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On the Co-Ordinated Convex Functions

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Abstract: In this paper, some new integral inequalities are given for convex functions on the co-ordinates. By using well-known classical inequalities and a new integral identity (Lemma 1), we obtain some general results for co-ordinated convex functions.

Keywords: Hadamard's inequality, co-ordinates, convexity, Hölder's inequality, Power mean inequality.

1 Introduction

Let us recall some known definitions and results which we will use in this paper. A function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is an interval, is said to be a convex function on I if

$$f(tx + (1-t)y) \le t f(x) + (1-t) f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1) holds, then f is concave.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave.

In [4], Dragomir defined convex functions on the co-ordinates as following;

Definition 1. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$.*

Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \lambda f(x,y) + (1-\lambda) f(z,w)$$

for all $(x,y), (z,w) \in \Delta$ and $\lambda \in [0,1]$.

In [4], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (2)\\
&\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\
&\le \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx\right.\\
&\quad \left. + \frac{1}{(d-c)}\int_c^d f(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)dy\right]\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

In [1], Bakula and Pečarić established several Jensen type inequalities for co-ordinated convex functions and in [5], Hwang *et al.* gave a mapping, discussed some properties of this mapping and proved some Hadamard-type inequalities for Lipschizian mapping in two variables. In [2], Özdemir *et al.* established new Hadamard-type inequalities for co-ordinated $m-$convex and $(\alpha, m)-$convex functions. Several new results can be

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found related to convex functions on the coordinates in the papers [9]-[13]. In [3], Sarıkaya *et al.* proved some Hadamard-type inequalities for co-ordinated convex functions as followings;

Theorem 2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|\\ \le&\frac{(b-a)(d-c)}{16}\\&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right)\end{aligned}\tag{3}$$

where

$$\begin{aligned}A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b[f(x,c)+f(x,d)]dx\right.\\\left.+\frac{1}{(d-c)}\int_c^d[f(a,y)dy+f(b,y)]dy\right].\end{aligned}$$

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|\\ \le&\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\\&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right)^{\frac{1}{q}}\end{aligned}\tag{4}$$

where

$$\begin{aligned}A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b[f(x,c)+f(x,d)]dx\right.\\\left.+\frac{1}{(d-c)}\int_c^d[f(a,y)dy+f(b,y)]dy\right]\end{aligned}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\ge1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|\\ \le&\frac{(b-a)(d-c)}{16}\\&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right)^{\frac{1}{q}}\end{aligned}\tag{5}$$

where

$$\begin{aligned}A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b[f(x,c)+f(x,d)]dx\right.\\\left.+\frac{1}{(d-c)}\int_c^d[f(a,y)dy+f(b,y)]dy\right].\end{aligned}$$

In [6], Özdemir *et al.* proved following inequalities for co-ordinated convex functions.

Theorem 5. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|\\ \le&\frac{(b-a)(d-c)}{64}\\&\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].\end{aligned}\tag{6}$$

Theorem 6. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right.\\&\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right|\\ \le&\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\\&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.\end{aligned}\tag{7}$$

Theorem 7. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\ge1$, is a convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right.\\&\left.-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\right|\\ \le&\frac{(b-a)(d-c)}{16}\\&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.\end{aligned}\tag{8}$$

In [7], Özdemir *et al.* proved the following Theorem which involves an inequality of Simpson's type;

Theorem 8. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice partially differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\frac{\partial^2 f}{\partial t\partial s}$ is a* *convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| \frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{9} \\
&+\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9} \\
&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36} \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\Bigg| \\
\le\ &\frac{25(b-a)(d-c)}{72} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{72}\right)
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
A=&\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx \\
&+\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy.
\end{aligned}$$

The main purpose of this paper is to establish a new lemma which gives more general results and different type inequalities for special values of λ and to prove several inequalities.

2 Main Results

For the simplicity, we will denote:

$$\begin{aligned}
&F(x,y;\Delta,\lambda) \\
=\ &(1-\lambda)^2 f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&+\lambda^2\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+\frac{\lambda(1-\lambda)}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right. \\
&\left.+f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right] \\
&-(1-\lambda)\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\
&-(1-\lambda)\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
&-\lambda\frac{1}{2(b-a)}\int_a^b[f(x,d)+f(x,c)]\,dx \\
&-\lambda\frac{1}{2(d-c)}\int_c^d[f(a,y)+f(b,y)]\,dy.
\end{aligned}$$

In order to prove our main theorems, we need the following lemma:

Lemma 1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice partially differentiable function on Δ where $a<b$, $c<d$ and $\lambda\in[0,1]$. If $\frac{\partial^2 f}{\partial x\partial y}\in L_1(\Delta)$, then the following equality holds:*

$$\begin{aligned}
&F(x,y;\Delta,\lambda)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
=\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d K(x)M(y)\frac{\partial^2 f}{\partial x\partial y}(x,y)dydx,
\end{aligned}$$

where

$$K(x)=\begin{cases} x-\left(a+\lambda\frac{b-a}{2}\right), & x\in\left[a,\frac{a+b}{2}\right] \\ x-\left(b-\lambda\frac{b-a}{2}\right), & x\in\left[\frac{a+b}{2},b\right]\end{cases}$$

and

$$M(y)=\begin{cases} y-\left(c+\lambda\frac{d-c}{2}\right), & y\in\left[c,\frac{c+d}{2}\right] \\ y-\left(d-\lambda\frac{d-c}{2}\right), & y\in\left[\frac{c+d}{2},d\right]\end{cases}.$$

Proof. Integrating by parts, we obtain

$$\begin{aligned}
&\int_a^b\int_c^d K(x)M(y)\frac{\partial^2 f}{\partial x\partial y}(x,y)dydx \\
=\ &\int_a^b K(x)\left[\int_c^{\frac{c+d}{2}}\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)\frac{\partial^2 f}{\partial x\partial y}(x,y)dy\right. \\
&\left.+\int_{\frac{c+d}{2}}^d\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)\frac{\partial^2 f}{\partial x\partial y}(x,y)dy\right]dx \\
=\ &\int_a^b K(x)\left[\left(y-\left(c+\lambda\frac{d-c}{2}\right)\right)\frac{\partial f}{\partial x}(x,y)\Big|_c^{\frac{c+d}{2}}\right. \\
&-\int_c^{\frac{c+d}{2}}\frac{\partial f}{\partial x}(x,y)dy \\
&\left.+\left(y-\left(d-\lambda\frac{d-c}{2}\right)\right)\frac{\partial f}{\partial x}(x,y)\Big|_{\frac{c+d}{2}}^d-\int_{\frac{c+d}{2}}^d\frac{\partial f}{\partial x}(x,y)dy\right]dx \\
=\ &\int_a^b K(x)\left[(1-\lambda)(d-c)\frac{\partial f}{\partial x}\left(x,\frac{c+d}{2}\right)\right. \\
&\left.+\left(\lambda\frac{d-c}{2}\right)\left(\frac{\partial f}{\partial x}(x,c)+\frac{\partial f}{\partial x}(x,d)\right)-\int_c^d\frac{\partial f}{\partial x}(x,y)dy\right]dx.
\end{aligned}$$

Integrating by parts again, we obtain

$$\begin{aligned}
&\int_a^b \int_c^d K(x)M(y)\frac{\partial^2 f}{\partial x \partial y}(x,y)dydx \\
&= (1-\lambda)^2 (b-a)(d-c)\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right] \\
&\quad +\lambda^2 (b-a)(d-c)\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right] \\
&\quad +\frac{\lambda(1-\lambda)(b-a)(d-c)}{2}\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\right. \\
&\quad \left. +f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right] \\
&\quad -(1-\lambda)(b-a)\int_c^d f(\frac{a+b}{2},y)dy \\
&\quad -(1-\lambda)(d-c)\int_a^b f(x,\frac{c+d}{2})dx \\
&\quad -\lambda\frac{(d-c)}{2}\int_a^b [f(x,d)+f(x,c)]dx \\
&\quad -\lambda\frac{(b-a)}{2}\int_c^d [f(a,y)+f(b,y)]dy+\int_a^b\int_c^d f(x,y)dxdy.
\end{aligned}$$

Dividing both sides of the above equality by $(b-a)(d-c)$, we get the required result.

Theorem 9. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partially differentiable function on Δ and $\lambda\in[0,1]$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequality:*

$$\begin{aligned}
&\left|F(x,y;\Delta,\lambda)\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{(b-a)(d-c)}{16}\left[2\lambda^2-2\lambda+1\right]^2 \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|(a,d)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|(b,d)}{4}\right).
\end{aligned} \tag{10}$$

Proof. From Lemma 1 and property of the modulus, we can write

$$\begin{aligned}
&\left|F(x,y;\Delta,\lambda)\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |K(x)M(y)|\left|\frac{\partial^2 f}{\partial x\partial y}\right|(x,y)dydx.
\end{aligned}$$

Using the change of variables $y=sd+(1-s)c$, $(d-c)ds=dy$, we obatin

$$\begin{aligned}
&\left|F(x,y;\Delta,\lambda)\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{d-c}{b-a}\int_a^b |K(x)|\left\{\int_0^{\frac{\lambda}{2}}\left(\frac{\lambda}{2}-s\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,sd+(1-s)c)\right|ds\right. \\
&\quad +\int_{\frac{\lambda}{2}}^{\frac{1}{2}}\left(s-\frac{\lambda}{2}\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,sd+(1-s)c)\right|ds \\
&\quad +\int_{\frac{1}{2}}^{1-\frac{\lambda}{2}}\left(1-\frac{\lambda}{2}-s\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,sd+(1-s)c)\right|ds \\
&\quad \left.+\int_{1-\frac{\lambda}{2}}^{1}\left(s-1+\frac{\lambda}{2}\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,sd+(1-s)c)\right|ds\right\}dx.
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial x\partial y}\right|$ is a convex function on the co-ordinates on Δ, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |K(x)M(y)|\left|\frac{\partial^2 f}{\partial x\partial y}\right|(x,y)dydx \\
&\leq \frac{d-c}{b-a}\int_a^b |K(x)|\left\{\int_0^{\frac{\lambda}{2}} s\left(\frac{\lambda}{2}-s\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,d)\right|ds\right. \\
&\quad +\int_0^{\frac{\lambda}{2}}(1-s)\left(\frac{\lambda}{2}-s\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,c)\right|ds \\
&\quad +\int_{\frac{\lambda}{2}}^{\frac{1}{2}} s\left(s-\frac{\lambda}{2}\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,d)\right|ds+\int_{\frac{\lambda}{2}}^{\frac{1}{2}}(1-s)\left(s-\frac{\lambda}{2}\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,c)\right|ds \\
&\quad +\int_{\frac{1}{2}}^{1-\frac{\lambda}{2}} s\left(1-\frac{\lambda}{2}-s\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,d)\right|ds \\
&\quad +\int_{\frac{1}{2}}^{1-\frac{\lambda}{2}}(1-s)\left(1-\frac{\lambda}{2}-s\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,c)\right|ds \\
&\quad +\int_{1-\frac{\lambda}{2}}^{1} s\left(s-1+\frac{\lambda}{2}\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,d)\right|ds \\
&\quad \left.+\int_{1-\frac{\lambda}{2}}^{1}(1-s)\left(s-1+\frac{\lambda}{2}\right)\left|\frac{\partial^2 f}{\partial x\partial y}(x,c)\right|ds\right\}dx.
\end{aligned}$$

By calculating the above integrals, we obtain

$$\begin{aligned}
&\left|F(x,y;\Delta,\lambda)\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{d-c}{b-a}\int_a^b |K(x)|\left\{\left[\frac{2\lambda^2-2\lambda+1}{8}\right]\left(\left|\frac{\partial^2 f}{\partial x\partial y}(x,c)\right|+\left|\frac{\partial^2 f}{\partial x\partial y}(x,d)\right|\right)\right\}.
\end{aligned}$$

By a similar argument for other integrals and using the change of variable $x=tb+(1-t)a$, $(b-a)dt=dx$ and convexity of $\left|\frac{\partial^2 f}{\partial x\partial y}(x,y)\right|$ on the co-ordinates on Δ, we deduce the result which is the desired.

Remark. If we choose $\lambda=1$ in (10), we have the inequality (3).

Remark. If we choose $\lambda=0$ in (10), we have the inequality (6).

Remark. If we choose $\lambda = \frac{1}{3}$ in (10), we have the inequality (9).

Theorem 10. *Let $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ be a twice partially differentiable function on Δ and $\lambda \in [0,1]$. If $\left|\frac{\partial^2 f}{\partial x \partial y}\right|^{\frac{p}{p-1}}$ is a convex function on the co-ordinates on Δ, then one has the inequality:*

$$
\begin{aligned}
&\left|F(x,y;\Delta,\lambda) + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[2\lambda^2-2\lambda+1\right]^2 \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}
\end{aligned} \tag{11}
$$

for $q > 1$, where $q = \frac{p}{p-1}$.

Proof. Let $p > 1$. From Lemma 1 and using the Hölder inequality (see [8]) for double integrals, we get

$$
\begin{aligned}
&\left|F(x,y;\Delta,\lambda) + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d |K(x)M(y)|^p\,dydx\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_a^b\int_c^d \left|\frac{\partial^2 f}{\partial x\partial y}(x,y)\right|^q dydx\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^{\frac{p}{p-1}}$ is a convex function on the co-ordinates on Δ, by taking into account the change of variable $x = tb+(1-t)a$, $(b-a)dt = dt$ and $y = sd+(1-s)c$, $(d-c)ds = dy$, we have

$$
\begin{aligned}
\left|\frac{\partial^2 f}{\partial x\partial y}(tb+(1-t)a,y)\right|^q &\leq t\left|\frac{\partial^2 f}{\partial x\partial y}(b,y)\right|^q \\
&\quad + (1-t)\left|\frac{\partial^2 f}{\partial x\partial y}(a,y)\right|^q
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{\partial^2 f}{\partial x\partial y}(tb+(1-t)a, sd+(1-s)c)\right|^q \\
&\leq ts\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)+t(1-s)\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c) \\
&\quad + s(1-t)\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\left|F(x,y;\Delta,\lambda) + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right| \\
&\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[2\lambda^2-2\lambda+1\right]^2 \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)}{4}\right. \\
&\left.+\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}},
\end{aligned}
$$

this completes the proof.

Remark. Under the assumptions of Theorem 10, if we choose $\lambda = 1$ in (11), we have the inequality (4).

Remark. Under the assumptions of Theorem 10, if we choose $\lambda = 0$ in (11), we have the inequality (7).

Corollary 1. *Under the assumptions of Theorem 10, if we choose $\lambda = \frac{1}{3}$ in (11), we have the inequality:*

$$
\begin{aligned}
&\left|\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{9}\right. \\
&+\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9} \\
&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36} \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
&\leq \frac{25(b-a)(d-c)}{324(p+1)^{\frac{2}{p}}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)}{4}\right. \\
&\left.+\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx \\
&\quad+\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy.
\end{aligned}
$$

*Remark.*In Corollary 1, since $\frac{1}{4}<\frac{1}{(p+1)^{\frac{2}{p}}}<1$, for $p>1$, we have the following inequality:

$$\begin{aligned}
&\Bigg|\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{9}\\
&+\frac{f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9}\\
&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\Bigg|\\
\leq\ &\frac{25(b-a)(d-c)}{324}\\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)}{4}\right.\\
&\left.+\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
A=\ &\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx\\
&+\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy.
\end{aligned}$$

Theorem 11.*Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partially differentiable function on Δ and $\lambda\in[0,1]$. If $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is a convex function on the co-ordinates on Δ and $q\geq 1$, then one has the inequality:*

$$\begin{aligned}
&\Bigg|F(x,y;\Delta,\lambda)\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\Bigg|\\
\leq\ &\frac{(b-a)(d-c)}{16}\left[2\lambda^2-2\lambda+1\right]^2\\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)}{4}\right.\\
&\left.+\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}.
\end{aligned}\tag{12}$$

*Proof.*From Lemma 1 and using the well-known Power-mean inequality (see [8]), we get

$$\begin{aligned}
&\Bigg|F(x,y;\Delta,\lambda)\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\Bigg|\\
\leq\ &\frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d|K(x)M(y)|\,dydx\right)^{1-\frac{1}{q}}\\
&\times\left(\int_a^b\int_c^d|K(x)M(y)|\left|\frac{\partial^2 f}{\partial x\partial y}(x,y)\right|^q dydx\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q$ is a convex function on the co-ordinates on Δ, by taking into account the change of variable $x=tb+(1-t)a$, $(b-a)dt=dt$ and $y=sd+(1-s)c$, $(d-c)ds=dy$, we have

$$\left|\frac{\partial^2 f}{\partial x\partial y}(tb+(1-t)a,y)\right|^q\leq t\left|\frac{\partial^2 f}{\partial x\partial y}(b,y)\right|^q+(1-t)\left|\frac{\partial^2 f}{\partial x\partial y}(a,y)\right|^q$$

and

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial x\partial y}(tb+(1-t)a,sd+(1-s)c)\right|^q\\
\leq\ &ts\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)+t(1-s)\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)\\
&+s(1-t)\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\Bigg|F(x,y;\Delta,\lambda)\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\Bigg|\\
\leq\ &\frac{(b-a)(d-c)}{16}\left[2\lambda^2-2\lambda+1\right]^2\\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)}{4}\right.\\
&\left.+\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

this completes the proof.

*Remark.*Under the assumptions of Theorem 11, if we choose $\lambda=1$ in (12)we have the inequality (5).

*Remark.*Under the assumptions of Theorem 11, if we choose $\lambda=0$ in (12), we have the inequality (8).

Corollary 2. *Under the assumptions of Theorem 11, if we choose $\lambda = \frac{1}{3}$ in (12), we have*

$$\begin{aligned}
&\left| \frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9} \right.\\
&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right|\\
\leq\ &\frac{25(b-a)(d-c)}{36^2}\\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial x\partial y}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
A = &\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx\\
&+\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy.
\end{aligned}$$

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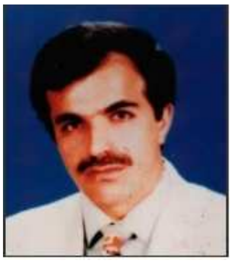

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