

for $\lambda\in[0,1]$ and $x\in\left[a+\lambda\frac{b-a}{2},\frac{a+b}{2}\right]$. Then since

$$F_x(x,\lambda)=\frac{(a+b-2x)[4x-(3-\lambda)a-(1+\lambda)b]}{4(b-a)^3}$$

and

$$F_\lambda(x,\lambda) = \frac{(a^2+b^2)\lambda(2-\lambda)-2x(b-x)-2a(x-b(1-\lambda)^2)}{8(b-a)^2},$$

we may obtain the critical points $\left(\frac{b+a}{2},1-\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{b+a}{2}-\frac{\sqrt{7}(b-a)}{14},1-\frac{2\sqrt{7}}{7}\right)$. Now

$$F_{xx}(x,\lambda)=\frac{(5-\lambda)a+(3+\lambda)b-8x}{2(b-a)^3},$$

$$F_{\lambda\lambda}(x,\lambda)=\frac{1-\lambda}{4},\quad F_{x\lambda}(x,\lambda)=\frac{2x-a-b}{4(b-a)^2}.$$

It's easy to know from the Second Derivative Test for functions of two variables (see [30]) that $F(x,\lambda)$ obtains its minimal value at $x=\frac{b+a}{2}-\frac{\sqrt{7}(b-a)}{14}$ and $\lambda=1-\frac{2\sqrt{7}}{7}$. Therefore, in (9), the point $\left(\frac{b+a}{2}-\frac{\sqrt{7}(b-a)}{14},1-\frac{2\sqrt{7}}{7}\right)$ gives the best estimator, i.e.,

$$\left|\left[\frac{\sqrt{7}}{7}\frac{f\left(\frac{b+a}{2}-\frac{\sqrt{7}(b-a)}{14}\right)+f\left(\frac{b+a}{2}+\frac{\sqrt{7}(b-a)}{14}\right)}{2} + \left(1-\frac{\sqrt{7}}{7}\right)\frac{f(a)+f(b)}{2}\right] + \frac{b-a}{2}\left[\frac{f'\left(\frac{b+a}{2}-\frac{\sqrt{7}(b-a)}{14}\right)-f'\left(\frac{b+a}{14}+\frac{\sqrt{7}(b-a)}{14}\right)}{2} - \left(\frac{1}{2}-\frac{\sqrt{7}}{7}\right)\frac{f'(b)-f'(a)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{7-2\sqrt{7}}{168}(b-a)^2\|f''\|_\infty, \tag{17}$$

the right hand side of which is smaller than that of both (3) and (5).

A new inequality of Ostrowski's type may be stated as follows:

Corollary 1. *Let f be as in Theorem 6. Additionally, if f is symmetric about the line $x=\frac{a+b}{2}$, i.e., $f(a+b-x)=f(x)$, then the following inequality*

$$\left|\frac{(1-\lambda)f(x)+(1+\lambda)f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le\left[\frac{1}{48}\lambda^2(3-\lambda)+\frac{1}{3}\frac{\left(\frac{a+b}{2}-x\right)^3}{(b-a)^3} + \frac{1}{3}\frac{\left[\frac{(1+\lambda)a+(3-\lambda)b}{4}-x\right]\left[x-\left(a+\lambda\frac{b-a}{2}\right)\right]^2}{(b-a)^3}\right] \times(b-a)^2\|f''\|_\infty \tag{18}$$

holds, for all $\lambda\in[0,1]$ and $x\in\left[a+\lambda\frac{b-a}{2},\frac{a+b}{2}\right]$.

3. Application to Composite Quadrature Rules

In [2,3], the authors utilized inequalities (3) and (4) to give estimates of composite quadrature rules which were pointed out have a markedly smaller error than that which may be obtained by the classical results. In this section, we apply our previous inequality (17) to give us estimates of new composite quadrature rules which have a further smaller error.

Theorem 7. *Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $\nu(h) := \max\{h_i : i = 1, \cdots, n\}$ and*

$$\begin{aligned}
&S(f, I_n) \\
&= \sum_{i=0}^{n-1} \left[\frac{\sqrt{7}}{7} \frac{f\left(\frac{x_{i+1}+x_i}{2} - \frac{\sqrt{7}h_i}{14}\right) + f\left(\frac{x_{i+1}+x_i}{2} + \frac{\sqrt{7}h_i}{14}\right)}{2} \right. \\
&\quad \left. + \left(1 - \frac{\sqrt{7}}{7}\right) \frac{f(x_i) + f(x_{i+1})}{2} \right] h_i \\
&\quad + \sum_{i=0}^{n-1} \frac{h_i^2}{2} \left[\frac{f'\left(\frac{x_{i+1}+x_i}{2} - \frac{\sqrt{7}h_i}{14}\right) - f'\left(\frac{x_{i+1}+x_i}{14} + \frac{\sqrt{7}h_i}{14}\right)}{2} \right. \\
&\quad \left. - \left(\frac{1}{2} - \frac{\sqrt{7}}{7}\right) \frac{f'(x_{i+1}) - f'(x_i)}{2} \right],
\end{aligned}$$

then the equality

$$\int_a^b f(x)dx = S(f, I_n) + R(f, I_n)$$

holds and the remainder $R(f, I_n)$ satisfies the estimate

$$|R(f, I_n)| \leq \frac{7 - 2\sqrt{7}}{168} \|f''\|_\infty \sum_{i=0}^{n-1} {h_i}^3. \tag{19}$$

Proof. Applying (17) on $[x_i, x_{i+1}]$, we have

$$\begin{aligned}
&\left| \int_{x_i}^{x_{i+1}} f(t)dt \right. \\
&\quad - \left[\frac{\sqrt{7}}{7} \frac{f\left(\frac{x_{i+1}+x_i}{2} - \frac{\sqrt{7}h_i}{14}\right) + f\left(\frac{x_{i+1}+x_i}{2} + \frac{\sqrt{7}h_i}{14}\right)}{2} \right. \\
&\quad \left. + \left(1 - \frac{\sqrt{7}}{7}\right) \frac{f(x_i) + f(x_{i+1})}{2} \right] h_i \\
&\quad - \frac{h_i^2}{2} \left[\frac{f'\left(\frac{x_{i+1}+x_i}{2} - \frac{\sqrt{7}h_i}{14}\right) - f'\left(\frac{x_{i+1}+x_i}{14} + \frac{\sqrt{7}h_i}{14}\right)}{2} \right. \\
&\quad \left. \left. - \left(\frac{1}{2} - \frac{\sqrt{7}}{7}\right) \frac{f'(x_{i+1}) - f'(x_i)}{2} \right] \right| \\
&\leq \frac{7 - 2\sqrt{7}}{168} {h_i}^3 \|f''\|_\infty.
\end{aligned}$$

Now summing over i from 0 to $n-1$ and utilizing the triangle inequality, we get (19).

Remark. It is obvious that inequality (19) is better than [2, inequality (19)] and the inequality in [3, Remark 3] due to a smaller error.

4. Application to probability density functions

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f : [a,b] \to [0,1]$ and with the cumulative distribution function

$$F(x) = Pr(X \leq x) = \int_a^x f(t)dt.$$

The following result holds:

Theorem 8. *With the above assumptions, we have the inequality*

$$\begin{aligned}
&\left| \frac{1}{2} \left[(1-\lambda) \frac{F(x) + F(a+b-x)}{2} + \frac{1+\lambda}{2} \right] \right. \\
&\quad - \frac{1}{2} \left[(1-\lambda) \left(x - \frac{a+b}{2}\right) \frac{f(x) - f(a+b-x)}{2} \right. \\
&\quad \left. \left. + \lambda \frac{b-a}{2} \frac{f(b) - f(a)}{2} \right] - \frac{b - E(X)}{b-a} \right| \\
&\leq \left[\frac{1}{48} \lambda^2 (3-\lambda) \frac{1}{3} \frac{\left(\frac{a+b}{2} - x\right)^3}{(b-a)^3} \right. \\
&\quad \left. + \frac{1}{3} \frac{\left[\frac{(1+\lambda)a + (3-\lambda)b}{4} - x\right] \left[x - \left(a + \lambda \frac{b-a}{2}\right)\right]^2}{(b-a)^3} + \right] \\
&\quad \times (b-a)^2 \|f'\|_\infty
\end{aligned} \tag{20}$$

for all $\lambda \in [0,1]$ and $x \in [a, \frac{a+b}{2}]$, where $E(X)$ is the expectation of X. The first constant $\frac{1}{3}$ in the right hand side of (20) is sharp in the sense that it can not be replaced by a smaller one provided that $x \neq \frac{(1+\lambda)a + (3-\lambda)b}{4}$ and $x \neq a + \lambda \frac{b-a}{2}$.

Proof. Follows by (9) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (20).

In particular, we have:

Corollary 2. *With the above assumptions, we have the inequality*

$$\begin{aligned}
&\left| \left[\frac{\sqrt{7}}{7} \frac{F\left(\frac{b+a}{2} - \frac{\sqrt{7}(b-a)}{14}\right) + F\left(\frac{b+a}{2} + \frac{\sqrt{7}(b-a)}{14}\right)}{2} \right. \right. \\
&\quad \left. + \left(1 - \frac{\sqrt{7}}{7}\right) \frac{1}{2} \right] \\
&\quad + \frac{b-a}{2} \left[\frac{f\left(\frac{b+a}{2} - \frac{\sqrt{7}(b-a)}{14}\right) - f\left(\frac{b+a}{14} + \frac{\sqrt{7}(b-a)}{14}\right)}{2} \right.
\end{aligned}$$

$$
\left. - \left(\frac{1}{2} - \frac{\sqrt{7}}{7} \right) \frac{f(b) - f(a)}{2} \right] - \frac{b - E(X)}{b - a} \Bigg|
$$
$$
\le \frac{7 - 2\sqrt{7}}{168} (b-a)^2 \|f'\|_\infty.
$$

Remark. If we choose $\lambda = 0$ in (20), we recapture [21, Theorem 4.1].

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