

# Strong Separation Axioms in Supra Topological Ordered Spaces

M. E. El-Shafei<sup>1</sup>, M. Abo-Elhamayel<sup>1</sup> and T. M. Al-shami<sup>2,\*</sup>

<sup>1</sup> Department of Mathematics, Faculty of Science, Mansoura University, Dakahlia, Egypt

<sup>2</sup> Department of Mathematics, Faculty of Education, Sana'a University, Sana'a, Yemen

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**Abstract:** In this paper, we introduce new separation axioms called strong supra regularly ordered spaces, strong supra normally ordered spaces and strong supra  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ) on supra topological ordered spaces and investigate the main properties of them. The relationships among these new separation axioms are studied and illustrated with the help of examples. Also, we point out the equivalent between  $T_1$ -ordered and strong  $T_1$ -ordered spaces which are introduced in [10].

**Keywords:** Strong supra  $T_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ), Strong supra regularly ordered spaces and strong supra normally ordered spaces.

## 1 Introduction

In 1965, Nachbin [11] studied topological ordered spaces by adding a partial ordered relation to a topological space. In 1968, McCartan [10] introduced  $T_i$ -separation axioms ( $i = 1, 2, 3, 4$ ) in topological ordered spaces and investigated some properties. Also, he obtained strong order separation axioms by replacing neighbourhood with open neighbourhood. Mashhour et al. [9] introduced a concept of supra topological spaces by dropping only the intersection condition in 1983. In 1991, Arya and Gupta [4] introduced and studied some new separation axioms in topological ordered spaces, namely semi  $T_1$ -ordered and semi  $T_2$ -ordered. In 2002, Kumar [8] introduced a concept of homeomorphism maps between topological ordered spaces. In 2004, Das [5] presented some ordered separation axioms in ordered spaces and supra topological ordered spaces. A concept of supra homeomorphism maps [1] in supra topological ordered spaces are studied in detail. By using a concept of supra  $R$ -open sets [6], El-Shafei et al. [7] introduced and investigated supra  $R$ -homeomorphism maps between supra topological ordered spaces in 2017. Recently, Abo-elhamayel and Al-shami [2] introduced new separation axioms called  $T_{ci}$  ( $i = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2$ ) by utilizing limit points of the closed sets with respect to the topological ordered spaces.

In this paper, we introduce new separation axioms called

strong supra regularly spaces, strong supra normally spaces and strong supra  $T_i$ -ordered spaces, for  $i = 0, 1, 2, 3, 4$ , on supra topological ordered spaces and investigate several properties for them. In particular, we illustrate their relationships with supra  $T_i$ -ordered spaces [5] and supra  $T_i$ -spaces [8] ( $i = 0, 1, 2$ ). Apart from that, McCartan's assertion [9] which state that every strong  $T_i$ -order space is a  $T_i$ -order space, for  $i = 0, 1, 2, 3, 4$ , is modified in the cases of  $i = 0, 1$ . At the end, we present a notion of supra topological ordered subspaces and verify that the property of being strong supra  $T_i$ -ordered spaces is hereditary, for  $i = 0, 1, 2$ .

## 2 Preliminaries

Throughout this paper, A triple  $(X, \mu, \preceq)$ , where  $\mu$  is a supra topology on  $X$  and  $\preceq$  is a partial order relation on  $X$ , is said to be a supra topological ordered space. A diagonal relation is denoted by  $\Delta$ .

**Definition 2.1.**[11] Let  $(X, \preceq)$  be a partially ordered set. Then

(i)  $i(a) = \{b \in X : a \preceq b\}$  and  $d(a) = \{b \in X : b \preceq a\}$ , for each  $a \in X$ .

(ii)  $i(A) = \bigcup\{i(a) : a \in A\}$  and  $d(A) = \bigcup\{d(a) : a \in A\}$ ,

\* Corresponding author e-mail: [tareqalshami83@gmail.com](mailto:tareqalshami83@gmail.com)

for each  $A \subseteq X$ .

(iii) A set  $A$  is called increasing (decreasing), If  $A = i(A)$  ( $A = d(A)$ ).

**Definition 2.2.** [10] A topological ordered space  $(X, \tau, \preceq)$  is called:

(i) Lower (Upper)  $T_1$ -ordered if for each  $a, b \in X$  such that  $a \not\preceq b$ , there exists an increasing (a decreasing) neighborhood  $W$  of  $a(b)$  such that  $b(a)$  belongs to  $W^c$ .

(ii)  $T_2$ -ordered if for each  $a, b \in X$  such that  $a \not\preceq b$ , there exist disjoint neighborhoods  $W_1$  and  $W_2$  of  $a$  and  $b$ , respectively, such that  $W_1$  is increasing and  $W_2$  is decreasing.

(iii) Lower (upper) regularly ordered if for every decreasing (increasing) closed set  $F$  and for each  $a \notin F$ , there exist disjoint neighborhoods  $W_1$  and  $W_2$  of  $F$  and  $a$ , respectively, such that  $W_1$  is decreasing (increasing) and  $W_2$  is increasing (decreasing).

(iv) Normally ordered if for every disjoint closed sets  $F_1$  and  $F_2$  such that  $F_1$  is decreasing and  $F_2$  is increasing, there exist disjoint neighborhoods  $W_1$  and  $W_2$  of  $F_1$  and  $F_2$ , respectively, such that  $W_1$  is decreasing and  $W_2$  is increasing.

(v) Lower (Upper)  $T_3$ -ordered if it is both lower (upper)  $T_1$ -ordered and lower (upper) regularly ordered.

(vi)  $T_4$ -ordered if it is both  $T_1$ -ordered and normally ordered.

**Remark 1.** McCartan defined strong separation axioms by replacing the word 'neighborhood' by the expression 'open neighbourhood' in Definition 2.2 and inferred that every strong  $T_i$ -order space is a  $T_i$ -order space, for  $i = 0, 1, 2, 3, 4$ .

**Definition 2.3.** [5] A supra topological ordered space  $(X, \mu, \preceq)$  is called:

(i) Lower (Upper) supra  $T_1$ -ordered if for each  $a, b \in X$  such that  $a \not\preceq b$ , there exists an increasing (a decreasing) supra open set  $G$  containing  $a(b)$  such that  $b(a)$  belongs to  $G^c$ .

(ii) Supra  $T_2$ -ordered if for every  $a, b \in X$  such that  $a \not\preceq b$ , there exist disjoint supra neighborhoods  $W_1$  and  $W_2$  of  $a$  and  $b$ , respectively, such that  $W_1$  is increasing and  $W_2$  is decreasing.

(iii) Lower (upper) regularly ordered if for every decreasing (increasing) supra closed set  $F$  and for each  $a \notin F$ , there exist disjoint supra neighborhoods  $W_1$  and  $W_2$  of  $F$  and  $a$ , respectively, such that  $W_1$  is decreasing (increasing) and  $W_2$  is increasing (decreasing).

(iv) Normally ordered if for every disjoint supra closed sets  $F_1$  and  $F_2$  such that  $F_1$  is decreasing and  $F_2$  is increasing, there exist disjoint supra neighborhoods  $W_1$  and  $W_2$  of  $F_1$  and  $F_2$ , respectively, such that  $W_1$  is decreasing and  $W_2$  is increasing.

A map  $f$  of  $(X, \preceq_1)$  into  $(Y, \preceq_2)$  is called order embedding if for each  $a, b \in X$ , then  $a \preceq_1 b$  iff  $f(a) \preceq_2 f(b)$ . A preordered set  $(X, \preceq)$  is called directed

set iff every two elements in  $X$  have an upper bound. A subset  $B$  of a directed set  $(X, \preceq)$  is called residual (or eventual) if there is an element  $b \in B$  such that  $x \preceq a$  implies  $x \in B$ , for all  $x \in X$ . A net is a map of a directed set into a non-empty set.

**Definition 2.4.** [9] A map  $f$  of  $X$  into  $Y$  is called  $S^*$ -continuous if the inverse image of each supra open subset of  $Y$  is a supra open subset of  $X$ .

**Definition 2.5.** [1] A subset  $G$  of  $(X, \mu, \preceq)$  is said to be:

(i) I-supra open if  $G$  is supra open and increasing.

(ii) D-supra open if  $G$  is supra open and decreasing.

Throughout this paper,  $SST_i$ -order space is abbreviation for strong supra  $T_i$ -order space. The set of natural number is denoted by  $\mathcal{N}$ .

### 3 Some remarks in McCartan's work and Das's work

In this section, we correct McCartan's assertion in the cases of  $i = 0, 1$ . Also, we introduce the concepts of  $SST_0$ -ordered and  $SST_1$ -ordered spaces and illustrate a relationship between them.

**Remark 2.** In [10], McCartan mentioned that every strong  $T_1$ -ordered space is a  $T_1$ -ordered space. We point out that strong  $T_1$ -ordered spaces equivalent to  $T_1$ -ordered spaces as follows

Let  $(X, \tau, \preceq)$  be a  $T_1$ -ordered space and  $a, b \in X$  such that  $a \not\preceq b$ . By Theorem 1 in [10], we obtain that  $i(a)$  and  $d(b)$  are closed sets. Now,  $i(a)$  is increasing and  $d(b)$  is increasing. Then  $(d(b))^c$  is an increasing open set containing  $a$  and  $(i(a))^c$  is a decreasing open set containing  $b$ . Obviously,  $b \notin (d(b))^c$  and  $a \notin (i(a))^c$ . Hence  $(X, \tau, \preceq)$  is a strong  $T_1$ -ordered space.

**Remark 3.** In [4], Das defined lower (upper) supra  $T_1$ -ordered spaces utilizing supra open neighbourhoods. If we replace supra open neighbourhoods by supra neighbourhoods and utilize Theorem 2.2 in [4], we get the same result which mentioned in Remark(3). Then we rename lower (upper) supra  $T_1$ -ordered to lower (upper)  $SST_1$ -ordered space.

In the following, we introduce the notions of  $SST_0$ -ordered and  $SST_1$ -ordered spaces.

**Definition 3.1.** A supra topological ordered space  $(X, \mu, \preceq)$  is called

(i)  $SST_0$ -ordered if it is lower  $SST_1$ -ordered or upper  $SST_1$ -ordered.

(ii)  $SST_1$ -ordered if it is both lower  $SST_1$ -ordered and upper  $SST_1$ -ordered.

**Theorem 3.1.** Every  $SST_1$ -ordered space is an  $SST_0$ -ordered space.

**Proof.** It is clear.  $\square$

The converse of the above theorem need not be true in general as illustrated in the following two examples.

**Example 3.1.** Let  $\mu = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$  be a supra topology on  $X = \{a, b, c\}$ , and  $\preceq = \Delta \cup \{(a, c), (b, c)\}$  be a partial order relation on  $X$ .

Since  $c \not\preceq a$ , then  $c \in i(\{b, c\}) = \{b, c\}$ .

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Since  $a \not\preceq b$ , then  $a \in i(\{a, c\}) = \{a, c\}$ .

Since  $b \not\preceq a$ , then  $b \in i(\{b, c\}) = \{b, c\}$ .

Thus  $(X, \mu, \preceq)$  is a lower  $SST_1$ -ordered space. But  $(X, \mu, \preceq)$  is not an upper  $SST_1$ -ordered space, as  $a \not\preceq b$  and there does not exist a decreasing supra neighbourhood  $W$  of  $b$  such that  $a \notin W$ .

**Example 3.2.** Let  $\mu = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$  be a supra topology on  $X = \{a, b, c\}$ , and  $\preceq = \Delta \cup \{(a, b), (b, c), (a, c)\}$  be a partial order relation on  $X$ .

Since  $b \not\preceq a$ , then  $a \in d(\{a\}) = \{a\}$ .

Since  $c \not\preceq b$ , then  $b \in d(\{a, b\}) = \{a, b\}$ .

Since  $c \not\preceq a$ , then  $a \in d(\{a\}) = \{a\}$ .

Thus  $(X, \mu, \preceq)$  is an upper  $SST_1$ -ordered space. But  $(X, \mu, \preceq)$  is not a lower  $SST_1$ -ordered space, as  $c \not\preceq b$  and there does not exist a supra neighbourhood  $W$  of  $c$  such that  $b \notin W$ .

**Theorem 3.2.** If a supra topological ordered space  $(X, \mu, \preceq)$  is lower (upper)  $SST_1$ -ordered, then  $scl(d(a)) \neq scl(d(b))$  ( $scl(i(a)) \neq scl(i(b))$ ), for each  $a, b \in X$ .

**Proof.** If  $(X, \tau, \preceq)$  is a lower  $SST_1$ -ordered space.

Let  $a, b \in X$ . Then  $a \not\preceq b$  or  $b \not\preceq a$ . Say,  $a \not\preceq b$ . Obviously,  $a \notin d(b)$  and  $a \in d(a)$ . Therefore  $d(a) \neq d(b)$ . Since  $(X, \tau, \preceq)$  is a lower  $SST_1$ -ordered space, then  $d(a)$  and  $d(b)$  are supra closed sets. Thus  $scl(d(a)) \neq scl(d(b))$ .

A similar proof can be given in the case of upper  $SST_1$ -ordered space.  $\square$

**Corollary 3.1.** If a supra topological ordered space  $(X, \mu, \preceq)$  is  $SST_1$ -ordered, then  $scl(d(a)) \neq scl(d(b))$  and  $scl(i(a)) \neq scl(i(b))$ , for each  $a, b \in X$ .

The converse of the above theorem generally is not true as shown in the following example.

**Example 3.3.** Let  $\mu = \{\emptyset, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$  be a supra topology on  $X = \{a, b, c\}$ , and  $\preceq = \Delta \cup \{(a, c), (b, c)\}$  be a partial order relation on  $X$ . On the one hand, we have the following:

(i)  $i(a) = \{a, c\}$  and  $scl(\{a, c\}) = \{a, c\}$ .

(ii)  $i(b) = \{b, c\}$  and  $scl(\{b, c\}) = X$ .

(iii)  $i(c) = \{c\}$  and  $scl(\{c\}) = \{c\}$ .

Therefore  $scl(i(a)) \neq scl(i(b)) \neq scl(i(c))$ . On the other

hand,  $(X, \mu, \preceq)$  is not an upper  $SST_1$ -ordered space, as  $b \not\preceq a$  and there does not exist a decreasing supra open set  $W$  containing  $a$  such that  $b \notin W$ .

## 4 New separation axioms in supra topological ordered spaces

In this section, we use a notion of monotone supra open neighbourhoods to introduce strong supra regularly spaces, strong supra normally spaces and strong supra  $T_i$ -ordered spaces ( $i = 2, 3, 4$ ) on supra topological ordered spaces. Also, we investigate several properties for these separation axioms.

**Definition 4.1.** A supra topological ordered space  $(X, \mu, \preceq)$  is called  $SST_2$ -ordered if for every  $a, b \in X$  such that  $a \not\preceq b$ , there exist disjoint supra open sets  $G$  and  $H$  containing  $a$  and  $b$ , respectively, such that  $G$  is increasing and  $H$  is decreasing.

**Theorem 4.1.** Every  $SST_2$ -ordered space is an  $SST_1$ -ordered space.

**Proof.** It is clear.  $\square$

The converse of the above theorem need not be true as illustrated in the following example

**Example 4.1.** Let  $\mu = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$  be a supra topology on  $X = \{a, b, c\}$ , and  $\preceq = \Delta \cup \{(a, c)\}$  be a partial order relation on  $X$ .

Then  $(X, \mu, \preceq)$  is an  $SST_1$ -ordered space. On the other hand,  $c \not\preceq a$  and the intersection of any supra open set containing  $c$  and any supra open set containing  $a$  is non-empty. Therefore  $(X, \mu, \preceq)$  is not an  $SST_2$ -ordered space.

In the following example, we illustrate that  $SST_2$ -ordered spaces is strictly stronger than supra  $T_2$ -ordered spaces.

**Example 4.2.** Let  $\mu = \{\emptyset, X, \{a\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$  be a supra topology on  $X = \{a, b, c, d\}$ , and  $\preceq = \Delta \cup \{(a, b), (c, b)\}$  be a partial order relation on  $X$ .

Then we have the following ten cases:

(i)  $b \not\preceq a$ ,  $b \in i(\{b, c\}) = \{b, c\}$  and  $a \in d(\{a\}) = \{a\}$ .

(ii)  $b \not\preceq c$ ,  $b \in i(\{b, d\}) = \{b, d\}$  and  $c \in d(\{a, c\}) = \{a, c\}$ .

(iii)  $b \not\preceq d$ ,  $b \in i(\{b, c\}) = \{b, c\}$  and  $d \in d(\{d\}) = \{d\}$ .

(iv)  $d \not\preceq b$ ,  $d \in i(\{d\}) = \{d\}$  and  $b \in d(\{a, b, c\}) = \{a, b, c\}$ .

(v)  $a \not\preceq c$ ,  $a \in i(\{a\}) = \{a, b\} \notin \mu$  and  $c \in d(\{c, d\}) = \{c, d\}$ .

(vi)  $c \not\preceq a$ ,  $c \in i(\{b, c\}) = \{b, c\}$  and  $a \in d(\{a\}) = \{a\}$ .

(vii)  $a \not\preceq d$ ,  $a \in i(\{a, b, c\}) = \{a, b, c\}$  and  $d \in d(\{d\}) = \{d\}$ .

(viii)  $d \not\preceq a$ ,  $d \in i(\{d\}) = \{d\}$  and  $a \in d(\{a\}) = \{a\}$ .

(ix)  $c \not\leq d$ ,  $c \in i(\{b, c\}) = \{b, c\}$  and  $d \in d(\{d\}) = \{d\}$ .  
 (x)  $d \not\leq c$ ,  $d \in i(\{d\}) = \{d\}$  and  $c \in d(\{b, c\}) = \{b, c\}$ .

Therefore  $(X, \mu, \preceq)$  is a supra  $T_2$ -ordered space. On the other hand,  $(X, \mu, \preceq)$  is not an  $SST_2$ -ordered space, as  $a \not\leq c$  and each increasing supra open set containing  $a$  intersects each supra open set containing  $c$ .

**Theorem 4.2.** If  $(X, \mu, \preceq)$  is an  $SST_2$ -ordered space, then the graph of the partial order relation  $\preceq$  on  $X$  is a supra closed subset of the product supra space  $X \times X$ .

**Proof.** Consider  $(a, b) \subseteq X \times X \setminus \preceq$ . Since  $(X, \mu, \preceq)$  is an  $SST_2$ -ordered space, then there exist disjoint supra open sets  $G$  and  $H$  containing  $a$  and  $b$ , respectively, such that  $G$  is increasing and  $H$  is decreasing. Therefore  $G \times H$  is a supra open set with respect to the product supra topology on  $X \times X$  containing  $(a, b)$ . Assume that  $\preceq \cap G \times H \neq \emptyset$ . In this case, there exists  $(x, y) \in \preceq \cap G \times H$  and this implies that  $(x, y) \in G \times H$ . But this contradicts the disjointness of  $G$  and  $H$ . Then  $\preceq \cap G \times H = \emptyset$ . Therefore  $G \times H \subseteq X \times X \setminus \preceq$ . Thus  $X \times X \setminus \preceq$  is a supra open set in the product supra spaces.  $\square$

**Corollary 4.1.** Let  $(X, \mu, \preceq)$  be  $SST_2$ -ordered and the nets  $\{x_\alpha : \alpha \in \Lambda\}$  and  $\{y_\alpha : \alpha \in \Lambda\}$  in  $X$  converge to  $a$  and  $b$ , respectively. If  $x_\alpha \preceq y_\alpha$  for all  $\alpha \in \Lambda$ , then  $a \preceq b$ .

**Proof.** Consider the nets  $\{x_\alpha\}$  converges to  $a$  and  $\{y_\alpha\}$  converges to  $b$  such that  $x_\alpha \preceq y_\alpha$  for all  $\alpha \in \Lambda$ . Obviously,  $(x_\alpha, y_\alpha)$  converge to  $(a, b)$ . Let  $A = \{(x_\alpha, y_\alpha) : \alpha \in \Lambda\}$ . Then  $A \subseteq \preceq$ . Since  $(X, \mu, \preceq)$  is an  $SST_2$ -ordered space, then  $\preceq$  is supra closed. Therefore  $(a, b) \in \preceq$ . This completes the proof.  $\square$

**Theorem 4.3.** Every  $SST_i$ -ordered space is a supra  $T_i$ -space, for  $i = 0, 1, 2$ .

**Proof.** It is clear.  $\square$

The following two examples illustrate that the converse of the above theorem does not hold in general.

**Example 4.3.** Let  $\mu = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$  be a supra topology on  $X = \{a, b, c\}$ , and  $\preceq = \Delta \cup \{(a, b), (c, b)\}$  be a partial order relation on  $X$ . Then  $(X, \mu, \preceq)$  is a supra  $T_2$ -space. But it is not an  $SST_1$ -ordered space, Since  $a \not\leq c$  and there does not exists a decreasing supra open set  $W$  containing  $c$  such that  $a \notin W$ .

**Example 4.4.** Let  $\mu = \{\emptyset, X, \{a, b\}, \{a, c\}\}$  be a supra topology on  $X = \{a, b, c\}$  and  $\preceq = \Delta \cup \{(a, b), (b, c), (a, c)\}$  be a partial order relation on  $X$ .

Then  $(X, \mu, \preceq)$  is a supra  $T_0$ -space. But is not a supra  $T_0$ -ordered space, Since  $b \not\leq a$  and there do not exist decreasing supra open set  $W_1$  containing  $a$  and increasing supra open set  $W_2$  containing  $b$  such that  $b \notin W_1$  and

$a \notin W_2$ .

**Definition 4.2.** A supra topological ordered space  $(X, \mu, \preceq)$  is called

- (i) Lower (upper) strong supra regularly ordered if for every decreasing (increasing) supra closed set  $F$  and for each  $a \notin F$ , there exist disjoint supra open sets  $G$  and  $H$  containing  $F$  and  $a$ , respectively, such that  $G$  is decreasing (increasing) and  $H$  is increasing (decreasing).
- (ii) Strong supra regularly ordered if it is both lower and upper strong supra regularly ordered.

In the following two examples, we show that the concepts of lower strong supra regularly ordered and upper strong supra regularly ordered are non-coincidence.

**Example 4.5.** Let  $\mu = \{\emptyset, X, \{c\}, \{a, b\}, \{b, c\}\}$  be a supra topology on  $X = \{a, b, c\}$ , and  $\preceq = \Delta \cup \{(a, c)\}$  be a partial order relation on  $X$ .

Then  $(X, \mu, \preceq)$  is an upper strong supra regularly ordered space. Now,  $\{a\}$  is a decreasing supra closed set and  $b \notin \{a\}$ . Since the intersection of all decreasing supra open sets containing  $\{a\}$  and all increasing supra open sets containing  $b$  is non-empty. Then  $(X, \mu, \preceq)$  is not strong lower supra regularly ordered.

**Example 4.6.** If the partial order relation  $\preceq$  in the above example is replaced by  $\preceq = \Delta \cup \{(c, a)\}$ . Then  $(X, \mu, \preceq)$  is a lower strong supra regularly ordered space. Now,  $\{a\}$  is an increasing supra closed set and  $b \notin \{a\}$ . Since the intersection of all increasing supra open sets containing  $\{a\}$  and all decreasing supra open sets containing  $b$  is non-empty. Then  $(X, \mu, \preceq)$  is not upper strong supra regularly ordered.

**Theorem 4.4.** Consider  $(X, \mu, \preceq)$  is a supra topological ordered space. Then the following statements are equivalent:

- (i)  $(X, \mu, \preceq)$  is lower (upper) strong supra regularly ordered;
- (ii) For all  $x \in X$  and every increasing (decreasing) supra open set  $V$  containing  $x$ , there is an increasing (a decreasing) supra open set  $U$  containing  $x$  such that  $scl(U) \subseteq V$ ;
- (iii) If a net  $\{x_\alpha : \alpha \in \Lambda\}$  is residually included in every increasing (decreasing) supra open set containing an element  $a$  and a net  $\{y_\alpha : \alpha \in \Lambda\}$  is residually included in every decreasing (increasing) supra open set containing an increasing (decreasing) supra closed set  $F$  such that  $x_\alpha \preceq y_\alpha$ , then  $a \in F$ .

**Proof.** (i)  $\rightarrow$  (ii): Consider  $(X, \mu, \preceq)$  is a strong lower supra regularly ordered space and  $V$  is an increasing supra open set containing  $x$ . Then  $V^c$  is a decreasing supra closed set and  $x \notin V^c$ . Therefore there exist disjoint an increasing supra open set  $U$  containing  $x$  and a decreasing supra open set  $W$  containing  $V^c$ . Thus  $U \subseteq W^c \subseteq V$ . Since  $W^c$  is a supra closed set, then  $scl(U) \subseteq V$ .

(ii)  $\rightarrow$  (iii): Consider  $a, F$ ,  $\{x_\alpha : \alpha \in \Lambda\}$  and  $\{y_\alpha : \alpha \in \Lambda\}$

are as given in (iii). Suppose that  $a \notin F$ . Then  $a \in F^c$  which is an increasing supra open set. By (ii) there exist an increasing supra open set  $V$  of  $a$  such that  $scl(V) \subseteq F^c$ . Therefore  $F \subseteq (scl(V))^c \subseteq V^c$ . Since  $x_\alpha \preceq y_\alpha$  for each  $\alpha \in \Lambda$  and  $V^c$  is a decreasing supra closed set including  $F$ , then  $\{x_\alpha : \alpha \in \Lambda\}$  is residually included in both  $V$  and  $V^c$ . But this is a contradiction. Thus  $a \in F$ .

(iii)  $\rightarrow$  (i): Assume that  $X$  is not strong lower supra regularly ordered. Then there is a decreasing supra closed  $F$  subset of  $X$  and  $a \in F^c$  such that every decreasing supra open set  $V$  containing  $F$  and every increasing supra open set  $U$  containing  $a$  have a non-empty intersection. Consider  $W_1$  is the family of all decreasing supra open set containing  $F$ ,  $W_2$  is the family of all increasing supra open set containing  $a$  and let  $W_1$  and  $W_2$  be directed by set inclusion. Let  $W_1 \times W_2$  be directed by the rule: for all  $(U_1, V_1), (U_2, V_2) \in W_1 \times W_2$  such that  $(U_1, V_1) \preceq (U_2, V_2)$  if and only if  $U_2 \subseteq U_1, V_2 \subseteq V_1$ . For all  $(U, V) \in W_1 \times W_2$ , there exists  $x_{(U,V)} \in U \cap V$ . Then the net  $\{x_{(U,V)} : (U, V) \in W_1 \times W_2\}$  is residually included in every increasing supra open set  $U$  containing  $a$  and every decreasing supra open set  $V$  containing  $F$ . But  $a \notin F$  which contradicts (iii). Thus  $(X, \mu, \preceq)$  is a lower strong regularly ordered.

A similar proof can be given for the case between parentheses.  $\square$

**Theorem 4.5.** Consider  $(X, \mu, \preceq)$  is a lower (upper) strong supra regularly ordered space. Then the following statements are equivalent:

- (i)  $(X, \mu, \preceq)$  is  $SST_2$ -ordered;
- (ii)  $(X, \mu, \preceq)$  is  $SST_1$ -ordered;
- (iii)  $(X, \mu, \preceq)$  is lower (upper)  $SST_1$ -ordered.

**Proof.** Let  $(X, \mu, \preceq)$  be lower strong supra regularly ordered.

Obviously, (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii).

(iii)  $\rightarrow$  (i) Let  $a, b \in X$  such that  $a \not\preceq b$ . Since  $(X, \mu, \preceq)$  is lower  $SST_1$ -ordered, then  $d(b)$  is a supra closed set. Obviously,  $d(b)$  is decreasing closed and  $a \notin d(b)$ . Since  $(X, \mu, \preceq)$  is lower strong supra regularly ordered, then there exist disjoint supra open sets  $G$  and  $H$  containing  $a$  and  $d(b)$ , respectively, such that  $G$  is increasing and  $H$  is decreasing. Thus  $(X, \mu, \preceq)$  is  $SST_2$ -ordered.

A similar proof can be given for the case between parentheses.  $\square$

**Corollary 4.2.** Consider  $(X, \mu, \preceq)$  is a strong supra regularly ordered space. Then the following statements are equivalent:

- (i)  $(X, \mu, \preceq)$  is  $SST_2$ -ordered;
- (ii)  $(X, \mu, \preceq)$  is  $SST_1$ -ordered;
- (iii)  $(X, \mu, \preceq)$  is  $SST_0$ -ordered.

**Definition 4.3.** A supra topological ordered space  $(X, \mu, \preceq)$  is called

(i) Lower (upper)  $SST_3$ -ordered if it is both lower (upper) supra regularly ordered and lower (upper)  $SST_1$ -ordered.

(ii)  $SST_3$ -ordered if it is both lower supra  $T_3$ -ordered and upper supra  $T_3$ -ordered.

**Theorem 4.6.** Every lower (upper)  $SST_3$ -ordered space is an  $SST_2$ -ordered space.

**Proof.** Consider  $(X, \mu, \preceq)$  is a lower (upper)  $SST_3$ -ordered. Then  $(X, \mu, \preceq)$  is both lower (upper) supra regularly ordered and lower (upper)  $SST_1$ -ordered space. From Theorem 3.6,  $(X, \mu, \preceq)$  is  $SST_2$ -ordered.  $\square$

**Corollary 4.3.** Every  $SST_3$ -ordered space is an  $SST_2$ -ordered space.

The following example illustrates that the converse of the above theorem and corollary does not hold in general.

**Example 4.7.** Consider the supra topology  $\mu = \{\emptyset, X, \{a\}, \{b\}, \{e\}, \{b, e\}, \{a, b\}, \{a, e\}, \{a, c\}, \{c, e\}, \{a, b, e\}, \{a, b, c\}, \{b, c, e\}, \{b, d, e\}, \{a, c, e\}, \{a, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$  on  $X = \{a, b, c, d, e\}$ , and  $\preceq = \Delta \cup \{(a, b), (a, c), (a, d)\}$  is a partial order relation on  $X$ .

Then  $(X, \mu, \preceq)$  is a strong supra  $T_2$ -ordered space.

Obviously,  $\{a, d\}$  is a decreasing supra closed set and  $c \notin \{a, d\}$ . Since the intersection of all decreasing supra open sets containing  $\{a, d\}$  and all increasing supra open sets containing  $c$  is non-empty. Then  $(X, \mu, \preceq)$  is not a lower  $SST_3$ -ordered. Obviously,  $\{b, c\}$  is an increasing supra closed set and  $d \notin \{b, c\}$ . Since the intersection of all increasing supra open sets containing  $\{b, c\}$  and all decreasing supra open sets containing  $d$  is non-empty. Then  $(X, \mu, \preceq)$  is not an upper  $SST_3$ -ordered. Thus  $(X, \mu, \preceq)$  is not an  $SST_3$ -ordered space.

**Definition 4.4.** A supra topological ordered space  $(X, \mu, \preceq)$  is called

(i) Strong supra normally ordered if for every disjoint supra closed sets  $F_1$  and  $F_2$  such that  $F_1$  is decreasing and  $F_2$  is increasing, there exist disjoint supra open sets  $G$  and  $H$  containing  $F_1$  and  $F_2$ , respectively, such that  $G$  is decreasing and  $H$  is increasing.

(ii)  $SST_4$ -ordered if it is both strong supra normally ordered and  $SST_1$ -ordered.

**Remark 4.** Let  $(X, \mu, \preceq)$  be a supra topological ordered space. Then the smallest decreasing (increasing) closed set containing  $F$  denote by  $D(F)(I(F))$ .

**Theorem 4.7.** A supra topological ordered space  $(X, \mu, \preceq)$  is strong supra normally ordered iff for all decreasing (increasing) supra closed set  $F$  and every decreasing (increasing) supra open neighborhood  $U$  including  $F$ , there exists a decreasing (increasing) supra neighborhood  $V$  of  $F$  such that  $D(V) \subseteq U(I(V) \subseteq U)$ .

**Proof.** Necessity: let  $F$  be a decreasing supra closed subset of  $X$  and  $U$  be an increasing supra open

neighborhood including  $F$ . Then  $U^c$  is a decreasing supra closed set and  $F \cap U^c = \emptyset$ . Since  $(X, \mu, \preceq)$  is a strong supra normally ordered space, then there exist disjoint decreasing supra open set  $V$  including  $F$  and increasing supra open set  $G$  including  $U^c$ . Now,  $U^c \subseteq G$  and  $V \cap G = \emptyset$  imply  $G^c \subseteq U$  and  $V \subseteq G^c$ . Therefore  $F \subseteq V \subseteq G^c \subseteq U$ . Thus  $F \subseteq D(V) \subseteq G^c \subseteq U$ . Hence  $D(V) \subseteq U$ .

**Sufficiency:** Let  $F_1$  and  $F_2$  be disjoint closed subsets of  $X$  such that  $F_1$  is decreasing and  $F_2$  is increasing. Then  $F_1 \subseteq F_2^c$ . Now,  $F_2^c$  is a decreasing supra open neighborhood including  $F_1$ . Therefore there exists a decreasing supra neighborhood  $U$  including  $F_2^c$  such that  $F_1 \subseteq D(U) \subseteq F_2^c$ . Putting  $H = X \setminus D(U)$  implies  $H$  is an increasing supra open set including  $F_2$ . Thus  $F_2 \subseteq H$ ,  $F_1 \subseteq U$  and  $H \cap U = \emptyset$ . Hence  $(X, \mu, \preceq)$  is a strong supra normally ordered space.  $\square$

**Theorem 4.8.** Every  $SST_4$ -ordered space is an  $SST_3$ -ordered space.

**Proof.** Consider  $(X, \mu, \preceq)$  is strong supra  $T_4$ -ordered. Let  $a \in X$  and  $F$  be decreasing or increasing closed such that  $a \notin F$ . Suppose  $F$  is decreasing. Since  $X$  is  $SST_1$ -ordered, then  $i(a)$  is an increasing supra closed set. Since  $X$  is supra normally ordered, there exist disjoint supra open sets  $G$  and  $H$  containing  $i(a)$  and  $F$ , respectively, such that  $G$  is increasing and  $H$  is decreasing. Thus  $(X, \mu, \preceq)$  is a lower strong supra regularly ordered. Similarity, when  $F$  is increasing we prove that  $(X, \mu, \preceq)$  is an upper strong supra regularly ordered.

Hence  $(X, \mu, \preceq)$  is a strong supra regularly ordered.  $\square$

The following example illustrates that the converse of the above theorem does not hold in general.

**Example 4.8.** Let  $(X, \mu)$  be the Niemytzki space. We define a partial order relation on  $X$  as follows  $\preceq = \Delta \cup \{(1, 2), (1, 3)\}$ . Then  $(X, \mu, \preceq)$  is an  $SST_3$ -space, but is not an  $SST_4$ -ordered space.

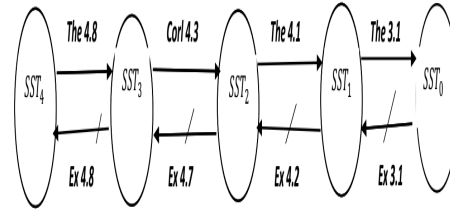
The following example ensure the existence of an  $SST_4$ -ordered space.

**Example 4.9.** Let  $\mu = \{G \subseteq \mathcal{N} \text{ such that } 1 \in G \text{ or } 2 \in G\}$  be a supra topology on  $\mathcal{N}$  and  $\preceq = \Delta \cup \{(1, a) : \text{for all } a \in \mathcal{N}\}$  be a partial order relation on  $\mathcal{N}$ . Then Since  $a \not\preceq 1$ , then  $a \in i(\{a, 2\}) = \{a, 2\}$ , and  $1 \in d(\{1\}) = \{1\}$ .

For all  $b \neq c \neq 1 \in \mathcal{N}$  such that  $b \not\preceq c$ , then  $b \in i(\{b, 2\}) = \{b, 2\}$ , and  $c \in d(\{c, 1\}) = \{c, 1\}$ . Then  $(\mathcal{N}, \mu, \preceq)$  is an  $SST_2$ -ordered space. Also, it is a strong supra normally ordered space. Therefore  $(\mathcal{N}, \mu, \preceq)$  is an  $SST_4$ -ordered space.

We summarize the relationships among  $SST_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ) in the following figure.

**Definition 4.5.** Let  $Y \subseteq X$  and  $(X, \mu, \preceq)$  be a supra topological ordered space. Then  $(Y, \mu_Y, \preceq)$  is called supra



**Fig. 1:** the relationships among  $SST_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ )

topological ordered subspace of  $(X, \mu, \preceq)$  provided that  $(Y, \mu_Y)$  is subspace of  $(X, \mu)$  and  $\preceq = \preceq \cap A \times A$ .

**lemma 4.1.** If  $U$  is an increasing (a decreasing) subset of an ordered set  $(X, \preceq)$  and  $A \subseteq X$ , then  $U \cap A$  is an increasing (a decreasing) subset of an ordered set  $(A, \preceq \cap A \times A)$ .

**Proof.** Let  $U$  be an increasing subset of an ordered set  $(X, \preceq)$  and  $A \subseteq X$ . In an ordered set  $(A, \preceq \cap A \times A)$ , let  $a \in i(U \cap A)$ . Since  $i(U \cap A) \subseteq i(U) \cap i(A) \subseteq U \cap A$ , then  $a \in U \cap A$ . Therefore  $i(U \cap A) = U \cap A$ . Thus  $U \cap A$  is an increasing subset of an ordered set  $(A, \preceq \cap A \times A)$ .

The proof is similar, when  $U$  is a decreasing set.  $\square$

**Theorem 4.9.** Every supra topological ordered subspace  $(Y, \mu_Y, \preceq)$  of  $SST_i$ -ordered space  $(X, \mu, \preceq)$  is an  $SST_i$ -ordered space, for  $i = 0, 1, 2$ .

**Proof.** We prove theorem when  $i = 2$  and other cases are similar.

Let  $(Y, \tau_Y, \preceq)$  be a supra topological ordered subspace of an  $SST_2$ -ordered space  $(X, \tau, \preceq)$  and let  $a, b \in Y \subseteq X$  such that  $a \not\preceq b$ . Then  $a \not\preceq b$ . Therefore there exist disjoint supra open sets  $U$  and  $V$  containing  $a$  and  $b$ , respectively, such that  $U$  is increasing and  $V$  is decreasing. Thus  $U \cap Y$  and  $V \cap Y$  are disjoint supra open subsets of  $(Y, \tau_Y, \preceq)$  containing  $a$  and  $b$ , respectively, such that  $U \cap Y$  is increasing and  $V \cap Y$  is decreasing. Hence  $(Y, \tau_Y, \preceq)$  is an  $SST_2$ -ordered space.  $\square$

**Theorem 4.10.** Let a bijective map  $f : (X, \mu, \preceq_1) \rightarrow (Y, \nu, \preceq_2)$  be ordered embedding  $S^*$ -continuous. If  $(Y, \nu, \preceq_2)$  is an  $SST_i$ -ordered space, then  $(X, \mu, \preceq_1)$  is an  $SST_i$ -ordered space, for  $i = 0, 1, 2$ .

**Proof.** When  $i = 1$ .

Let  $a, b \in X$  such that  $a \not\preceq_1 b$ . Then there exist  $x, y \in Y$  such that  $x = f(a)$  and  $y = f(b)$ . Since  $f$  is ordered embedding, then  $x \not\preceq_2 y$ . Since  $(Y, \nu, \preceq_2)$  is an  $SST_1$ -ordered space, then there exist an increasing supra open set  $G_1$  containing  $x$  and a decreasing supra open set  $G_2$  containing  $y$  such that  $y \notin G_1$  and  $x \notin G_2$ . Since  $f$  is  $S^*$ -continuous and order preserving, then  $f^{-1}(G_1)$  is an increasing supra open containing  $a$  and  $f^{-1}(G_2)$  is a

decreasing supra open containing  $b$  such that  $b \notin f^{-1}(G_1)$  and  $a \notin f^{-1}(G_2)$ . Thus  $(X, \mu, \preceq_1)$  is an  $SST_1$ -ordered space.

A similar proof can be given for  $i = 0, 2$ .  $\square$

## 5 Conclusion

In the present paper, we have introduced and studied new separation axioms called strong supra regularly ordered spaces, strong supra normally ordered spaces and  $SST_i$ -ordered spaces ( $i = 0, 1, 2, 3, 4$ ) on supra topological ordered spaces. We point out that  $T_1$ -ordered (supra  $T_1$ -ordered) equivalent to strong  $T_1$ -ordered ( $SST_1$ -ordered) and  $SST_2$ -ordered is a proper extension of supra  $T_2$ -ordered. Several examples are established to illustrate relationships among these separation axioms. The concepts initiated in this work can be applied in the field of ordered information system. Using these separation axioms will help to distinguish between connected objects or attributes in information system. In an upcoming paper, we shall study the relationships between supra compact spaces [3] and  $SST_1$ -ordered spaces.

## Conflict of interest

The authors declare that they have no conflict of interest.

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**M. E. El-Shafer** is a professor and Head of Department of Mathematics, Faculty of Science, Mansoura University, Egypt. He received his B.Sc. Degree in mathematics in 1977 from Assiut University, M.Sc. Degree in 1983 From Assiut University and Ph.D. Degree in 1987 from Mansoura University. His research interested are : Fuzzy topology, Fuzzy bitopology, Soft topology, Order topology and the theory of rough sets. In these areas he has published over 40 technical papers in refereed international journals or conference proceedings.



**M. Abo-Elhamayel** is a lecture of Mathematics, Faculty of Science, Mansoura University, Egypt. He received his B.Sc. degree in mathematics in 2003, M.Sc. degree in 2010, and Ph.D. degree in 2014 from Mansoura university, Faculty of Science, Egypt. His research interested are: Bitopology, Soft topology, Order topology and theory of rough sets. In these areas he has published over 11 technical papers in refereed international journals or conference proceedings.



**T. M. Al-shami** is a Ph.D student in Mathematics, Faculty of Science, Mansoura university, Egypt. He received his B.Sc. degree in mathematics in 2009 from Sana'a university, Yemen and M.Sc. degree in Mathematics in 2015 from Mansoura university, Egypt. He is an assistant professor in Department of Mathematics, Faculty of Education, Sana'a University, Yemen. His research interested are: General topology, Ordered topology, Soft topology, Theory of rough sets and digital topology. In these areas he has published over 9 technical papers in refereed international journals or conference proceedings