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Approximation Properties for Stancu Type $q-$Baskakov-Kantorovich Operators

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Abstract: In this paper, we give an interesting generalization of the Stancu type Baskakov-Kantorovich operators based on the q-integers and investigate their approximation properties. Also, we obtain the estimates for the rate of convergence for a sequence of them by the weighted modulus of smoothness.

Keywords: $q-$ integer, $q-$ Baskakov-Kantorovich operators, Baskakov-Kantorovich-Stancu operators, weighted spaces, rate of convergence, weighted modulus of smoothness.

1 Introduction

In recent years, due to the intensive development of $q-$calculus, generalizations of some operators related to $q-$calculus have emerged (see [2, 3, 7, 12–16]). Aral and Gupta defined $q-$ generalization of the Baskakov operator and investigated approximation properties of these operators in [3]. In [13], Gupta and Radu introduced the Baskakov- Kantorovich operators based on $q-$integers and investigated their weighted statistical approximation properties. They also proved some direct estimations for error using weighted modulus of smoothness in case $0 < q < 1$. In recent study Büyükyazıcı and Atakut [7] introduced a new Stancu type generalization of $q-$ Baskakov operator is defined as

$$L_n^{\alpha,\beta}(f;q,x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \; f\left(\frac{1}{q^{k-1}}\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right) \tag{1}$$

where $0 \le \alpha \le \beta,\, q \in (0,1),\; f \in C[0,\infty)$ and the following conditions are provided:

Let $\{\varphi_n\}$ $(n = 1,2,\ldots)$ $\varphi_n : \mathbb{R} \to \mathbb{R}$ be a sequence which is satisfying following conditions,

(i) φ_n $(n = 1,2,\ldots)$, $k-$ times continuously $q-$differentiable any closed interval $[0,A]$,

(ii) $\varphi_n(0) = 1, (n = 1,2,\ldots)$,

(iii) for all $x \in [0,A]$, and $(k = 0,1,\ldots;n = 1,2,\ldots)$, $(-1)^k D_q^k(\varphi_n(x)) \ge 0$,

(iv) there exists a positive integer $m(n)$, such that

$$D_q^k(\varphi_n(x)) = -[n]_q D_q^{k-1}\varphi_{m(n)}(x),\;\; (k = 1,2,\ldots;n = 1,2,\ldots), \tag{2}$$

(v) $\lim_{n\to\infty} \frac{[n]_q}{[m(n)]_q} = 1.$

Now, to explain the construction of the new $q-$operators, we mention some basic definitions of $q-$calculus and Lemma.

Let $q > 0$. For each nonnegative integer n, we define the $q-$ integer $[n]_q$ as

$$[n]_q = \begin{cases} (1-q^n)/(1-q) & \text{if} \quad q \ne 1 \\ n & \text{if} \quad q = 1 \end{cases}$$

and the $q-$ factorial $[n]_q!$ as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q\cdots[1]_q & \text{if} \quad n \ge 1 \\ 1 & \text{if} \quad n = 0 \end{cases}$$

For the integers n and k, with $0 \le k \le n$, the $q-$ binomial coefficients are then defined as follows (see [15]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Note that the following relation is satisfied

$$[n]_q = [n-1]_q + q^{n-1}.$$

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Definition 1.1. The $q-$ derivative of a function f with respect to x is

$$D_q(f(x)) = \frac{f(qx)-f(x)}{qx-x} \quad , \quad x \neq 0 \ ; D_q(f(0)) = \lim_{x\to 0} D_q(f(x))$$

which is also known as the Jackson derivative. High $q-$ derivatives are

$$D_q^0(f(x)) = f(x) \ , \ D_q^n(f(x)) = D_q\left(D_q^{n-1}(f(x))\right) \ , \ n = 1,2,3,...$$

Note that as $q \to 1$, the $q-$ derivative approach the usual derivative.

Definition 1.2. The $q-$integration is defined as

$$\int_0^a f(t)d_qt = (1-q)a\sum_{j=0}^{\infty} f(q^ja)q^j \quad , \quad a > 0.$$

Over a general interval $[a,b]$, $0 < a < b$, one defines

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Definition 1.3. Let $f(x)$ be a continuous function on some interval $[a,b]$ and $c \in (a,b)$. Jackson's $q-$ Taylor formula (*see* [14, 15]) is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{(D_q^k f)(c)}{[k]_q!}(x-c)_q^k$$

where $(x-c)_q^k = \prod_{i=0}^{k-1}(x-cq^i)$.

First we need the following auxiliary result. Throughout the paper, we use e_i the test functions defined by $e_i(t) := t^i$ for every integer $i \geq 0$.

Lemma 1.4. ([7]) Let $L_n^{\alpha,\beta}$ be defined by (1). Then the following identities hold:

$$L_n^{\alpha,\beta}(e_0;q,x) = 1, \tag{3}$$

$$L_n^{\alpha,\beta}(e_1;q,x) = \frac{[n]_q}{[n]_q+\beta}x + \frac{\alpha}{[n]_q+\beta}, \tag{4}$$

$$L_n^{\alpha,\beta}(e_2;q,x) = \frac{[n]_q[m(n)]_q}{q\left([n]_q+\beta\right)^2}x^2 + \frac{[n]_q(2\alpha+1)}{\left([n]_q+\beta\right)^2}x + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}. \tag{5}$$

2 Some properties of Stancu type q-Baskakov-Kantorovich operators

Let $\{\varphi_n\}$ be a sequence of real functions on $\mathbb{R}_+ = [0,\infty)$ which are $k-$ times continuously $q-$ differentiable on $\mathbb{R}_+$ satisfying following conditions:

$(c1)$ $\varphi_n(0) = 1, (n = 1,2,...)$,

$(c2)$ for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, $(-1)^k D_q^k(\varphi_n(x)) \geq 0$, $x \in \mathbb{R}_+$,

$(c3)$ there exists a positive integer $m(n)$, such that

$$D_q^k(\varphi_n(x)) = -[n]_q D_q^{k-1}\varphi_{m(n)}(x); \quad k,n \in \mathbb{N}$$

$(c4)$ $\lim_{n\to\infty} \frac{[n]_q}{[m(n)]_q} = 1$.

In this paper, under the conditions $(c1)-(c4)$, we definition a new generalization of Stancu type $q-$Baskakov-Kantorovich operators as following

$$L_n^{*(\alpha,\beta)}(f;q,x) = \left([n]_q+\beta\right)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} f\left(q^{-k+1}t\right)d_qt\ , \tag{6}$$

where $x \in \mathbb{R}_+$, $n \in \mathbb{N}$, $0 \leq \alpha \leq \beta$.

Remark 2.1. Let $x \in \mathbb{R}_+$. If $\varphi_n(x) = e_q^{-[n]_qx}$, then for all $k,n \in \mathbb{N}$ we have $D_q^k(\varphi_n(x)) = (-1)^k[n]_q^k\, e_q^{-[n]_qx}$. In this case the operators $L_n^{*(\alpha,\beta)}$ reduce to Stancu type $q-$Szász$-$Kantorovich operators given as follows:

$$L_n^{*(\alpha,\beta)}(f,q,x) = \left([n]_q+\beta\right)e_q^{-[n]_qx}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{\left([n]_qx\right)^k}{[k]_q!} \int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} f\left(q^{-k+1}t\right)d_qt\ ,$$

where $e_q^{-[n]_qx}$ is $q-$analogues of the exponential function defined by

$$e_q^x = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{x^k}{[k]_q!}.$$

In each of the following theorems, we assume that $q = q_n$, where $\{q_n\}$ is a sequence of real numbers such that $0 < q_n < 1$ for all n and $\lim_{n\to\infty} q_n = 1$.

Now we give the following Lemmas, which are necessary to prove our theorems:

Lemma 2.2. The following relations are satisfied:

$$\int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} d_qt = \frac{1}{[n]_q+\beta}, \tag{7}$$

$$\int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} t d_q t = \frac{[2]_q [k]_q + q^k(1+2\alpha)}{[2]_q \left([n]_q+\beta\right)^2}, \qquad (8)$$

$$\int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} t^2 d_q t = \frac{[3]_q [k]_q^2 + q^k[k]_q\left((1+3\alpha)[2]_q+1\right)+\left(1+3\alpha+3\alpha^2\right)q^{2k}}{[3]_q\left([n]_q+\beta\right)^3}. \qquad (9)$$

Proof. From properties of $q-$analogue integration, by simple computation we obtain (7-9).

By the following Lemma Korovkin's conditions are satisfied.

Lemma 2.3. For all $x \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\alpha,\beta \geq 0$ and $0<q<1$, we have

$$L_n^{*(\alpha,\beta)}(e_0;q,x) = 1, \qquad (10)$$

$$L_n^{*(\alpha,\beta)}(e_1;q,x) = \frac{[n]_q}{[n]_q+\beta}x + \frac{q(1+2\alpha)}{[2]_q\left([n]_q+\beta\right)}, \qquad (11)$$

$$L_n^{*(\alpha,\beta)}(e_2;q,x) = \frac{[n]_q[m(n)]_q}{q\left([n]_q+\beta\right)^2}x^2 + \frac{[n]_q\left[[3]_q+q\left((1+3\alpha)[2]_q+1\right)\right]}{[3]_q\left([n]_q+\beta\right)^2}x + \frac{q^2\left(1+3\alpha+3\alpha^2\right)}{[3]_q\left([n]_q+\beta\right)^2}. \qquad (12)$$

Proof. From definition (6) and the identities(3) and (7), we can easily obtain

$$L_n^{*(\alpha,\beta)}(e_0;q,x) = \left([n]_q+\beta\right)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} d_q t = L_n^{\alpha,\beta}(e_0;q,x) = 1.$$

Now for e_1,from (3), (4) and (8) we can write

$$\begin{aligned}
L_n^{*(\alpha,\beta)}(e_1;q,x) &= \left([n]_q+\beta\right)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} q^{-k+1} t d_q t \\
&= \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{[k]_q+q^{k-1}\alpha}{q^{k-1}\left([n]_q+\beta\right)} \\
&\quad - \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}\frac{q^{k-1}\alpha}{q^{k-1}\left([n]_q+\beta\right)} \\
&\quad + \frac{q(1+2\alpha)}{[2]_q\left([n]_q+\beta\right)}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \\
&= L_n^{\alpha,\beta}(e_1;q,x) - \frac{\alpha}{[n]_q+\beta}L_n^{\alpha,\beta}(e_0;q,x) + \frac{q(1+2\alpha)}{[2]_q\left([n]_q+\beta\right)}L_n^{\alpha,\beta}(e_0;q,x) \\
&= \frac{[n]_q}{[n]_q+\beta}x + \frac{q(1+2\alpha)}{[2]_q\left([n]_q+\beta\right)}.
\end{aligned}$$

The finally, for e_2, we use (3), (4), (5) and (9), one has

$$\begin{aligned}
L_n^{*(\alpha,\beta)}(e_2;q,x) &= \left([n]_q+\beta\right)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} q^{-2k+2} t^2 d_q t \\
&= \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k\left(\frac{[k]_q+q^{k-1}\alpha}{q^{k-1}\left([n]_q+\beta\right)}\right)^2 \\
&\quad -2\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k q^{-2k+2}q^{k-1}\alpha\frac{[k]_q+q^{k-1}\alpha}{\left([n]_q+\beta\right)^2} \\
&\quad + \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k q^{-2k+2}\frac{q^{2k-2}\alpha^2}{\left([n]_q+\beta\right)^2} \\
&\quad + \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k q^{-2k+2}\left((1+3\alpha)[2]_q+1\right)\frac{[k]_q+q^{k-1}\alpha}{[3]_q\left([n]_q+\beta\right)^2} \\
&\quad - \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k q^{-2k+2}q^k q^{k-1}\alpha\frac{\left((1+3\alpha)[2]_q+1\right)}{[3]_q\left([n]_q+\beta\right)^2} \\
&\quad + \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k\frac{q^2\left(1+3\alpha+3\alpha^2\right)}{[3]_q\left([n]_q+\beta\right)^2} \\
&= L_n^{\alpha,\beta}(e_2;q,x) - \frac{2\alpha}{[n]_q+\beta}L_n^{\alpha,\beta}(e_1;q,x) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2}L_n^{\alpha,\beta}(e_0;q,x) \\
&\quad + \frac{q\left((1+3\alpha)[2]_q+1\right)}{[3]_q\left([n]_q+\beta\right)}L_n^{\alpha,\beta}(e_1;q,x) - q\alpha\frac{\left((1+3\alpha)[2]_q+1\right)}{[3]_q\left([n]_q+\beta\right)^2}L_n^{\alpha,\beta}(e_0;q,x) \\
&\quad + \frac{q^2\left(1+3\alpha+3\alpha^2\right)}{[3]_q\left([n]_q+\beta\right)^2}L_n^{(\alpha,\beta)}(e_0;q,x) \\
&= \frac{[n]_q[m(n)]_q}{q\left([n]_q+\beta\right)^2}x^2 \\
&\quad + \frac{[n]_q\left([3]_q+q\left((1+3\alpha)[2]_q+1\right)\right)}{[3]_q\left([n]_q+\beta\right)^2}x + \frac{q^2\left(1+3\alpha+3\alpha^2\right)}{[3]_q\left([n]_q+\beta\right)^2}.
\end{aligned}$$

This completes the proof of Lemma 2.3.

Using above Lemma, we can obtain following theorem.

Theorem 2.4. Let $q_n \in (0,1)$ and $\lim_{n\to\infty} q_n = 1$. Then the sequence $\left\{L_n^{*(\alpha,\beta)}(f;q_n,.)\right\}$ converges to f uniformly on $[0,A]$ for each $f \in C(\mathbb{R}_+)$ and $A > 0$.

3 Rate of convergence

$B_{\rho_\gamma}(\mathbb{R}_+)$, the weighted space of real valued functions f defined on $\mathbb{R}_+$ with the property $|f(x)| \le M_f \rho_\gamma(x)$ where $\rho_\gamma(x) = 1 + x^{\gamma+2}$ and M_f is constant depending on the function f. We also consider the weighted subspace $C_{\rho_\gamma}(\mathbb{R}_+)$ of $B_{\rho_\gamma}(\mathbb{R}_+)$ given by

$$C_{\rho_\gamma}(\mathbb{R}_+) := \left\{ f \in B_{\rho_\gamma}(\mathbb{R}_+) : f \text{ continuous on } \mathbb{R}_+ \right\}.$$

The norm in B_{ρ_γ} is defined as

$$\|f\|_{\rho_\gamma} = \sup_{x\in\mathbb{R}_+} \frac{|f(x)|}{\rho_\gamma(x)}.$$

We can give some estimations of the errors $\left|L_n^{*(\alpha,\beta)}(f;q,x) - f(x)\right|$, $n \in \mathbb{N}$, for unbounded functions by using a weighted modulus of smoothness associated to the space $B_{\rho_\gamma}(\mathbb{R}_+)$.

We consider

$$\Omega_{\rho_\gamma}(f;\delta) = \sup_{x\ge 0,\ 0<h\le\delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^{2+\gamma}}, \quad \delta > 0, \quad \gamma \ge 0. \tag{13}$$

It is evident that for each $f \in B_{\rho_\gamma}(\mathbb{R}_+)$, $\Omega_{\rho_\gamma}(f;.)$ is well defined and

$$\Omega_{\rho_\gamma}(f;\delta) \le 2\|f\|_{\rho_\gamma}.$$

The weighted modulus of smoothness $\Omega_{\rho_\gamma}(f;.)$ possesses the following properties.

$$\Omega_{\rho_\gamma}(f;\lambda\delta) \le (\lambda+1)\Omega_{\rho_\gamma}(f;\delta), \ \delta > 0, \ \lambda > 0, \tag{14}$$

$$\Omega_{\rho_\gamma}(f;n\delta) \le n\Omega_{\rho_\gamma}(f;\delta), \ n \in N,$$

$$\lim_{\delta\to 0^+} \Omega_{\rho_\gamma}(f;\delta) = 0.$$

As it is known, weighted Korovkin type theorems have been proven by Gadjiev (see [10]).

Theorem 3.1. Let $q \in (0,1)$ and $\gamma \ge 0$. For all non-decreasing $f \in B_{\rho_\gamma}(\mathbb{R}_+)$ we have

$$\left|L_n^{*(\alpha,\beta)}(f;q,x) - f(x)\right| \le \sqrt{L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}^2;q,x)} \left(1 + \frac{1}{\delta}\sqrt{L_n^{*(\alpha,\beta)}(\psi_x^2;q,x)}\right) \Omega_{\rho_\gamma}(f;\delta),$$

$x \ge 0$, $\delta > 0$, $n \in \mathbb{N}$, where $\mu_{x,\gamma}(t) := 1 + (x + |t-x|)^{2+\gamma}$, $\psi_x(t) := |t-x|$, $t \ge 0$.

Proof. Let $n \in \mathbb{N}$ and $f \in B_{\rho_\gamma}(\mathbb{R}_+)$. From (13) and (14), we can write

$$\begin{aligned} |f(t) - f(x)| &\le \left(1 + (x+|t-x|)^{2+\gamma}\right)\left(1 + \frac{1}{\delta}|t-x|\right)\Omega_{\rho_\gamma}(f;\delta) \\ &= \mu_{x,\gamma}(t)\left(1 + \frac{1}{\delta}\psi_x(t)\right)\Omega_{\rho_\gamma}(f;\delta). \end{aligned}$$

Taking into account the definition of $q-$ integration, we get

$$\int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} f\left(q^{-k+1}t\right) d_q t = q^{k-1} \int_{\frac{[k]_q+q^{k-1}\alpha}{q^{k-2}\left([n]_q+\beta\right)}}^{\frac{[k+1]_q+q^k\alpha}{q^{k-1}\left([n]_q+\beta\right)}} f(t)\, d_q t. \tag{15}$$

Consequently, the operators $L_n^{*(\alpha,\beta)}$ can be expressed as follows

$$L_n^{*(\alpha,\beta)}(f;q,x) = \left([n]_q + \beta\right) \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_n(x))}{[k]_q!} (-x)^k q^{k-1} \int_{\frac{[k]_q+q^{k-1}\alpha}{q^{k-2}\left([n]_q+\beta\right)}}^{\frac{[k+1]_q+q^k\alpha}{q^{k-1}\left([n]_q+\beta\right)}} f(t)\, d_q t.$$

By using the Cauchy-Schwartz inequality and (15), we obtain

$$\begin{aligned} &\left|L_n^{*(\alpha,\beta)}(f;q,x) - f(x)\right| \\ &\le \left([n]_q + \beta\right) \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_n(x))}{[k]_q!} (-x)^k q^{k-1} \int_{\frac{[k]_q+q^{k-1}\alpha}{q^{k-2}\left([n]_q+\beta\right)}}^{\frac{[k+1]_q+q^k\alpha}{q^{k-1}\left([n]_q+\beta\right)}} |f(t) - f(x)|\, d_q t \\ &\le \left(L_n^{*(\alpha,\beta)}(\mu_{x,\gamma};q,x) + \frac{1}{\delta} L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}\psi_x;q,x)\right)\Omega_{\rho_\gamma}(f;\delta) \\ &\le \sqrt{L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}^2;q,x)}\left(1 + \frac{1}{\delta}\sqrt{L_n^{*(\alpha,\beta)}(\psi_x^2;q,x)}\right)\Omega_{\rho_\gamma}(f;\delta). \end{aligned}$$

Lemma 3.2. For $m \in \mathbb{N}$ and $q \in (0,1)$ we have

$$L_n^{*(\alpha,\beta)}(e_m;q,x) \le A_{m,q}(1 + x^m), \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N},$$

where $A_{m,q}$ is a positive constant depending only on m, α and q.

Proof. For $k \in \mathbb{N}$ and $0 < q < 1$ the following inequality holds true

$$1 \le [k+1]_q \le 2[k]_q. \tag{16}$$

Thus, for $m \in \mathbb{N}$, from (1) and (16) we get

$$\begin{aligned} L_n^{\alpha,\beta}(e_m;q,x) &= \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \frac{1}{q^{km-m}}\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)^m \\ &= \frac{x[n]_q}{[n]_q+\beta}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_{m(n)}(x))}{[k]_q!}(-x)^k \frac{1}{q^{k(m-1)}}\left(\frac{[k]_q+q^k\alpha}{[n]_q+\beta}\right)^{m-1} \\ &+\frac{\alpha}{[n]_q+\beta}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \left(\frac{[k]_q+q^{k-1}\alpha}{q^{k-1}\left([n]_q+\beta\right)}\right)^{m-1} \\ &\le \frac{x[n]_q}{[n]_q+\beta}\varphi_{m(n)}(x)\left(\frac{1+\alpha}{[n]_q+\beta}\right)^{m-1} \\ &+\frac{x[n]_q}{[n]_q+\beta}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{D_q^k(\varphi_{m(n)}(x))}{[k]_q!}(-x)^k \left(\frac{2[k]_q+q^k\alpha}{q^k\left([n]_q+\beta\right)}\right)^{m-1} \\ &+\frac{\alpha}{[n]_q+\beta}L_n^{\alpha,\beta}(e_{m-1};q,x) \\ &= \frac{x[n]_q}{[n]_q+\beta}\varphi_{m(n)}(x)\left(\frac{1+\alpha}{[n]_q+\beta}\right)^{m-1} \\ &+\frac{x[n]_q}{[n]_q+\beta}\left(\frac{2}{q}\right)^{m-1} L_{n+1}^{\alpha,\beta}(e_{m-1};q,x)+\frac{\alpha}{[n]_q+\beta}L_n^{\alpha,\beta}(e_{m-1};q,x) \end{aligned}$$

and we have

$$\begin{aligned} L_n^{\alpha,\beta}(e_m;q,x) &\le x+\left(\frac{2}{q}\right)^{m-1}\frac{1}{[n]_q+\beta}\left(x[n]_q+\alpha q^{m-1}\right)L_{n+1}^{\alpha,\beta}(e_{m-1};q,x) \\ &\le 2m\left(\frac{2}{q}\right)^{m-1}\frac{1}{[n]_q+\beta}\left([n]_q(1+x^m)+\alpha^m q^{\frac{m(m-1)}{2}}\right). \end{aligned}$$

based on the above inequality and by using the mathematical induction over $m \in \mathbb{N}$, we obtain

$$L_n^{\alpha,\beta}(e_m;q,x) \le B_{m,q}(1+x^m),$$

$x \in \mathbb{R}_+$, $n \in \mathbb{N}$, where

$$B_{m,q} := 2m\left(\frac{2}{q}\right)^{\frac{m(m-1)}{2}}\left(1+\alpha^m q^{\frac{m(m-1)}{2}}\right). \tag{17}$$

On the other hand,

$$\begin{aligned} L_n^{*(\alpha,\beta)}(e_m;q,x) &= \left([n]_q+\beta\right)\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k \int_{q\left(\frac{[k]_q+q^{k-1}\alpha}{[n]_q+\beta}\right)}^{\frac{[k+1]_q+q^k\alpha}{[n]_q+\beta}} e_m\left(q^{-k+1}t\right)d_qt \\ &= \frac{[n]_q+\beta}{\left([n]_q+\beta\right)^{m+1}}\sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{D_q^k(\varphi_n(x))}{[k]_q!}(-x)^k\frac{q^{-km+m}}{[m+1]_q} \\ &\times\left\{\left([k+1]_q+q^k\alpha\right)^{m+1}-q^{m+1}\left([k]_q+q^{k-1}\alpha\right)^{m+1}\right\}. \end{aligned}$$

We consider

$$\left([k+1]_q+q^k\alpha\right)^{m+1}-q^{m+1}\left([k]_q+q^{k-1}\alpha\right)^{m+1}.$$

For $k \in \mathbb{N}$, one obtain

$$\begin{aligned} &\left([k+1]_q+q^k\alpha\right)^{m+1}-q^{m+1}\left([k]_q+q^{k-1}\alpha\right)^{m+1} \\ &= \left(\left([k+1]_q+q^k\alpha\right)^m+q\left([k+1]_q+q^k\alpha\right)^{m-1}\right. \\ &\left.\left([k]_q+q^{k-1}\alpha\right)+\ldots+q^m\left([k]_q+q^{k-1}\alpha\right)^m\right) \\ &\le (m+1)\left([k+1]_q+q^k\alpha\right)^m \\ &\le (m+1)2^m\left([k]_q+q^k\alpha\right)^m, \end{aligned}$$

hence, we can write

$$\begin{aligned} L_n^{*(\alpha,\beta)}(e_m;q,x) &\le \frac{\varphi_n(x)(m+1)(1+\alpha)^m q^m}{\left([n]_q+\beta\right)^m[m+1]_q} \\ &+\frac{2^m(m+1)}{[m+1]_q}L_n^{\alpha,\beta}(e_m;q,x) \\ &\le A_{m,q}(1+x^m), \end{aligned}$$

where $A_{m,q} := \frac{(m+1)(1+\alpha)^m q^m}{[m+1]_q}+\frac{2^m(m+1)}{[m+1]_q}B_{m,q}$ and $B_{m,q}$ is given by (17).

Remark 3.3. Since any linear positive operator is monotone, from Lemma 3.2 we can easily see that $L_n^{*(\alpha,\beta)}(f;q,.) \in B_{\rho_\gamma}(\mathbb{R}_+)$ for each $f \in B_{\rho_\gamma}(\mathbb{R}_+)$, $\gamma \in \mathbb{N}_0$.

Theorem 3.4. Let $f \in B_{\rho_\gamma}(\mathbb{R}_+)$ be a non-decreasing function, then

$$\left\|L_n^{*(\alpha,\beta)}(f;q_n,.)-f\right\|_{\rho_{\gamma+1}} \le K_{\gamma,q_0}\Omega_{\rho_\gamma}(f;\delta_n),$$

where $\delta_n := \sqrt{\frac{[n]_{q_n}+1}{q_n\left([n]_{q_n}+\beta\right)}}$ and K_{γ,q_0} is a positive constant independent on f and n.

Proof. The identities (3)-(5) imply

$$\begin{aligned} L_n^{*(\alpha,\beta)}(\psi_x^2;q_n,x) &= L_n^{*(\alpha,\beta)}\left((t-x)^2;q_n,x\right) \\ &= \frac{[n]_{q_n}[m(n)]_{q_n}}{q_n\left([n]_{q_n}+\beta\right)^2}x^2 \\ &+\frac{[n]_{q_n}\left([3]_{q_n}+q_n\left((1+3\alpha)[2]_{q_n}+1\right)\right)}{[3]_{q_n}\left([n]_{q_n}+\beta\right)^2}x \\ &+\frac{q_n^2(1+3\alpha+3\alpha^2)}{[3]_{q_n}\left([n]_{q_n}+\beta\right)^2}-2x\left\{\frac{[n]_{q_n}}{[n]_{q_n}+\beta}x+\frac{q_n(1+2\alpha)}{[2]_{q_n}\left([n]_{q_n}+\beta\right)}\right\}+x^2 \\ &\le \frac{\left([n]_{q_n}+1+\beta\right)x^2}{q_n\left([n]_{q_n}+\beta\right)}+\frac{2(3\alpha+3)}{q_n\left([n]_{q_n}+\beta\right)}x+\frac{1+3\alpha+3\alpha^2}{q_n\left([n]_{q_n}+\beta\right)} \\ &\le \frac{9(1+\beta)^2\rho_0(x)}{q_n\left([n]_{q_n}+\beta\right)}\left\{[n]_{q_n}+1\right\} \end{aligned}$$

Let $\gamma \in \mathbb{N}_0$ and $f \in B_{\rho_\gamma}(\mathbb{R}_+)$ be a fixed function. From

Theorem 3.1 and above inequality, we can write

$$\begin{aligned}
&\frac{\left|L_n^{*(\alpha,\beta)}(f;q_n,x)-f(x)\right|}{\rho_{\gamma+1}(x)} \\
&\leq \sqrt{\frac{L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}^2;q_n,x)}{\rho_{\gamma+1}^2(x)}}\left(1+\frac{1}{\delta_n}\sqrt{L_n^{*(\alpha,\beta)}(\psi_x^2;q_n,x)}\right)\Omega_{\rho_\gamma}(f;\delta_n) \\
&\leq \sqrt{\frac{L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}^2;q_n,x)\rho_0(x)}{\rho_{\gamma+1}^2(x)}}\left(1+\frac{1}{\delta_n}\sqrt{\frac{9(1+\beta)^2\rho_0(x)}{q_n\left([n]_{q_n}+\beta\right)}\left\{[n]_{q_n}+1\right\}}\right)\Omega_{\rho_\gamma}(f;\delta_n) \\
&\leq 12(1+\beta)\sqrt{\frac{L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}^2;q_n,x)}{\rho_{2(\gamma+1)}(x)}}\left(1+\frac{1}{\delta_n}\sqrt{\frac{[n]_{q_n}+1}{q_n\left([n]_{q_n}+\beta\right)}}\right)\Omega_{\rho_\gamma}(f;\delta_n).
\end{aligned}$$

Since

$$\begin{aligned}
\mu_{x,\gamma}^2(t) &= \left(1+(x+|t-x|)^{2+\gamma}\right)^2 \leq 2\left(1+(2x+t)^{4+2\gamma}\right) \\
&\leq 2\left(1+2^{4+2\gamma}\left((2x)^{4+2\gamma}+t^{4+2\gamma}\right)\right),
\end{aligned}$$

from Lemma 3.2, we get

$$L_n^{*(\alpha,\beta)}(\mu_{x,\gamma}^2;q_n,x) \leq \lambda_{\gamma,q_n}^2\rho_{2(\gamma+1)}(x),$$

where $\lambda_{\gamma,q_n}^2 = 2^{5+2\gamma}\left(2^{4+2\gamma}+A_{4+2\gamma,q_n}\right)$. Choosing $\delta_n := \sqrt{\frac{[n]_{q_n}+1}{q_n\left([n]_{q_n}+\beta\right)}}$ and $K_{\gamma,q_0} := 24(1+\beta)\lambda_{\gamma,q_0}$, where $q_0 := \min_{n\in\mathbb{N}} q_n$, the proof is finished.

Remark 3.5. If $\lim_{n\to\infty} q_n = 1$, then $\lim_{n\to\infty}\delta_n = 0$, which yields that $\lim_{n\to\infty}\Omega_{\rho_\gamma}(f;\delta_n) = 0$. Therefore Theorem 3.4 gives the rate of convergence of $L_n^{*(\alpha,\beta)}$ to f.

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