Hurwitz Type Results for Sum of Two Triangular Numbers

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Received: 25 May 2015, Revised: 2 Jun. 2015, Accepted: 3 Jun. 2015
Published online: 1 Jul. 2015

Abstract: Let $t_2(n)$ denote the number of representations of $n$ as a sum of two triangular numbers and $t_{(a,b)}(n)$ denote number of representations of $n$ as a sum of $a$ times triangular number and $b$ times triangular number. In this paper, we prove number of results in which generating functions of $t_2(n)$ and $t_{(1,3)}(n)$ are infinite product. We also establish relations between $t_{(1,3)}(n)$, $t_{(1,12)}(n)$, $t_{(3,4)}(n)$, $t_2(n)$ and $t_{(1,4)}(n)$.

Keywords: Representation of triangular numbers, generating functions, theta functions

Throughout the paper, we employ the standard notation

$$ (a; q)_\infty := \prod_{n=0}^\infty (1 - aq^n), \quad |q| < 1. $$

Ramanujan’s general theta function is defined as

$$ f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. $$

For convenience, we denote $f(q, q^3)$ by $\varphi(q)$, $f(q, q^4)$ by $\psi(q)$ and $f(-q, -q^2)$ by $f(-q)$. The Jacobi triple product identity [1] is defined by

$$ f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. $$

By Jacobi identity each $\varphi(q)$, $\psi(q)$ and $f(-q)$ is a product. In fact

$$ \varphi(q) = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, $$

$$ \psi(q) = (-q; q^4)_\infty (-q^3; q^4)_\infty (q^4; q^4)_\infty, $$

$$ f(-q) = (q; q^3)_\infty (q^2; q^3)_\infty (q^3; q^3)_\infty. $$

Let $r_k(n)$ denote the number of representations of $n$ as a sum of $k$ squares and $t_k(n)$ denote the number of representations of $n$ as a sum of $k$ triangular numbers. Let $t_{(a,b)}(n)$ denote the number of solutions in non negative integer of the equation

$$ a \frac{x_1(x_1 + 1)}{2} + b \frac{x_2(x_2 + 1)}{2} = n. $$

There is a remarkable relation between $r_k(n)$ and $t_k(n)$ [2]:

$$ r_k(8n + k) = 2^{k-1} \left( 2 + \left( \frac{k}{4} \right) \right) t_k(n), \quad \text{for } 1 \leq k \leq 7. $$

A. Hurwitz [4] proved several results in which generating function of $r_3(an + b)$ is a simple infinite product. For example

$$ \sum_{n \geq 0} r_3(4n + 1)q^n = 6\varphi^2(q)\psi(q^2), $$

$$ \sum_{n \geq 0} r_3(4n + 2)q^n = 12\varphi(q)\psi^2(q^2), $$

$$ \sum_{n \geq 0} r_3(8n + 1)q^n = 6\varphi^2(q)\psi(q). $$

These results have been proved by S. Cooper and M. D. Hirschhorn [3] and they have also established eighty infinite families of similar results.

The main purpose of this paper is to prove number of results in which generating functions of $t_2(n)$ and $t_{(1,3)}(n)$, when $n$ is restricted to an arithmetic sequence are infinite products.

In fact, we prove the following results.

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Theorem 1. We have
\[
\sum_{n=0}^{\infty} t_2(8n+1)q^n = 2\psi(q)f(q^3, q^9), 
\] (1)
\[
\sum_{n=0}^{\infty} t_2(8n+3)q^n = 2\psi(q)f(q^5, q^{11}), 
\] (2)
\[
\sum_{n=0}^{\infty} t_2(8n+5)q^n = 2\psi(q)f(q, q^{15}), 
\] (3)
\[
\sum_{n=0}^{\infty} t_2(8n+7)q^n = 2\psi(q)f(q^3, q^{13}). 
\] (4)

Putting \(a=q\) and \(b=q^2\) in (17), we obtain
\[
\psi(q) = \psi(q^4) + 2q\psi(q^8). 
\] (18)

Employing (18) in (16), we see that
\[
\sum_{n=0}^{\infty} t_2(n)q^n = \psi(q^2)\{\psi(q^4) + 2q\psi(q^8)\}. 
\] (19)

Immediately, it follows that
\[
\sum_{n=0}^{\infty} t_2(2n+1)q^n = 2\psi(q)f(q^4). 
\] (20)

Theorem 2. We have
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+2)q^n = 2\psi(q^3)f(q^3, q^{11}), 
\] (5)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+3)q^n = 2\psi(q)f(q^{21}, q^{27}), 
\] (6)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+6)q^n = 2\psi(q^3)f(q^7, q^9), 
\] (7)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+7)q^n = 2q^2\psi(q)f(q^9, q^{39}), 
\] (8)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+10)q^n = 2\psi(q^5)f(q^5, q^{11}), 
\] (9)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+11)q^n = 2q^4\psi(q)f(q^3, q^{45}), 
\] (10)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+14)q^n = 2q\psi(q^3)f(q, q^{15}), 
\] (11)
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n+15)q^n = 2\psi(q)f(q^{15}, q^{33}). 
\] (12)

Putting \(a=q\) and \(b=q^3\) in (17), we obtain
\[
\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}). 
\] (21)

Employing (21) in (20) and then extracting those terms in which the power of \(q\) is 0 \((\text{mod} \ 2)\) and replacing \(q^2\) by \(q\), we find that
\[
\sum_{n=0}^{\infty} t_{(1,3)}(4n+1)q^n = 2\psi(q^3)f(q^3, q^5). 
\] (22)

Putting \(a=q^3\) and \(b=q^5\) in (17), we get
\[
f(q^3, q^5) = f(q^{14}, q^{18}) + q^3f(q^2, q^{30}). 
\] (23)

Employing (23) in (22), it immediately follows that
\[
\sum_{n=0}^{\infty} t_{(1,3)}(2n+1)q^n = 2\psi(q)f(q^7, q^9) 
\] and
\[
\sum_{n=0}^{\infty} t_{(1,3)}(8n+5)q^n = 2q\psi(q)f(q, q^{15}). 
\]

This completes the proofs of (1) and (3). The proofs of (2) and (4) are similar.

1 Proof of Theorem 1

From [1, Entry 25(iv), p. 36], we have
\[
\sum_{n=0}^{\infty} t_2(n)q^n = \psi^2(q) 
\] (16)
\[= \psi(q^2)\psi(q). 
\]
Adding Entries 30(ii) and 30(iii) in [1, p. 43], we obtain
\[
f(a, b) = f(a^3b, ab^3) + af(b/a, a^2b^3). 
\] (17)
Extracting the terms in which the power of $q$ is 0 (mod 2) and replacing $q^2$ by $q$, we obtain
\[
\sum_{n=0}^{\infty} t_{(1,3)}(2n)q^n = \varphi(q^3)\psi(q^2)
\]
\[
= \psi(q^2)\{\varphi(q^{12}) + 2q^3\psi(q^{24})\}. \quad (26)
\]
Again, extracting those terms in which the power of $q$ is 1 (mod 2), divide by $q$ and replacing $q^3$ by $q$, we find that
\[
\sum_{n=0}^{\infty} t_{(1,3)}(4n + 2)q^n = 2q\psi(q)\psi(q^{12}). \quad (27)
\]
Employing (21) in (27), we immediately see that
\[
\sum_{n=0}^{\infty} t_{(1,3)}(8n + 2)q^n = 2q\psi(q^6)f(q, q^7),
\]
\[
= 2q\psi(q^6)\{f(q^{10}, q^{12}) + qf(q^6, q^{26})\}.
\]
Hence,
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n + 2)q^n = 2q\psi(q^3)f(q^3, q^{13}),
\]
\[
\sum_{n=0}^{\infty} t_{(1,3)}(16n + 10)q^n = 2\psi(q^3)f(q^5, q^{11}).
\]
This completes the proofs of (5) and (9). The proofs of remaining identities are similar to the proofs of (5) and (9).

### 3 Proof of Theorem 3

By (27), we have
\[
\sum_{n=0}^{\infty} t_{(1,3)}(4n + 2)q^n = 2q\psi(q)\psi(q^{12})
\]
\[
= 2q \sum_{n=0}^{\infty} t_{(1,12)}(n)q^n.
\]
Now, comparing the coefficients of $q^n$ in both sides of the above identity, we get (13).

Proofs of (14) and (15) are similar to that of (13).

### Acknowledgement

The first author is thankful to the University Grants Commission, Government of India for the financial support under the grant F.510/2/SAP-DRS/2011. The second author is thankful to UGC-BSR fellowship. The third author is thankful to DST, New Delhi for awarding INSPIRE Fellowship [No. DST/INSPIRE Fellowship/2012/122], under which this work has been done.

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