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A Structure-Preserving Modified Exponential Method for the Fisher–Kolmogorov Equation

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Abstract: In this work, we propose an exponential-type discretization of the well-known Fisher's equation from population dynamics. Only non-negative, bounded and monotone solutions are physically relevant in this note, and the discretization that we provide is able to preserve these properties. The method is a modified explicit exponential technique which has the advantage of requiring a small amount of computational resources and computer time. It is worthwhile to notice that our technique has the advantage over other exponential-like methodologies that it yields no singularities. In addition, the preservation of the properties of non-negativity, boundedness and monotonicity are distinctive features of our method. As consequences of the analytical properties of the technique, the method is capable of preserving the spatial and the temporal monotonicity of solutions. Qualitative and quantitative numerical simulations assess the convergence properties of the finite-difference scheme proposed in this manuscript.

Keywords: Structure-preserving method, Exponential technique, Correction for non-singularity

1 Introduction

Throughout, we let Ω be an open and bounded interval of $\mathbb{R}$. Let $u = u(x,t)$ be a real function defined on the closure of $\Omega \times \mathbb{R}^+$, which is twice differentiable in the interior of its domain and which satisfies the initial-boundary-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)\,(1-u(x,t)), \\ u(x,0) = \phi(x),\ \forall x \in \overline{\Omega}, \\ \dfrac{\partial u}{\partial \mathbf{n}}(x,t) = 0, \quad \forall x \in \partial\Omega, \forall t \in \mathbb{R}^+ \cup \{0\}, \end{cases} \tag{1}$$

for some continuous function $\phi : \overline{\Omega} \to \mathbb{R}$ that satisfies $0 \leq \phi(x) \leq 1$ at each $x \in \overline{\Omega}$. Clearly, the partial differential equation of (1) is the classical Fisher's equation, which was investigated simultaneously and independently in 1937 by R. A. Fisher [10] and A. N. Kolmogorov, I. G. Petrovsky and N. S. Piscounov [17]. Many generalizations of Fisher's equation are available in the literature nowadays [15,20,21,22,23,24].

Let κ be a positive number. It is well-known that the one-dimensional Fisher's equation has non-negative and bounded solutions. In fact, some of those solutions are traveling waves that connect monotonically and asymptotically the constant solutions of Fisher's equation [1]. In view of this fact, we will restrict our attention to solutions satisfying $u(x,t) \geq 0$ for each $x \in \overline{\Omega}$ and $t \geq 0$. After dividing both sides of (1) by the nonnegative function $u(x,t)+\kappa$ and using the chain rule, we obtain

$$\frac{\partial \ln(u(x,t)+\kappa)}{\partial t} = \frac{\frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)\,(1-u(x,t))}{u(x,t)+\kappa}, \tag{2}$$

for each $x \in \Omega$ and each $t \in \mathbb{R}^+$. Associated to this differential equation, we consider the same initial-boundary conditions as in the problem (1) for a continuous and nonnegative function ϕ.

In the present work, we are interested in developing a numerical method to approximate the solution of (1), with the following characteristics:

– The non-negativity and the boundedness of approximations are preserved.

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- The technique is computationally fast.
- The method is easy to implement in any computer language
- The computational implementation allows to employ fine grid meshes.

More precisely, we are interested in developing a variation of the exponential approach proposed in [4,5]. That family of exponential methods required for the approximations to be strictly positive at all times. Moreover, those techniques were sensitive to solutions close to zero and they exhibited instabilities in those circumstances. The correction proposed in the present work saves those shortcomings. In fact, we show that our modified exponential technique is capable of preserving the non-negativity, the boundedness and the monotonicity of the approximations. It is important to point out that these analytical characteristics of the solutions are indeed present in many mathematical models [14,16], especially in some traveling-wave solutions of nonlinear partial differential equations [25,12].

This note is sectioned as follows. In Section 2, we introduce the discrete nomenclature employed in this work and present the modified method of interest. An explicit presentation of our finite-difference scheme is proposed in that stage. Section 3 is devoted to prove the most relevant properties of the method, namely, the preservation of the non-negativity, the boundedness and the monotonicity. Some qualitative and quantitative simulations are presented in Section 4. In that section, we provide numerical results in support of the convergent character of our technique. Finally, we close this work with a section of concluding remarks and directions of future research.

2 Exponential method

Let M be a positive integer and suppose that $\overline{\Omega} = [a,b] \subset \mathbb{R}$, where $a < b$. Fix a uniform partition $\{x_m\}_{m=0}^{M}$ of $[a,b]$, with step-size equal to Δx. Also, let $\{t_k\}_{k=0}^{\infty}$ be a partition of the temporal interval $[0,\infty)$. For each $k \in \mathbb{Z}^+ \cup \{0\}$, let

$$\Delta t_k = t_{k+1} - t_k, \tag{3}$$

$$R_k = \frac{\Delta t_k}{(\Delta x)^2}. \tag{4}$$

Let w_m^k represent an estimation to the value $u(x_m,t_k)$, for each $m \in \{0,1,\ldots,M\}$ and $k \in \mathbb{Z}^+ \cup \{0\}$. Throughout, we use the following operators:

$$\delta_t w_m^k = \frac{w_m^{k+1} - w_m^k}{\Delta t_k}, \tag{5}$$

$$\delta_{xx} w_m^k = \frac{w_{m+1}^k - 2w_m^k + w_{m-1}^k}{(\Delta x)^2}, \tag{6}$$

for each $m \in \{1,\ldots,M-1\}$ and $k \in \mathbb{Z}^+ \cup \{0\}$. Obviously, these operators approximate the values of the functions u_t and u_{xx} at the point (x_m,t_k) with order of consistency

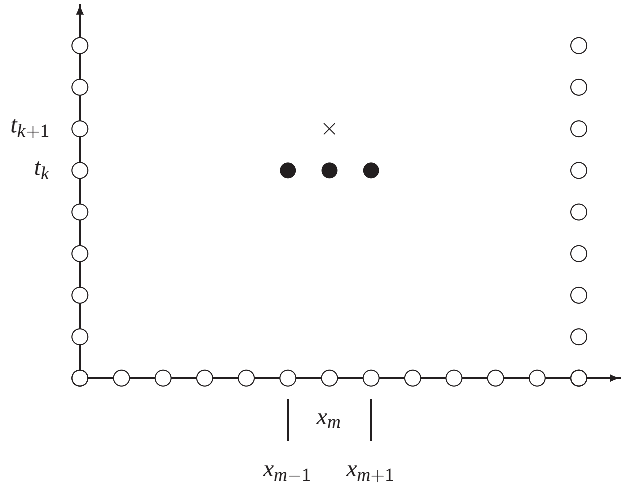

Fig. 1: Forward-difference stencil of the finite-difference method (7) at the time t_k around the node x_m. The black circles represent the known approximations to the exact solution at time t_k, and the cross denotes the unknown approximation at time t_{k+1}.

equal to Δt and $(\Delta x)^2$, respectively. Finally, we will impose exact discrete conditions at the time $t = 0$, and discrete homogeneous Neumann conditions at the boundary of the spatial domain.

Let $m \in \{1,\ldots,M-1\}$ and $k \in \mathbb{Z}^+ \cup \{0\}$, and let $(\kappa_k)_{k=0}^{\infty}$ be a sequence of positive numbers. We discretize the partial differential equation (2) at the point (x_m,t_k) as follows:

$$\delta_t \ln(w_m^k + \kappa_k) = \frac{\delta_{xx} w_m^k + w_m^k(1-w_m^k)}{w_m^k + \kappa_k}. \tag{7}$$

Equivalently,

$$w_m^{k+1} = (w_m^k + \kappa_k)\exp\left[\frac{\Delta t_k \delta_{xx} w_m^k + \Delta t_k w_m^k (1-w_m^k)}{w_m^k + \kappa_k}\right] - \kappa_k, \tag{8}$$

which evinces the explicit nature of (7). For convenience, the forward-difference stencil of this technique is presented in Figure 1. An alternative expression of this method is readily at hand if we consider the following notation. Let

$$A_k = -\Delta t_k, \tag{9}$$

$$B_k = \Delta t_k - 2R_k, \tag{10}$$

$$C_{m,w}^k = R_k(w_{m+1}^k + w_{m-1}^k). \tag{11}$$

Clearly, the method (7) can be rewritten iteratively as

$$w_m^{k+1} = F_{m,w}^k(w_m^k), \tag{12}$$

where

$$F_{m,w}^k(w) = (w+\kappa_k)\exp\left[\frac{A_k w^2 + B_k w + C_{m,w}^k}{w+\kappa_k}\right] - \kappa_k. \tag{13}$$

Before closing the present section we would like to highlight the easiness to implement the finite-difference scheme (7) in a computer program. There are reports in the literature which describe discretizations similar to this method, most notably [3,13]. However, those approaches use values of $\kappa_k = 0$ for each $k \in \mathbb{Z}^+ \cup \{0\}$. Those techniques become unstable when the numerical solutions are close or equal to zero at any point of the grid. Indeed, notice that (13) becomes

$$F^k_{m,w}(w) = w\exp\left[\frac{A_k w^2 + B_k w + C^k_{m,w}}{w}\right]. \qquad (14)$$

in that case. So, values of w too close or equal to zero may result in the well-known computational instabilities. From that perspective, the inclusion of the positive parameters κ_k in the finite-difference discretization (7) avoids divisions by zero when the approximations w^k_m may take on that value.

3 Dynamical properties

The present section is devoted to showing that the finite-difference method (7) is a dynamically consistent technique in the Mickens' sense, that is, that the method presented in this work preserves many mathematical features of some relevant solutions of the classical Fisher's equation [6,18,19]. More precisely, we establish conditions that guarantee the preservation of the non-negativity, the boundedness and the monotonicity of approximations obtained through (7). To that end, it suffices to bound the range of the function $F^k_{m,w} : [0,1] \to \mathbb{R}$ of Equation (13) within $[0,1]$. We will use $\mathbf{w}^k$ to represent the ordered vector of approximations at each time t_k, that is, we let

$$\mathbf{w}^k = (w^k_0, w^k_1, \ldots, w^k_M) \qquad (15)$$

for each $k \in \mathbb{Z}^+ \cup \{0\}$.

Lemma 1. *Let $k \in \mathbb{Z}^+ \cup \{0\}$, and let $0 \le \mathbf{w}^k \le 1$. Then the function $F^k_{m,w}$ is increasing in $[0,1]$ for each $m \in \{1,\ldots,M-1\}$ when $\kappa_k(\Delta x)^2 > 2$ and*

$$\Delta t_k\left(\frac{2}{(\Delta x)^2} + 1\right) < 1. \qquad (16)$$

Proof. For each $m \in \{1,\ldots,M-1\}$ and each $w \in [0,1]$, we define the function

$$v = g^k_{m,w}(w) = A_k w^2 + (1 + 2\kappa_k A_k)w + D^k_{m,w}, \qquad (17)$$

where

$$D^k_{m,w} = \kappa_k(1 - B_k) - C^k_{m,w}. \qquad (18)$$

Graphically, the functions $g^k_{m,w}$ are parabolas in the u-v plane that open in the negative direction of the v-axis. The inequality (16) implies that $2R_k < 1$. Moreover,

$$\begin{aligned} g^k_{m,w}(0) &= \kappa_k(1-2R_k) + \kappa_k\Delta t_k - R_k(w^k_{m+1} + w^k_{m-1}) \\ &\ge \kappa_k(1-2R_k) + \left[\kappa_k - \frac{2}{(\Delta x)^2}\right]\Delta t_k \end{aligned} \qquad (19)$$

and

$$\begin{aligned} g^k_{m,w}(1) &= 1 - R_k(w^k_{m+1} + w^k_{m-1}) - (1+\kappa_k)\Delta t_k \\ &\quad + \kappa_k(1-2R_k) \\ &\ge (1+\kappa_k)(1 - 2R_k - \Delta t_k) \\ &= (1+\kappa_k)\left[1 - \Delta t_k\left(\frac{2}{(\Delta x)^2} + 1\right)\right]. \end{aligned} \qquad (20)$$

Under the hypotheses both $g^k_{m,w}(0)$ and $g^k_{m,w}(1)$ are positive, and this implies that $g^k_{m,w}$ is positive in the entire interval $[0,1]$. Note now that the derivative of $F^k_{m,w}$ in $[0,1]$ is given by

$$\frac{dF^k_{m,w}}{dw} = \frac{g^k_{m,w}(w)}{w + \kappa_k}\exp\left[\frac{A_k w^2 + B_k w + C^k_{m,w}}{w + \kappa_k}\right], \qquad (21)$$

and that this function is positive in $[0,1]$. We conclude that the function $F^k_{m,w}$ is increasing in that interval.

The following is the main result on the existence and uniqueness of positive and bounded solutions of (7).

Proposition 1. *Let $0 \le \mathbf{w}^0 \le 1$, and suppose that $\kappa_k(\Delta x)^2 > 2$ and (16) hold for each $k \in \mathbb{Z}^+ \cup \{0\}$. Then there exists a unique sequence $(\mathbf{w}_k)_{k=0}^{\infty}$ satisfying (7) with discrete homogeneous Neumann boundary conditions, such that*

$$0 \le \mathbf{w}^k \le 1 \qquad (22)$$

for each $k \in \mathbb{Z}^+ \cup \{0\}$.

Proof. By hypothesis, the conclusion of the proposition is valid for $k = 0$. Suppose then that the conclusion is true for some $k \in \mathbb{Z}^+ \cup \{0\}$. By Lemma 1, the function $F^k_{m,w}$ is increasing in $[0,1]$ for each $m \in \{1,\ldots,M-1\}$. Moreover, each $C^k_{m,w}$ is nonnegative, so

$$F^k_{m,w}(0) = \kappa_k \exp(C^k_{m,w}/\kappa_k) - \kappa_k \ge 0. \qquad (23)$$

On the other hand,

$$\begin{aligned} F^k_{m,w}(1) &= (1+\kappa_k)\exp\left[-\frac{R_k(2 - w^k_{m+1} - w^k_{m-1})}{1+\kappa_k}\right] - \kappa_k \\ &\le (1+\kappa_k) - \kappa_k = 1. \end{aligned} \qquad (24)$$

As a consequence,

$$0 \le F^k_{m,w}(0) \le F^k_{m,w}(w) \le F^k_{m,w}(1) \le 1 \qquad (25)$$

hold for each $w \in [0,1]$. So $0 \le w^{k+1}_m \le 1$ for each $m \in \{1,\ldots,M-1\}$, and the conclusion follows now by induction.

We wish to establish now that the finite-difference scheme (7) is actually a monotonicity-preserving method under the same hypotheses of the previous theorem. In the next result, we will employ the notation $\mathbf{v} \leq \mathbf{w}$ to represent two real vectors $\mathbf{v}$ and $\mathbf{w}$ of the same dimension, such that each of the components of $\mathbf{v}$ is less than or equal to the corresponding component of $\mathbf{w}$.

Proposition 2. *Let $0 \leq \mathbf{v}^0 \leq \mathbf{w}^0 \leq 1$, and let $\kappa_k(\Delta x)^2 > 2$ and (16) hold for each $k \in \mathbb{Z}^+ \cup \{0\}$. Let $(\mathbf{v}_k)_{k=0}^{\infty}$ and $(\mathbf{w}_k)_{k=0}^{\infty}$ be the unique solutions of (7) with discrete homogeneous Neumann boundary conditions for $\mathbf{v}^0$ and $\mathbf{w}^0$, respectively. Then*

$$0 \leq \mathbf{v}^k \leq \mathbf{w}^k \leq 1 \tag{26}$$

for each $k \in \mathbb{Z}^+ \cup \{0\}$.

Proof. Lemma 1 guarantees that all the functions $F_{m,w}^k$ are increasing, and Proposition 1 assures that there exist unique solutions $(\mathbf{v}_k)_{k=0}^{\infty}$ and $(\mathbf{w}_k)_{k=0}^{\infty}$ of the numerical method for the initial conditions $\mathbf{v}^0$ and $\mathbf{w}^0$, respectively, such that $0 \leq \mathbf{v}^k \leq 1$ and $0 \leq \mathbf{w}^k \leq 1$ for each $k \in \mathbb{Z}^+ \cup \{0\}$. The proposition is valid for $k = 0$ by hypothesis, so let us suppose that it is true for some $k \in \mathbb{Z}^+ \cup \{0\}$. This implies that the following identities and inequalities are satisfied for each $m \in \{1, \ldots, M-1\}$:

$$v_{m+1}^k = F_{m,v}^k(v_m^k) \leq F_{m,v}^k(w_m^k) \leq F_{m,w}^k(w_m^k) = w_m^{k+1}. \tag{27}$$

As a consequence,

$$0 \leq \mathbf{v}^{k+1} \leq \mathbf{w}^{k+1} \leq 1, \tag{28}$$

and the conclusion of the proposition follows by induction.

The following corollaries are easy consequences of Proposition 2. The first of them indicates that the method (7) is capable of preserving the temporal monotonicity of approximations. This is an important characteristic of our finite-difference scheme in view that monotonicity is a feature present in some of the solutions of the classical Fisher's equation.

Corollary 1. *Suppose that $0 \leq \mathbf{w}^0 \leq \mathbf{w}^1 \leq 1$, and let $\kappa_k(\Delta x)^2 > 2$ and (16) hold for each $k \in \mathbb{Z}^+ \cup \{0\}$. Let $(\mathbf{w}^k)_{k=0}^{\infty}$ be solution of (7) with discrete homogeneous Neumann boundary conditions satisfying $0 \leq \mathbf{w}^k \leq 1$ for each $k \in \mathbb{Z}^+ \cup \{0\}$. Then*

$$0 \leq \mathbf{w}^k \leq \mathbf{w}^{k+1} \leq 1 \tag{29}$$

for each $k \in \mathbb{Z}^+ \cup \{0\}$. □

A real vector $\mathbf{w} = (w_1, \ldots, w_M)$ is *spatially increasing* if $w_m \leq w_{m+1}$ for each $m \in \{1, \ldots, M-1\}$. If $-\mathbf{w}$ is increasing, then we say that $\mathbf{w}$ is *spatially decreasing*. The next corollary states that the finite-difference method (7) also preserves the spatial monotonicity of approximations.

Corollary 2. *Let $0 \leq \mathbf{w}^0 \leq 1$ be spatially increasing (decreasing), and suppose that $\kappa_k(\Delta x)^2 > 2$ and (16) hold for each $k \in \mathbb{Z}^+ \cup \{0\}$. Let $(\mathbf{w}_k)_{k=0}^{\infty}$ be the respective solution of (7) with discrete homogeneous Neumann boundary conditions, such that $0 \leq \mathbf{w}^k \leq 1$ for each $k \in \mathbb{Z}^+ \cup \{0\}$. Then $\mathbf{w}^k$ is spatially increasing (decreasing) for each $k \in \mathbb{Z}^+ \cup \{0\}$.* □

As we will see below, the conditions proposed in the propositions of the present section are only sufficient conditions to guarantee the non-negativity, the boundedness and the monotonicity of approximations. The numerical experiments described in Section 4 give testimony of this fact.

4 Illustrative simulations

In the present section, we provide numerical experiments in which we verify the main characteristics of our modified exponential method, namely, its capability to preserve non-negativity, boundedness and monotonicity, as well as qualitative and quantitative numerical support on the convergence of the method.

In our experiments, we will fix a spatial domain Ω, and consider the initial-boundary-value problem (1) with suitable parameters. For simplification purposes, the constants Δt_k and κ_k will be all equal to fixed positive values Δt and κ, respectively, for each $k \in \mathbb{Z}^+ \cup \{0\}$. Our simulations were carried out using ©Matlab 7.12.0.635 (R2011a) on a ©Sony Vaio PCG-5L1P laptop computer with Kubuntu 15.10 as operating system. In terms of computational times, we are aware that better results may be obtained with more modern high performance equipment and more modest Linux/Unix distributions.

It is well known [2] that Fisher's model has an exact traveling-wave solution of the form

$$u(x,t) = \left[1 + \exp\left\{\frac{1}{\sqrt{6}}\left(x - \frac{5}{\sqrt{6}}t\right)\right\}\right]^{-2}, \tag{30}$$

which is clearly positive, bounded and monotone. In our experiments, we will let $\phi(x) = u(x,0)$ for each $x \in \Omega$, and fix discrete homogeneous Neumann conditions at the endpoints of Ω. The next examples provide qualitative and quantitative comparisons against this exact solution of Fisher's equation in order to assess numerically the convergence of the method, and the capability to preserve the positivity, the boundedness and the monotonicity of the approximations. Some qualitative comparisons are provided firstly.

Example 1. Let $\overline{\Omega} = [-20, 150]$, and fix the computational constant $\Delta x = 2$. Additionally, we let $\Delta t = 0.01$ and $\kappa = 1$. Under these circumstances, Figure 2 shows snapshots of the exact solution (30) and the approximate solution computed through (2) at the times 0, 10, 20, 30, 40 and 50. In addition to the fact that the method

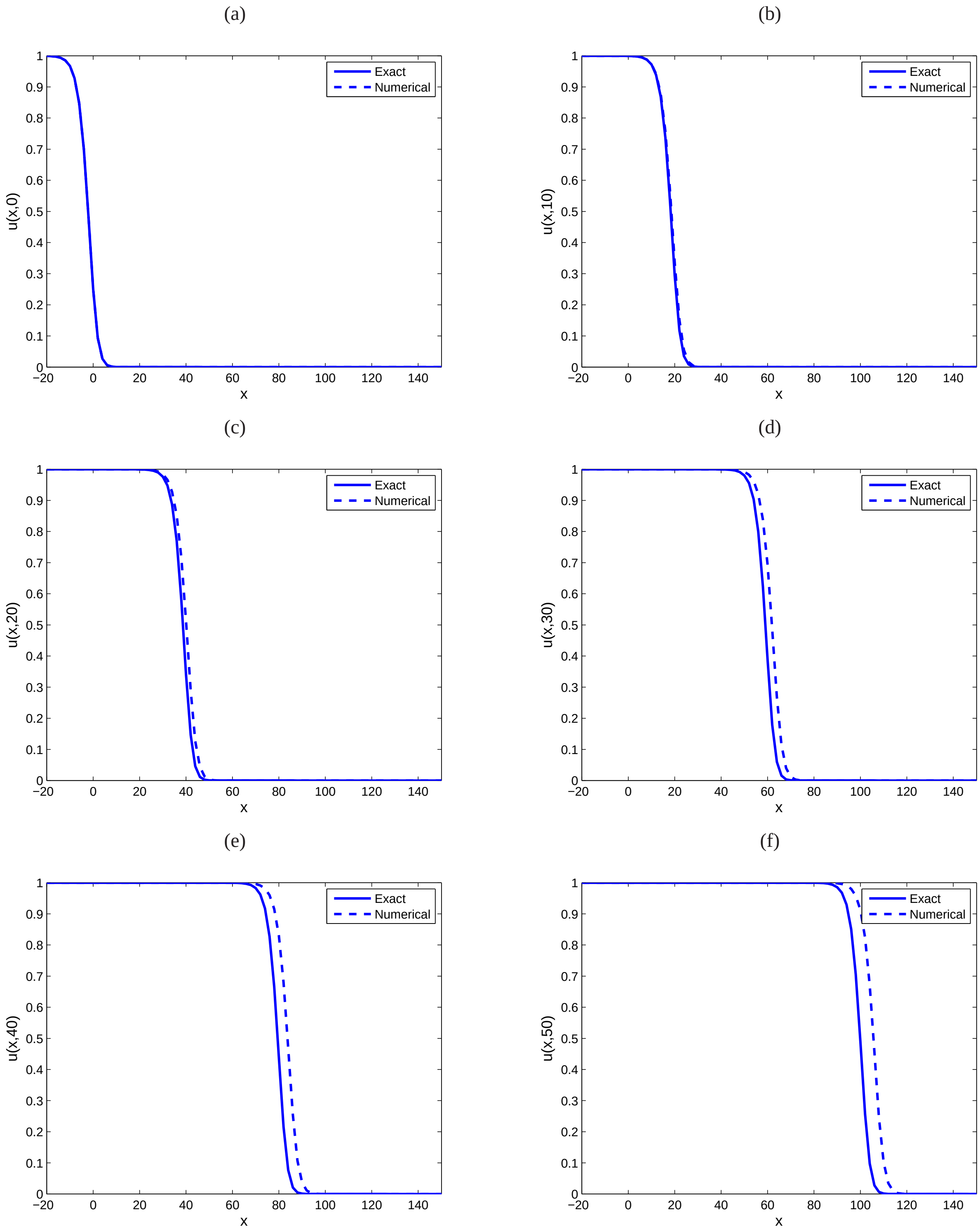

Fig. 2: Graphs of the exact (solid) and the approximate (dashed) solutions of (1). The exact solution is given by (30), and the approximations have been obtained through (7) at the times (a) $t = 0$, (b) $t = 10$, (c) $t = 20$, (d) $t = 30$, (e) $t = 40$ and (f) $t = 50$. The following computer parameters were used: $\Omega = [-20, 150]$, $\Delta x = 2$, $\Delta t = 0.01$, and $\kappa = 1$. Meanwhile, the initial profile was given by the exact solution at the time $t = 0$.

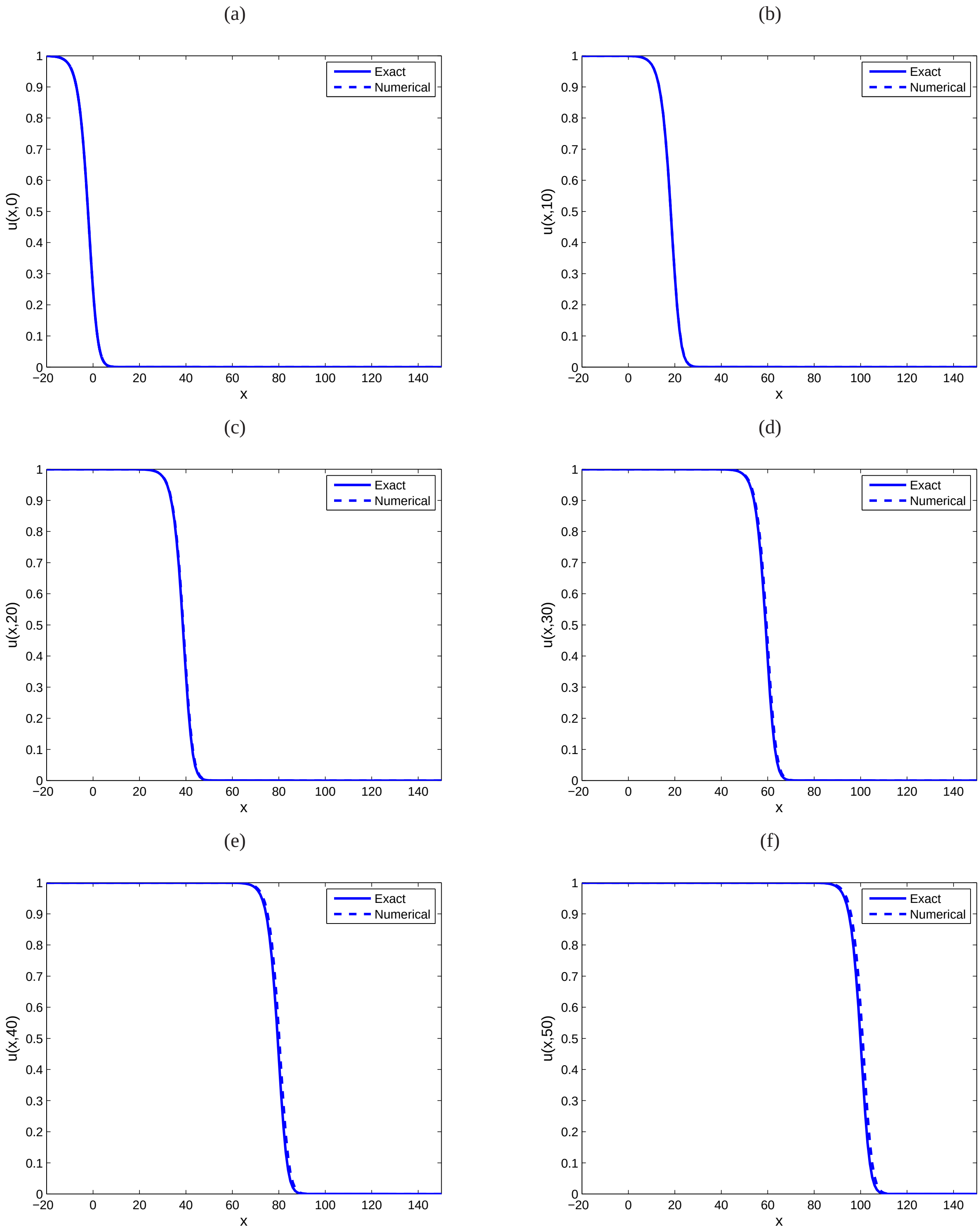

Fig. 3: Graphs of the exact (solid) and the approximate (dashed) solutions of (1). The exact solution is given by (30), and the approximations have been obtained through (7) at the times (a) $t = 0$, (b) $t = 10$, (c) $t = 20$, (d) $t = 30$, (e) $t = 40$ and (f) $t = 50$. The following computer parameters were used: $\Omega = [-20, 150]$, $\Delta x = 1$, $\Delta t = 0.01$, and $\kappa = 1$. Meanwhile, the initial profile was given by the exact solution at the time $t = 0$.

Table 1: Analysis of spatial convergence of the method (7), using $\kappa = 1$ and two fixed values of Δt, namely, 0.001 and 0.0005. The calculation of the absolute error was performed using the exact solution (30) of the classical Fisher's equation.

Spatial convergence analysis					
$\Delta t = 0.001$			$\Delta t = 0.0005$		
Δx	$\varepsilon_{\Delta x,\Delta t}$	$\rho^s_{\Delta x,\Delta t}$	Δx	$\varepsilon_{\Delta x,\Delta t}$	$\rho^s_{\Delta x,\Delta t}$
4×2^{0}	9.2300×10^{-1}	–	2×2^{0}	5.1511×10^{-1}	–
4×2^{-1}	5.1284×10^{-1}	0.8478	2×2^{-1}	1.5089×10^{-1}	1.7713
4×2^{-2}	1.4748×10^{-1}	1.7980	2×2^{-2}	3.6257×10^{-2}	2.0571
4×2^{-3}	3.2588×10^{-2}	2.1781	2×2^{-3}	7.7696×10^{-3}	2.2223

preserves the positivity, the boundedness and the monotonicity of approximations, a good qualitative agreement between the exact and the numerical solutions is readily noted from the graphs. Figure 3 is a repetition of the same experiment with $\Delta x = 1$. In this case, a better qualitative agreement between the exact and the numerical solutions is found. □

In order to provide a quantitative assessment of the performance of our method, we will compare numerically the approximations against the exact solution (30). Given a numerical approximation $\mathbf{w}^K$ at the time T, and the corresponding ordered set of exact solutions $\mathbf{u}^K$ on the same grid, we define the *absolute error* as

$$\varepsilon_{\Delta x,\Delta t} = \max\left\{|w^K_m - u^K_m| : m = 0,1,\ldots,M\right\}. \qquad (31)$$

We define the spatial and temporal rate of convergence, respectively, as

$$\rho^s_{\Delta x,\Delta t} = \log_2\left(\frac{\varepsilon_{2\Delta x,\Delta t}}{\varepsilon_{\Delta x,,\Delta t}}\right), \qquad (32)$$

$$\rho^t_{\Delta x,\Delta t} = \log_2\left(\frac{\varepsilon_{\Delta x,2\Delta t}}{\varepsilon_{\Delta x,\Delta t}}\right). \qquad (33)$$

The next example offers a brief quantitative analysis of the convergence property of the method (7).

*Example 2.*Let us fix now $\overline{\Omega} = [-20,100]$. Table 1 shows the calculated spatial rate of convergence for $\kappa = 1$ and two fixed values of Δt, namely, 0.001 and 0.0005. The results indicate that the method has a quadratic order of convergence in the spatial variable. A similar analysis of temporal convergence confirms that the method has linear order in time. □

Before we close this section, we would like to note that the simulations shown in the present section suggest that the method (7) is capable of preserving the non-negativity, the boundedness and the monotonicity of the numerical approximations even when the conditions established in the propositions of Section 3 are not satisfied. From that point of view, the hypotheses of those results are only sufficient conditions to guarantee the preservation of these mathematical features of the numerical solutions.

5 Conclusions and perspectives

In this work, we designed a discrete exponential scheme to approximate the solutions of the classical one-dimensional Fisher's equation. The method preserves the positivity and the boundedness of the solutions, and it is a variation of some exponential methods available in the literature which are computationally sensitive to approximate solutions that are close or equal to zero. The inclusion of a positive parameter to avoid singularities results in a family of new methods that present various advantages over other standard approaches in the literature. The following are some of the distinctive features of our method:

- It is an explicit technique.
- Its computational implementation is relatively easy.
- It is computationally fast.
- It contemplates the presence of a parameter to avoid singularities. This feature clearly improves similar approaches reported in the literature [3,13].
- It requires a smaller amount of computer memory.
- It preserves the non-negativity and the boundedness of approximations.

Of course, many research problems open after this work. For instance, a thorough analysis of convergence of the explicit exponential method proposed in this manuscript is a topic that deserves attention. This investigation would be motivated by the fact that the numerical simulations suggest that our technique converges to the exact solution. Another interesting problem would be the extension of this technique to more general parabolic partial differential equations, like the Burgers-Fisher and the Burgers-Huxley models [9,26]. In those cases, the respective discretizations may be obtained by applying the approach described in the present report, but the preservation of the non-negativity, the boundedness and the monotonicity are properties which are difficult to establish. In particular, extensions of this approach would be interesting in the context of more complicated systems of partial differential equations, like those describing the growth of biological films that interact with substrates of nutrients [7,8,11].

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