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Some New Algorithms for Solving a System of General Variational Inequalities

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Abstract: In this paper, we consider a new system of extended general variational inequalities involving six nonlinear operators. Using projection operator technique, we show that system of extended general variational inequalities is equivalent to a system of fixed point problems. Using this alternative equivalent formulation, some Gauss-Seidel type algorithms for solving a system of extended general variational inequalities are suggested and investigated. Convergence of these new methods is considered under some suitable conditions. Several special cases are discussed. Results obtained in this paper continue to hold for these problems. The ideas and techniques of this paper may stimulate further research in this field.

Keywords: Variational inequalities, Projection operator, Gauss-Seidel type algorithm, Convergence.
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1 Introduction

Variational inequality theory, which was introduced and considered by Stampacchia [28], provides us with a unified, innovative and general framework to study a wide class of problems, which arise in finance, economics, network analysis, transportation, elasticity, optimization and applied sciences. Variational inequalities have been generalized and extended in several directions using the novel and new techniques. For the applications and other techniques for solving variational inequalities, see $[1-31]$ and references therein.

Motivated by recent advances in this area, we introduce and consider a new system of extended general variational inequalities with six nonlinear operators. It is shown that nonconvex minimax problems can be studied via this system of extended general variational inequalities, see Example 1. This class of systems include many new and known systems of variational inequalities as special cases. Using the projection technique, we have shown that the new system of extended general variational inequalities is equivalent to fixed point problems. This alternative equivalent formulation is used to propose and investigate some new algorithms for solving a systems of variational inequalities. We would like to emphasize that new algorithms are quite different from the algorithms of Yang et al [30]. To implement the algorithms of Yang et al [30], one has to find the inverse of the operator, which is itself a difficult problem. To overcome this drawback, we suggest and analyze some new algorithms, which do not involve the inverse of operators. Convergence analysis of these new algorithms is considered under some suitable conditions. We have rewritten the equivalent formulation in a more convenient form using a suitable substitution. These equivalent formulations are used to suggest a wide class of new algorithms for solving a system of extended general variational inequalities. It is shown that these new iterative methods include several known and new methods for solving system of variational inequalities. Our results represent a refinement and improvement of the recent results of [30]. Our algorithms are much easier in implementation than algorithms in [30] and computational workload is also less than those of [30]. The interested readers are encouraged to find new, novel and innovative applications of variational inequalities and optimization problem in pure and applied sciences. The implementation of new proposed methods in this paper is another direction for further research.

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2 Preliminaries and Basic Results

Let $\mathscr{H}$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let Ω_1, Ω_2 be two closed and convex sets in $\mathscr{H}$.

For given nonlinear operators $T_1, T_2, g_1, g_2, h_1, h_2 : \mathscr{H} \to \mathscr{H}$, consider a problem of finding $x, y \in \mathscr{H} : h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ such that

$$\left.\begin{array}{l}\langle T_1x, g_1(v) - h_1(y)\rangle \geq 0, \quad \forall v \in \mathscr{H} : g_1(v) \in \Omega_1, \\ \langle T_2y, g_2(v) - h_2(x)\rangle \geq 0, \quad \forall v \in \mathscr{H} : g_2(v) \in \Omega_2.\end{array}\right\} \quad (1)$$

The system (1) is called a system of extended general variational inequalities with six operators.

We now list some special cases of the system of extended general variational inequalities (1).

I. If $g_1 = g_2 = g$, $h_1 = h_2 = h$ and $\Omega_1 = \Omega_2 = \Omega$, a closed convex set in $\mathscr{H}$, then problem (1) reduces to find $x, y \in \mathscr{H} : h(y) \in \Omega$, $h(x) \in \Omega$ such that

$$\left.\begin{array}{l}\langle T_1x, g(v) - h(y)\rangle \geq 0, \quad \forall v \in \mathscr{H} : g(v) \in \Omega, \\ \langle T_2y, g(v) - h(x)\rangle \geq 0, \quad \forall v \in \mathscr{H} : g(v) \in \Omega.\end{array}\right\} \quad (2)$$

The problem of type (2) is called a system of extended general variational inequalities with four nonlinear operators.

II. If $g_1 = h_1 = g$, $g_2 = h_2 = h$ and $\Omega_1 = \Omega_2 = \Omega$, a closed convex set in $\mathscr{H}$, then problem (1) collapse to find $x, y \in \mathscr{H} : g(y), h(x) \in \Omega$ such that

$$\left.\begin{array}{l}\langle T_1x, g(v) - g(y)\rangle \geq 0, \quad \forall v \in \mathscr{H} : g(v) \in \Omega, \\ \langle T_2y, h(v) - h(x)\rangle \geq 0, \quad \forall v \in \mathscr{H} : h(v) \in \Omega,\end{array}\right\} \quad (3)$$

is a system of general variational inequalities with four nonlinear operators.

III. If $T_1 = T_2 = T$, then problem (2) reduces to find $u \in \mathscr{H} : h(u) \in \Omega$ such that

$$\langle Tu, g(v) - h(u)\rangle \geq 0, \ \forall v \in \mathscr{H} : g(v) \in \Omega, \quad (4)$$

which is called extended general variational inequality, introduced and studied by Noor [21].

For suitable and appropriate choice of operators and spaces, one can obtain several new and known classes of variational inequalities. For recent applications, existence theory, iterative methods, sensitivity analysis and different aspects of problem (4), see [20,21,22] and references therein.

We now summarize some basic properties and related definitions which are essential in the following discussions.

Lemma 1. *Let Ω be a closed and convex set in $\mathscr{H}$. Then for a given $z \in \mathscr{H}$, $u \in \Omega$ satisfies*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in \Omega, \quad (5)$$

if and only if,

$$u = P_\Omega[z],$$

where P_Ω is the projection of $\mathscr{H}$ onto a closed and convex set Ω in $\mathscr{H}$.

It is well known that the projection operator P_Ω is nonexpansive, that is,

$$\|P_\Omega[u] - P_\Omega[v]\| \leq \|u - v\|, \ \forall \mathscr{H}. \quad (6)$$

Definition 1. *A nonlinear operator $T : \mathscr{H} \to \mathscr{H}$ is said to be:*

(i) strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in \mathscr{H}.$$

(ii) Lipschitz continuous if there exists a constant $\beta > 0$, such that

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in \mathscr{H}.$$

Note that, if T satisfies (i) and (ii), then $\alpha \leq \beta$.

Lemma 2. *[29] If $\{\delta_n\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n \ for all n \geq 0,$$

with $0 \leq \lambda_n \leq 1$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$, then $\lim_{n\to\infty} \delta_n = 0$.

Using the auxiliary principle technique of Glowinski et al [7], as developed by Noor [20,21], one can easily show that problem (1) is equivalent to that of finding $x, y \in \mathscr{H} : h_1(y) \in \Omega_1$, $h_2(x) \in \Omega_2$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1x + h_1(y) - g_1(x), g_1(v) - h_1(y)\rangle \geq 0, \\ \langle \rho_2 T_2y + h_2(x) - g_2(y), g_2(v) - h_2(x)\rangle \geq 0,\end{array}\right\}. \quad (7)$$

where $\forall v \in \mathscr{H}, g_1(v) \in \Omega_1, g_2(v) \in \Omega_2$, $\rho_1 > 0$ and $\rho_2 > 0$ are constants.

We use this equivalent formulation to develop some new iterative methods for solving the system of extended general variational inequalities and its variant forms.

3 Applications

In this section, it is shown that the optimality conditions of nonconvex minimax problem can be studied via system of extended general variational inequalities (1). For this purpose, we recall the following concepts.

Definition 2. *[21] Let Ω be any set in $\mathscr{H}$. The set Ω is said to be hg-convex, if there exist functions $g, h : \mathscr{H} \to \mathscr{H}$ such that*

$$h(u) + t(g(v) - h(u)) \in \Omega,$$
$$\forall u, v \in \mathscr{H} : h(u), g(v) \in \Omega, t \in [0, 1].$$

Clearly every convex set is hg-convex, but the converse is not true, see [5]. hg-convex sets are also called nonconvex sets. For the properties and other aspects of hg-convex sets, see Cristescu and Lupsa [5] and references therein. If $h = g$, then the hg-convex set Ω is called the g-convex set, which was introduced by Noor [18] in 1988 implicitly. For other properties of the g-convex set, see Youness [31] .

Definition 3.[21] *The function $F:\Omega\to\mathscr{H}$ is said to be hg-convex, if there exist two functions h,g such that*

$$F(h(u)+t(g(v)-h(u)))\le(1-t)F(h(u))+F(g(v)),$$

$$\forall u,v\in\mathscr{H}:h(u),g(v)\in\Omega,t\in[0,1].$$

Clearly every convex function is *hg*-convex, but the converse is not true, see [10,20]. In general *hg*-convex functions are nonconvex functions. For basic properties of *hg*-convex (nonconvex) functions, see [20,21,22]. For $g=h$, Definition 3 is due to Youness [31].

It is known that [21] the minimum of a differentiable *hg*-convex function on the *hg*-convex set Ω in $\mathscr{H}$ can be characterized by the extended general variational inequality (4). For the sake of completeness, we include it without proof.

Lemma 3.[21] *Let $F:\Omega\to\mathscr{H}$ be a differentiable nonconvex function. Then $u\in\mathscr{H}:h(u)\in\Omega$ is the minimum of nonconvex function F on Ω, if and only if, $u\in\mathscr{H}:h(u)\in\Omega$ satisfies the inequality*

$$\left\langle F'(h(u)),g(v)-h(u)\right\rangle\ge 0,\ \forall v\in\mathscr{H}:g(v)\in\Omega,$$

where $F'(\cdot)$ is the differential of F at $h(u)\in\Omega$.

Lemma 3 implies that *hg*-convex programming problem can be studied via extended general variational inequality (4) with $Tu=F'(h(u))$. In a similar way, one can show that the extended general variational inequality (4) is the Fritz-John condition of the inequality constrained optimization problem.

We now show that the nonconvex minimax problem can be characterized by a system of extended general variational inequalities of the type (1). This is the main motivation of the next example.

*Example 1.*Consider the following nonconvex minimax problem as

$$\min_{x\in\mathscr{H}:h_2(x)\in\Omega_2}\left\{\max_{y\in\mathscr{H}:h_1(y)\in\Omega_1}f(h_2(x),h_1(y))\right\},\tag{8}$$

where f is twice differentiable in $\mathscr{H}\times\mathscr{H}$. The solution of (8) is equivalent to the saddle point of $f(h_2(x),h_1(y))$, that is, a point $x^*,y^*\in\mathscr{H}:h_1(y^*)\in\Omega_1,h_2(x^*)\in\Omega_2$ satisfies

$$f(h_2(x^*),h_1(y))\le f(h_2(x^*),h_1(y^*))\le f(h_2(x),h_1(y^*)),$$

for all $x,y\in\mathscr{H}:h_1(y)\in\Omega_1,h_2(x)\in\Omega_2$.

Using the technique of Bazaraa et al [2], one can show that x^*,y^* is a saddle point of $f(h_2(x),h_1(y))$, if and only if, it satisfies

$$\left.\begin{array}{l}\left\langle\rho_1\nabla_x f(x^*,y^*),g_1(x)-h_1(y^*)\right\rangle\ge 0,\\ \left\langle\rho_2\nabla_y f(x^*,y^*),g_2(x)-h_2(x^*)\right\rangle\ge 0,\end{array}\right\}\tag{9}$$

where $\forall x\in\mathscr{H},g_1(x)\in\Omega_1,g_2(x)\in\Omega_2$, $\rho_1>0$ and $\rho_2>0$ are constants.

Clearly problem (9) is a special case of (1) with $\nabla_x f(x^*,y^*)=T_1x$ and $\nabla_y f(x^*,y^*)=T_2y$.

4 Main Results

In this section, we first show that system of extended general variational inequalities (7) is equivalent to a system of fixed point problems. This alternative equivalent formulation is used to suggest algorithms for solving problem (7), using the technique of Noor and Noor [24].

Lemma 4.*The system of extended general variational inequalities (7) has a solution, $x,y\in\mathscr{H}:h_1(y)\in\Omega_1\subset g_1(\mathscr{H}),h_1(\mathscr{H})$ and $h_2(x)\in\Omega_2\subset g_2(\mathscr{H}),h_2(\mathscr{H})$, if and only if, $x,y\in\mathscr{H}:h_1(y)\in\Omega_1,h_2(x)\in\Omega_2$ satisfies the relations*

$$h_1(y)=P_{\Omega_1}[g_1(x)-\rho_1T_1x],\tag{10}$$

$$h_2(x)=P_{\Omega_2}[g_2(y)-\rho_2T_2y],\tag{11}$$

where $\rho_1>0$ and $\rho_2>0$ are constants.

Lemma 4 implies that the system (7) is equivalent to the fixed point problems (10) and (11). This alternative equivalent formulation is very useful from numerical and theoretical point of view. Using the fixed point formulations (10) and (11), we suggest and analyze some iterative algorithms.

We can rewrite (10) and (11) in the following equivalent forms:

$$y=(1-\beta_n)y+\beta_n\left\{y-h_1(y)+P_{\Omega_1}[g_1(x)-\rho_1T_1x]\right\},\tag{12}$$

$$x=(1-\alpha_n)x+\alpha_n\left\{x-h_2(x)+P_{\Omega_2}[g_2(y)-\rho_2T_2y]\right\},\tag{13}$$

where $0\le\alpha_n,\beta_n\le 1$ for all $n\ge 0$.

This alternative formulation is used to suggest the following algorithms for solving system of extended general variational inequalities (7) and its variant forms.

Algorithm 1*For given $x_0,y_0\in\mathscr{H}:h_1(y_0)\in\Omega_1$ and $h_2(x_0)\in\Omega_2$, find x_{n+1} and y_{n+1} by the iterative schemes*

$$\begin{aligned}&y_{n+1}\\&=(1-\beta_n)y_n+\beta_n\left\{y_n-h_1(y_n)+P_{\Omega_1}[g_1(x_n)-\rho_1T_1x_n]\right\},\\&x_{n+1}\\&=(1-\alpha_n)x_n+\alpha_n\left\{x_n-h_2(x_n)+P_{\Omega_2}[g_2(y_{n+1})-\rho_2T_2y_{n+1}]\right\},\end{aligned}\tag{14}$$

where $0\le\alpha_n,\beta_n\le 1$ for all $n\ge 0$.

Algorithm 1 can be viewed as a Gauss-Seidel method for solving a system of extended general variational inequalities (7).

We now discuss some special cases of Algorithm 1.

I.If $g_1=g_2=g$, $h_1=h_2=h$ and $\Omega_1=\Omega_2=\Omega$, then Algorithm 1 reduces to following projection algorithm for solving the system (2).

Algorithm 2*For given $x_0,y_0\in\mathscr{H}:h(x_0),h(y_0)\in\Omega$, find x_{n+1} and y_{n+1} by the iterative schemes*

$$\begin{aligned}&y_{n+1}\\&=(1-\beta_n)y_n+\beta_n\{y_n-h(y_n)+P_\Omega[g(x_n)-\rho_1T_1x_n]\},\\&x_{n+1}\\&=(1-\alpha_n)x_n+\alpha_n\{x_n-h(x_n)+P_\Omega[g(y_{n+1})-\rho_2T_2y_{n+1}]\},\end{aligned}$$

where $0\le\alpha_n,\beta_n\le 1$ for all $n\ge 0$.

II. If $h=g$, then Algorithm 2 reduces to the following algorithm for solving system of extended general variational inequalities.

Algorithm 3 *For given $x_0, y_0 \in \mathscr{H} : g(x_0), g(y_0) \in \Omega$, compute sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes*

$$\begin{aligned} & y_{n+1} \\ & = (1-\beta_n) y_n + \beta_n \{y_n - g(y_n) + P_{\Omega}[g(x_n) - \rho_1 T_1 x_n]\}, \\ & x_{n+1} \\ & = (1-\alpha_n) x_n + \alpha_n \{x_n - g(x_n) + P_{\Omega}[g(y_{n+1}) - \rho_2 T_2 y_{n+1}]\}, \end{aligned}$$

where $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 0$.

For suitable and appropriate choice of operators and spaces, one can obtain several new and known iterative methods for solving system of extended general variational inequalities and related problems. It has been shown [23] that the problem (1) has a solution under some suitable conditions.

We now investigate the convergence analysis of Algorithm 1. This is the main motivation of our next result.

Theorem 4. *Let operators $T_1, T_2, g_1, g_2, h_1, h_2 : \mathscr{H} \to \mathscr{H}$ be strongly monotone with constants $\alpha_{T_1} > 0$, $\alpha_{T_2} > 0$, $\alpha_{g_1} > 0$, $\alpha_{g_2} > 0$, $\alpha_{h_1} > 0$, $\alpha_{h_2} > 0$ and Lipschitz continuous with constants $\beta_{T_1} > 0$, $\beta_{T_2} > 0$, $\beta_{g_1} > 0$, $\beta_{g_2} > 0$, $\beta_{h_1} > 0$, $\beta_{h_2} > 0$ respectively. If following conditions hold:*

(i) $\theta_{T_1} = \sqrt{1 - 2\rho_1 \alpha_{T_1} + \rho_1^2 \beta_{T_1}^2} < 1.$

(ii) $\theta_{T_2} = \sqrt{1 - 2\rho_2 \alpha_{T_2} + \rho_2^2 \beta_{T_2}^2} < 1.$

(iii) $0 \leq \alpha_n, \beta_n \leq 1$ *for all* $n \geq 0$,

$$\begin{aligned} \alpha_n (1-\theta_{h_2}) - \beta_n \left(\theta_{g_1} + \theta_{T_1}\right) &\geq 0 \\ \beta_n (1-\theta_{h_1}) &\geq 0 \\ \alpha_n \left(\theta_{g_2} + \theta_{T_2}\right) &\geq 0, \end{aligned}$$

such that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\alpha_n (1-\theta_{h_2}) - \beta_n (\theta_{g_1} + \theta_{T_1})\right) &= \infty \\ \sum_{n=0}^{\infty} \beta_n (1-\theta_{h_1}) &= \infty \\ \sum_{n=0}^{\infty} \alpha_n (\theta_{g_2} + \theta_{T_2}) &= \infty, \end{aligned}$$

where

$$\theta_{g_1} = \sqrt{1 - 2\alpha_{g_1} + \beta_{g_1}^2}, \quad \theta_{g_2} = \sqrt{1 - 2\alpha_{g_2} + \beta_{g_2}^2},$$

and

$$\theta_{h_1} = \sqrt{1 - 2\alpha_{h_1} + \beta_{h_1}^2}, \quad \theta_{h_2} = \sqrt{1 - 2\alpha_{h_2} + \beta_{h_2}^2},$$

then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 1 converge to x and y respectively.

Proof. Let $x, y \in \mathscr{H} : h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ be a solution of (7). Then from (13) and (14), we have

$$\begin{aligned} & \|x_{n+1} - x\| \\ & = \| (1-\alpha_n) x_n + \alpha_n \{x_n - h_2(x_n) \\ & + P_{\Omega_2}[g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}]\} \\ & - (1-\alpha_n) x - \alpha_n \{x - h_2(x) + P_{\Omega_2}[g_2(y) - \rho_2 T_2 y]\} \| \\ & \leq (1-\alpha_n) \|x_n - x\| + \alpha_n \|x_n - x - (h_2(x_n) - h_2(x))\| \\ & + \alpha_n \left\| P_{\Omega_2}[g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}] - P_{\Omega_2}[g_2(y) - \rho_2 T_2 y] \right\| \\ & \leq (1-\alpha_n) \|x_n - x\| + \alpha_n \|x_n - x - (h_2(x_n) - h_2(x))\| \\ & + \alpha_n \|y_{n+1} - y - (g_2(y_{n+1}) - g_2(y))\| \\ & + \alpha_n \|y_{n+1} - y - \rho_2 (T_2 y_{n+1} - T_2 y)\|. \end{aligned} \tag{15}$$

Since operator T_2 is strongly monotone and Lipschitz continuous with constants $\alpha_{T_2} > 0$ and $\beta_{T_2} > 0$, respectively. Then it follows that

$$\begin{aligned} & \|y_{n+1} - y - \rho_2 (T_2 y_{n+1} - T_2 y)\|^2 \\ & = \|y_{n+1} - y\|^2 - 2\rho_2 \langle T_2 y_{n+1} - T_2 y, y_{n+1} - y \rangle \\ & + \|T_2 y_{n+1} - T_2 y\|^2 \\ & \leq \left(1 - 2\rho_2 \alpha_{T_2} + \rho_2^2 \beta_{T_2}^2\right) \|y_{n+1} - y\|^2. \end{aligned} \tag{16}$$

In a similar way, we have

$$\begin{aligned} & \|x_n - x - (h_2(x_n) - h_2(x))\|^2 \\ & \leq \left(1 - 2\alpha_{h_2} + \beta_{h_2}^2\right) \|x_n - x\|^2, \end{aligned} \tag{17}$$

and

$$\begin{aligned} & \|y_{n+1} - y - (g_2(y_{n+1}) - g_2(y))\|^2 \\ & \leq \left(1 - 2\alpha_{g_2} + \beta_{g_2}^2\right) \|y_{n+1} - y\|^2, \end{aligned} \tag{18}$$

where we have used the strongly monotonicity and Lipschitz continuity of operators g_2, h_2 with constants $\alpha_{g_2} > 0$, $\alpha_{h_2} > 0$ and $\beta_{g_2} > 0$, $\beta_{h_2} > 0$, respectively.

Combining (15) – (18), we obtain

$$\begin{aligned} & \|x_{n+1} - x\| \\ & \leq (1-\alpha_n) \|x_n - x\| + \alpha_n \sqrt{1 - 2\alpha_{h_2} + \beta_{h_2}^2} \|x_n - x\| \\ & + \alpha_n \sqrt{1 - 2\alpha_{g_2} + \beta_{g_2}^2} \|y_{n+1} - y\| \\ & + \alpha_n \sqrt{1 - 2\rho_2 \alpha_{T_2} + \rho_2^2 \beta_{T_2}^2} \|y_{n+1} - y\| \\ & = \left(1 - \alpha_n (1 - \theta_{h_2})\right) \|x_n - x\| \\ & + \alpha_n (\theta_{g_2} + \theta_{T_2}) \|y_{n+1} - y\|. \end{aligned} \tag{19}$$

Similarly, using strongly monotonicity and Lipschitz continuity of operators T_1, g_1, h_1 with constants $\alpha_{T_1} > 0, \alpha_{g_1} > 0, \alpha_{h_1} > 0$ and $\beta_{T_1} > 0, \beta_{g_1} > 0, \beta_{h_1} > 0$, respectively. From (12) and (14), we have

$$\begin{aligned} & \|y_{n+1} - y\| \\ & = \| (1-\beta_n) y_n + \beta_n \{y_n - h_1(y_n) \\ & + P_{\Omega_1}[g_1(x_n) - \rho_1 T_1 x_n]\} \\ & - (1-\beta_n) y - \beta_n \{y - h_1(y) + P_{\Omega_1}[g_1(x) - \rho_1 T_1 x]\} \| \\ & \leq (1-\beta_n) \|y_n - y\| + \beta_n \|y_n - y - (h_1(y_n) - h_1(y))\| \\ & + \beta_n \left\| P_{\Omega_1}[g_1(x_n) - \rho_1 T_1 x_n] - P_{\Omega_1}[g_1(x) - \rho_1 T_1 x] \right\| \end{aligned}$$

$$\begin{aligned}
&\|y_{n+1}-y\| \\
&\leq (1-\beta_n)\|y_n-y\| + \beta_n \|y_n - y - (h_1(y_n) - h_1(y))\| \\
&+\beta_n \|x_n - x - (g_1(x_n) - g_1(x))\| \\
&+\beta_n \|x_n - x - \rho_1 (T_1 x_n - T_1 x)\| \\
&\leq (1-\beta_n)\|y_n - y\| + \beta_n \theta_{h_1} \|y_n - y\| \\
&+\beta_n \theta_{g_1} \|x_n - x\| + \beta_n \theta_{T_1} \|x_n - x\| \\
&= (1-\beta_n(1-\theta_{h_1}))\|y_n - y\| + \beta_n(\theta_{g_1}+\theta_{T_1})\|x_n - x\|. \quad (20)
\end{aligned}$$

Adding (19) and (20), we have

$$\begin{aligned}
&\|x_{n+1}-x\| + \|y_{n+1}-y\| \\
&\leq (1-\alpha_n(1-\theta_{h_2}))\|x_n - x\| + \alpha_n(\theta_{g_2}+\theta_{T_2})\|y_{n+1}-y\| \\
&+(1-\beta_n(1-\theta_{h_1}))\|y_n - y\| \\
&+\beta_n(\theta_{g_1}+\theta_{T_1})\|x_n - x\| \\
&= (1-\alpha_n(1-\theta_{h_2}) + \beta_n(\theta_{g_1}+\theta_{T_1}))\|x_n - x\| \\
&+\alpha_n(\theta_{g_2}+\theta_{T_2})\|y_{n+1}-y\| + (1-\beta_n(1-\theta_{h_1}))\|y_n - y\|.
\end{aligned}$$

From which, we have

$$\begin{aligned}
&\|x_{n+1}-x\| + (1-\alpha_n(\theta_{g_2}+\theta_{T_2}))\|y_{n+1}-y\| \\
&\leq (1-\alpha_n(1-\theta_{h_2}) + \beta_n(\theta_{g_1}+\theta_{T_1}))\|x_n - x\| \\
&+(1-\beta_n(1-\theta_{h_1}))\|y_n - y\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-x\| + \nu\|y_{n+1}-y\| \\
&\leq \max(\nu_1,\nu_2)(\|x_n - x\| + \|y_n - y\|) \\
&= \theta(\|x_n - x\| + \|y_n - y\|), \quad (21)
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \max(\nu_1,\nu_2) \\
\nu_1 &= 1-(\alpha_n(1-\theta_{h_2}) - \beta_n(\theta_{g_1}+\theta_{T_1})) \\
\nu_2 &= 1-(\beta_n(1-\theta_{h_1})) \\
\nu &= 1-\alpha_n(\theta_{g_2}+\theta_{T_2}).
\end{aligned}$$

From assumption (iii), we have $\theta < 1$. Thus, using Lemma 2, it follows from (21) that

$$\lim_{n\to\infty}[\|x_{n+1}-x\| + \nu\|y_{n+1}-y\|] = 0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x\| = 0 \text{ and } \lim_{n\to\infty}\|y_{n+1}-y\| = 0.$$

This is the desired result.

Using Lemma 4, one can easily show that $x,y \in \mathscr{H} : h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ is a solution of (7) if and only if, $x,y \in \mathscr{H} : h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ satisfies

$$h_1(y) = P_{\Omega_1}[z] \quad (22)$$
$$h_2(x) = P_{\Omega_2}[w] \quad (23)$$
$$z = g_1(x) - \rho_1 T_1 x \quad (24)$$
$$w = g_2(y) - \rho_2 T_2 y. \quad (25)$$

This alternative formulation can be used to suggest and analyze the following iterative methods for solving the system (7).

Algorithm 5_For given_ $x_0, y_0 \in \mathscr{H} : h_1(y_0) \in \Omega_1, h_2(x_0) \in \Omega_2$ _find_ x_{n+1} _and_ y_{n+1} _by the iterative schemes_

$$y_{n+1} = (1-\beta_n)y_n + \beta_n\{y_n - h_1(y_n) + P_{\Omega_1}[z_n]\} \quad (26)$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{x_n - h_2(x_n) + P_{\Omega_2}[w_n]\} \quad (27)$$
$$z_n = g_1(x_n) - \rho_1 T_1 x_n \quad (28)$$
$$w_n = g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}, \quad (29)$$

where $0 \leq \alpha_n, \beta_n \leq 1$ _for all_ $n \geq 0$.

For appropriate and suitable choice of operators and spaces, one can obtain several new and known iterative methods for solving system of extended general variational inequalities and related optimization problems.

We now consider the convergence analysis of Algorithm 5, using the technique of Theorem 4. For the sake of completeness and to convey an idea, we include all the details.

Theorem 6._Let operators_ $T_1, T_2, g_1, g_2, h_1, h_2 : \mathscr{H} \to \mathscr{H}$ _be strongly monotone with constants_ $\alpha_{T_1} > 0$, $\alpha_{T_2} > 0$, $\alpha_{g_1} > 0$, $\alpha_{g_2} > 0$, $\alpha_{h_1} > 0$, $\alpha_{h_2} > 0$ _and Lipschitz continuous with constants_ $\beta_{T_1} > 0$, $\beta_{T_2} > 0$, $\beta_{g_1} > 0$, $\beta_{g_2} > 0$, $\beta_{h_1} > 0$, $\beta_{h_2} > 0$ _respectively. If following conditions hold:_

$(i)\theta_{T_1} = \sqrt{1-2\rho_1\alpha_{T_1}+\rho_1^2\beta_{T_1}^2} < 1.$

$(ii)\theta_{T_2} = \sqrt{1-2\rho_2\alpha_{T_2}+\rho_2^2\beta_{T_2}^2} < 1.$

$(iii) 0 \leq \alpha_n, \beta_n \leq 1$ _for all_ $n \geq 0$,

$$\begin{aligned}
\alpha_n(1-\theta_{h_2}) - \beta_n\left(\theta_{g_1}+\theta_{T_1}\right) &\geq 0 \\
\beta_n(1-\theta_{h_1}) &\geq 0 \\
\alpha_n\left(\theta_{g_2}+\theta_{T_2}\right) &\geq 0,
\end{aligned}$$

such that

$$\begin{aligned}
\sum_{n=0}^{\infty}(\alpha_n(1-\theta_{h_2}) - \beta_n(\theta_{g_1}+\theta_{T_1})) &= \infty \\
\sum_{n=0}^{\infty}\beta_n(1-\theta_{h_1}) &= \infty \\
\sum_{n=0}^{\infty}\alpha_n(\theta_{g_2}+\theta_{T_2}) &= \infty,
\end{aligned}$$

where

$$\theta_{g_1} = \sqrt{1-2\alpha_{g_1}+\beta_{g_1}^2},\ \theta_{g_2} = \sqrt{1-2\alpha_{g_2}+\beta_{g_2}^2},$$

and

$$\theta_{h_1} = \sqrt{1-2\alpha_{h_1}+\beta_{h_1}^2},\ \theta_{h_2} = \sqrt{1-2\alpha_{h_2}+\beta_{h_2}^2},$$

then sequences $\{x_n\}$ _and_ $\{y_n\}$ _obtained from Algorithm 5 converge to_ x _and_ y _respectively._

*Proof.*Let $x,y \in \mathscr{H} : h_1(y) \in \Omega_1, h_2(x) \in \Omega_2$ be a solution of (7). Then from (17), (23) and (27), we have

$$\begin{aligned} &\|x_{n+1} - x\| \\ &\leq (1-\alpha_n)\|x_n - x\| + \alpha_n \|x_n - x - (h_2(x_n) - h_2(x))\| \\ &+\alpha_n \left\|P_{\Omega_2}[w_n] - P_{\Omega_2}[w]\right\| \\ \leq& (1-\alpha_n)\|x_n - x\| + \alpha_n\theta_{h_2}\|x_n - x\| + \alpha_n\|w_n - w\| \\ =& \left(1-\alpha_n\left(1-\theta_{h_2}\right)\right)\|x_n - x\| + \alpha_n\|w_n - w\|. \end{aligned} \tag{30}$$

Similarly, from (20), (22) and (26), we have

$$\begin{aligned} &\|y_{n+1} - y\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n \|y_n - y - (h_1(y_n) - h_1(y))\| \\ &+\beta_n \left\|P_{\Omega_1}[z_n] - P_{\Omega_1}[z]\right\| \\ \leq& (1-\beta_n)\|y_n - y\| + \beta_n\theta_{h_1}\|y_n - y\| + \beta_n\|z_n - z\| \\ =& \left(1-\beta_n\left(1-\theta_{h_1}\right)\right)\|y_n - y\| + +\beta_n\|z_n - z\|. \end{aligned} \tag{31}$$

From (16), (18), (25) and (29), we have

$$\begin{aligned} &\|w_n - w\| \\ &= \|g_2(y_{n+1}) - \rho_2 T_2 y_{n+1} - g_2(y) + \rho_2 T_2 y\| \\ &\leq \|y_{n+1} - y - (g_2(y_{n+1}) - g_2(y))\| \\ &+ \|y_{n+1} - y - \rho_2(T_2 y_{n+1} - T_2 y)\| \\ &\leq (\theta_{g_2} + \theta_{T_2})\|y_{n+1} - y\|. \end{aligned} \tag{32}$$

Similarly, from (20), (24) and (28), we have

$$\begin{aligned} &\|z_n - z\| \\ &= \|g_1(x_n) - \rho_1 T_1 x_n - g_1(x) + \rho_1 T_1 x\| \\ \leq& \|x_n - x - (g_1(x_n) - g_1(x))\| + \|x_n - x - \rho_1(T_1 x_n - T_1 x)\| \\ \leq& (\theta_{g_1} + \theta_{T_1})\|x_n - x\|. \end{aligned} \tag{33}$$

Combining (30), (32) and (31), (33), we have

$$\begin{aligned} &\|x_{n+1} - x\| \\ &\leq \left(1-\alpha_n\left(1-\theta_{h_2}\right)\right)\|x_n - x\| + \alpha_n(\theta_{g_2} + \theta_{T_2})\|y_{n+1} - y\| \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\|y_{n+1} - y\| \\ &\leq \left(1-\beta_n\left(1-\theta_{h_1}\right)\right)\|y_n - y\| + \beta_n(\theta_{g_1} + \theta_{T_1})\|x_n - x\| \end{aligned} \tag{35}$$

Adding (34) and (35), we have

$$\begin{aligned} &\|x_{n+1} - x\| + \|y_{n+1} - y\| \\ \leq& \left(1-\alpha_n\left(1-\theta_{h_2}\right)\right)\|x_n - x\| + \alpha_n(\theta_{g_2} + \theta_{T_2})\|y_{n+1} - y\| \\ &+ \left(1-\beta_n\left(1-\theta_{h_1}\right)\right)\|y_n - y\| + \beta_n(\theta_{g_1} + \theta_{T_1})\|x_n - x\| \\ =& \left(1-\alpha_n\left(1-\theta_{h_2}\right) + \beta_n(\theta_{g_1} + \theta_{T_1})\right)\|x_n - x\| \\ &+\alpha_n(\theta_{g_2} + \theta_{T_2})\|y_{n+1} - y\| \\ &+ \left(1-\beta_n\left(1-\theta_{h_1}\right)\right)\|y_n - y\|. \end{aligned}$$

From which, we have

$$\begin{aligned} &\|x_{n+1} - x\| + (1-\alpha_n(\theta_{g_2} + \theta_{T_2}))\|y_{n+1} - y\| \\ \leq& \left(1-\alpha_n\left(1-\theta_{h_2}\right) + \beta_n(\theta_{g_1} + \theta_{T_1})\right)\|x_n - x\| \\ &+ \left(1-\beta_n\left(1-\theta_{h_1}\right)\right)\|y_n - y\|, \end{aligned}$$

which implies that

$$\begin{aligned} &\|x_{n+1} - x\| + \nu\|y_{n+1} - y\| \\ &\leq \max(\nu_1, \nu_2)(\|x_n - x\| + \|y_n - y\|) \\ =& \theta(\|x_n - x\| + \|y_n - y\|), \end{aligned} \tag{36}$$

where

$$\begin{aligned} \theta &= \max(\nu_1, \nu_2) \\ \nu_1 &= 1 - \left(\alpha_n\left(1-\theta_{h_2}\right) - \beta_n(\theta_{g_1} + \theta_{T_1})\right) \\ \nu_2 &= 1 - \beta_n\left(1-\theta_{h_1}\right) \\ \nu &= 1 - \alpha_n(\theta_{g_2} + \theta_{T_2}). \end{aligned}$$

From assumption (iii), we have $\theta < 1$. Thus, using Lemma 2, it follows from (36) that

$$\lim_{n\to\infty}[\|x_{n+1} - x\| + \nu\|y_{n+1} - y\|] = 0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x\| = 0,$$

and

$$\lim_{n\to\infty}\|y_{n+1} - y\| = 0.$$

This is the required result.

5 Conclusion

In this paper, we have considered a new system of extended general variational inequalities. It is shown that the optimality conditions of the differentiable nonconvex minimax problem on the nonconvex sets can be characterized by a new system of extended general variational inequalities. We have proved that the system of extended general variational inequalities is equivalent to systems of fixed point problems. This equivalent formulation has been used to propose and analyze several Gauss-Seidel type algorithms for solving system of extended general variational inequalities. Convergence of these new Gauss-Seidel type algorithms is investigated under some suitable conditions. Several special cases are also discussed. The implementation of these algorithms and their comparison with other techniques need further research. The idea and technique of this paper may motivate for further research in this area.

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