Star Edge Coloring of Subcubic Graphs

Kavita Pradeep* and V. Vijayalakshmi

Department of Mathematics, Anna University, MIT Campus, Chennai - 600044, India

Received: 22 Dec. 2018, Revised: 2 Feb. 2019, Accepted: 15 Feb. 2019

Published online: 1 Mar. 2019

Abstract: A proper edge coloring of a graph G is called star edge coloring if there is no bi-colored path or cycle of length four in G. The minimum number of colors needed to star color the edges of G is called the star chromatic index of G, denoted by \( \chi_s(G) \). In 2013 [1], Dvořák et al. proved that for a subcubic graph G, \( \chi_s(G) \leq 7 \) and conjectured that it is less than or equal to 6. In this paper, we show that if a subcubic graph G has maximum average degree less than \( \frac{4}{3} \) then \( \chi_s(G) \leq 6 \).

Keywords: Star edge coloring, star chromatic index, subcubic graphs, maximum average degree

1 Introduction

All the graphs considered in this paper are finite and simple. For a graph G with vertex set V(G) and edge set E(G) a proper edge coloring of G is an assignment of colors to the edges of G so that no two adjacent edges receive the same color. A star edge coloring is a proper edge coloring with an additional condition that any path or cycle of length four is not bi-colored. The star chromatic index of G is the least number of colors needed for a star edge coloring of G, denoted by \( \chi_s(G) \). This coloring was introduced by Liu and Deng [2] in 2008. In 2013 [1], Dvořák, Mohar and Šámal found the following bound for the star chromatic index of complete graphs.

**Theorem 1.** The star chromatic index of the complete graph \( K_n \) satisfies

\[
(2 + o(1))n \leq \chi_s'(K_n) \leq n^{2\sqrt{n + 1} + o(1)} \log n.
\]

In particular, for every \( \varepsilon > 0 \) there exists a constant \( c \) such that \( \chi_s'(K_n) \leq cn^{1+\varepsilon} \) for every \( n \geq 1 \).

They also obtained a near-linear upper bound in terms of the maximum degree \( \Delta \) for general graphs. In addition, they considered subcubic graph - a graph in which each vertex has degree at most three and showed that the star chromatic index of every subcubic graph is at most 7. L. Bezegová et. al.[3] proved that the star chromatic index of every subcubic tree and subcubic outerplanar graph is at most 4 and 5 respectively.

Motivated by the strong list edge coloring of subcubic graphs [4], [5], list version of star edge coloring is also studied and bounds are given in terms of maximum degree of the graph. The maximum average degree of a graph G, denoted by \( mad(G) \) is defined as

\[
mad(G) = \max_{H \subseteq G, |V(H)| \geq 1} \left( \frac{2|E(H)|}{|V(H)|^2} \right).
\]

S. Kerdjoudj et.al. [6] proved that the list star chromatic index of a subcubic graph G with \( mad(G) < \frac{4}{3} \) and \( mad(G) < \frac{4}{7} \) is at most 5 and 6 respectively. They also showed that the list star chromatic index for every subcubic graph is at most 8. S. Kerdjoudj and A. Raspaud [7] proved that for subcubic graphs with \( mad(G) < \frac{30}{11} \) it is at most 7. Recently, B. Lužar et.al.[8] proved that 7 colors suffice for the list star edge coloring of every subcubic graph G.

There are examples for subcubic graphs with the star chromatic index equal to 6, (e.g. \( K_{3,3} \) and \( K_4 \) with one subdivided edge) but no example of a subcubic graph requiring 7 colors is known. Based on this fact, Dvořák et al.[1] proposed the following conjecture.

**Conjecture 1.** If G is a subcubic graph, then \( \chi_s(G) \leq 6 \).

In this paper we prove the following theorem in support of the Conjecture 1.

**Theorem 2.** Let G be a subcubic graph with \( mad(G) < \frac{4}{3} \)

Then, \( \chi_s(G) \leq 6 \).

The girth of a graph G is the length of a shortest cycle in G. For every planar graph with girth g, \( mad(G) < \frac{2e}{g-2} \). This, together with the theorem above gives the following corollary.

**Corollary 1.** Let G be a planar subcubic graph with girth \( g \geq 8 \). Then \( \chi_s(G) \leq 6 \).

* Corresponding author e-mail: kavita2428@gmail.com
Notations: Let $d(v)$ denote the degree of the vertex $v$ in $G$. A vertex of degree $k$ is called a $k$-vertex. A $k$-vertex adjacent to a vertex $v$ is a $k$-neighbor of $v$. A $3k$-vertex is a 3-vertex adjacent to exactly $k$ ($0 \leq k \leq 3$) vertices of degree 2. A 3-vertex adjacent to a 1-vertex is a light 3-vertex. For an edge coloring $\phi$ of a graph $G$, let $\phi(v)$ denote the set of colors used on the edges incident with the vertex $v \in V(G)$ in the coloring $\phi$. Similarly, for an edge $uv \in E(G)$, $\phi(uv)$ denotes the color used on the edge $uv$. We say that a color $c$ is an available color for an edge $uv$, if $c$ is not assigned to any of its neighbors and there is no bi-colored path of length four or cycle of length four involving $uv$ when colored with $c$. Otherwise, it is a forbidden color. The set of available colors for the edge $uv$ is denoted by $A(uv)$.

Proof of Theorem 2
Let $G'$ be a minimum counterexample minimising $|E(G) + V(G)|$. Then the edges of $G'$ are not star colorable with six colors and $mad(G') < 2$. By minimality of $G'$, we can assume that it is connected. Otherwise, we can star color independently the edges of each connected component of $G'$ with six colors. We list some structures or set of some subgraphs called reducible configurations in $G'$. We prove all the claims by contradiction. For each of the claim, we suppose that the described structure exists in $G'$, then we remove a certain number of edges from $G'$ to form a graph $H$, which by minimality of $G'$ is star edge colorable with six colors. Let $\phi$ be such a star edge coloring of $H$. We show a contradiction by extending the star edge coloring $\phi$ of $H$ with six colors to a star edge coloring of $G'$ with same number of colors. Before going into the proof we need the following observation.

Observation:
Let $u$ and $v$ be two adjacent 3-vertices in $G'$. Let $N(u) = \{v_1, v_2, u\}$ and $N(u) = \{u_1, u_2, u\}$. Let $\phi$ be a star edge coloring of $H = G' \setminus \{uu_i, i = 1, 2\}$ with six colors. Let $v_1$ not adjacent to $v_2$ (in worst case) be two 3-vertices. If $|\phi(v_1) \cap \phi(v_2)| \geq 1$, then there is at least one color that is not present on the edges incident to the vertices $v_1$ and $v_2$. If the edge $uv$ is not colored with this color, then we can recolor it with this color. If $|\phi(v_1) \cap \phi(v_2)| = 0$, then we can assume that any available color for the edge $uv$ is such that, either $\phi(uv) \in \phi(v_1)$ or $\phi(uv) \in \phi(v_2)$. In the worst case, in order to exclude greater number of colors for the edges incident with $u$, we always assume the latter part $|\phi(v_1) \cap \phi(v_2)| = 0$. When $v_1$ is adjacent to $v_2$ or either of $v_1$ or $v_2$ are 2-vertices then we can always recolor the edge $uv$ with a color that is not present on the edges incident to the vertices $v_1$ and $v_2$.

Claim 1. $G'$ does not contain a 1-vertex adjacent to a 2-vertex.
Suppose there is a 1-vertex $u$ adjacent to a 2-vertex $v$ in $G'$. As $G'$ is a minimum counterexample, $H = G' \setminus \{uv\}$ is star edge colorable with six colors. By counting the number of maximum possible forbidden colors for $uv$, it is easy to see that we have at least three colors available for it. So, the coloring of $H$ can be extended to $G'$, a contradiction.

Claim 2. $G'$ does not contain a 3-vertex adjacent to two 1-vertices.
Let a 3-vertex $u$ be adjacent to the 1-vertices $v$ and $w$ in $G'$. Consider $H = G' \setminus \{uv\}$, which by minimality of $G'$ is star edge colorable with six colors. By counting the number of maximum possible forbidden colors for the edge $uv$, it is easy to see that we have at least two colors available for it. So, the coloring of $H$ can be extended to $G'$, a contradiction.

Claim 3. $G'$ does not contain a 3-vertex adjacent to a 1-vertex and a 2-vertex.
Suppose $u$ be such a 3-vertex adjacent to the 1-vertex $v$ and a 2-vertex $w$. The graph $H = G' \setminus \{uw_i\}$ has a star edge coloring with six colors. Again, by counting the number of maximum possible forbidden colors for the edge $uw$ it is easy to see that there is at least one color available for $uw_i$, a contradiction.

Claim 4. $G'$ does not contain two adjacent light 3-vertices.
Let $u$ and $v$ be two adjacent light 3-vertices adjacent to the 1-vertices $u_1$ and $v_1$ respectively in $G'$. Let $x$ and $y$ be the other neighbors of $u$ and $v$ respectively. By claim 3, $x$ and $y$ are 3-vertices. Consider $H = G' \setminus \{uu_1\}$, which by minimality of $G'$ is star edge colorable with six colors. By counting the number of available colors it is easy to see that we have at least one color available for $uu_1$. So, the coloring of $H$ can be extended to $G'$, a contradiction.

Claim 5. $G'$ does not contain two adjacent 2-vertices.
Suppose $u_1$ and $u_2$ be the two adjacent 2-vertices in $G'$. For $i = 1, 2$, let $v_i$ be the other neighbor of $u_i$. By minimality of $G'$, $H = G' \setminus \{uu_{12}\}$ has a star edge coloring $\phi$ with six colors. When $v_1$ is adjacent to $v_2$ or when $v_1 = v_2$, by counting the number of available colors for $uu_{12}$, we can easily extend $\phi$ to $G'$. So let $v_1$ be distinct and not adjacent to $v_2$ with $N(v_1) = \{w_i, w'_i, u\}$, $i = 1, 2$. If there is an available color for the edge $uu_{12}$ we are done. Otherwise, all the six colors are forbidden for this edge, this means, $|\phi(v_1) \cap \phi(v_2)| = 0$ and $\phi(u_iu_i) \in \{\phi(w_i) \cap \phi(w'_i)\}$. So, we recolor the edge $uu_{12}$ using the Observation. This gives $|A(u_1u_2)| \geq 1$. Hence, $\phi$ can be extended to $G'$, a contradiction.

Claim 6. $G'$ does not contain a 3-vertex adjacent to three 2-vertices.
Let $u$ be such a 3-vertex in $G'$. Let $u_1, u_2$ and $u_3$ be the 2-neighbors of $u$. Let each $u_i$ be adjacent to $x_i$. By claim 1 and claim 5, $x_i$ is a 3-vertex for each $i$. By minimality of $G'$, $H = G' \setminus \{uu_i, i = 1, 2, 3\}$ has a star edge coloring $\phi$ with six colors. We extend this coloring to the edges $uu_1$, $uu_2$ and $uu_3$ in order. First, we color the edge $uu_1$ with
a color such that it does not appear on the colored edges incident to the vertices $x_1$, $u_2$ and $u_3$. Then, we color the edge $uu_2$ with a color which does not appear on the colored edges incident to the vertices $x_2$, $u$ and $u_3$. Finally, we have at least one color available for the edge $uu_3$. Hence, $\varphi$ is extended to $G$, a contradiction.

Claim 7. $G$ does not contain a cycle $uuwu$, where $u$ is either a light 3-vertex or a 2-vertex.

Suppose $uuwu$ be such a cycle in $G$. When $u$ is adjacent to a 1-vertex $u_1$, consider $H = G \setminus \{uu_1\}$, which by minimality of $G$ is star edge colorable with six colors. By counting the number of maximum possible forbidden colors it is easy to see that we have at least one color available for $uu_1$. When $u$ is a 2-vertex, consider $H = G \setminus \{uv\}$ having a star edge coloring $\varphi$ with six colors. Let $v_1$ and $v_2$ be the neighbors of $v$ and $w$ respectively. If $v_1$ is adjacent to $w_1$ or $v_1 = w_1$, then it is easy to see that $uv$ can be easily colored. Otherwise, (when $v_1$ and $w_1$ are non adjacent) if $uv$ cannot be colored, then $|\varphi(v_1) \cap \varphi(w)| = 0$. So, we can recolor $uw$ with a color other than $\varphi(uw)$ such that there is an available color for the edge $uv$, which is a contradiction.

Claim 8. $G$ does not contain a cycle $xuvwx$, where $v$ and $x$ are 2-vertices.

Suppose $G$ contains such a cycle $xuvwx$. Let $v$ and $x$ be the 2-vertices. Consider $H = G \setminus \{uv, uw, vw, wx\}$, which by minimality of $G$ has a star edge coloring $\varphi$ with six colors. Clearly, $u$ is not adjacent to $w$. Let $y_1$ and $y_2$ be the neighbors of $u$ and $w$ respectively (other than $v$ and $x$). If $y_1 = y_2$, it can be observed that $uv$ can be extended to the edges of $G$ easily. So, we can assume that $y_1$ is distinct from $y_2$. For the worst case, let $d(y_1) = 3, i = 1, 2$. Let $z_1$ and $z_2$ be the other neighbors of $y_i$. Recolor the edges $u1y_i$ and $w_2y_3$ using the Observation. In worst case, we can assume that $\varphi(u1y_i) \in \varphi(z_1)$ and $\varphi(w_2y_3) \in \varphi(z_2)$. So, we can color the edges $ixw$ and $uw$ with $c_1$ and $c_2$ respectively, such that $c_1 \notin \varphi(v_1y_i) \cup \varphi(wz_2y_3)$ and $c_2 \notin \varphi(y_i) \cup \varphi(wz_2y_3)$. We have, $|\varphi(v_1y_i) \cap \varphi(wz_2y_3)| \geq 2$ and $|\varphi(v_1y_i) \cap \varphi(wz_2y_3)| \geq 3$. So, let there be an appropriate color for $wx$ so that there is at least one color available for the edge $uw$. Hence, $\varphi$ can be extended to the edges of $G$, a contradiction.

Claim 9. $G$ does not contain a 3-vertex adjacent to two light 3-vertices.

Suppose $G$ contains such a vertex $u$. Let $N(u) = \{v_1, v_2, x_3\}$. Let $v_1$ and $v_2$ be the light 3-vertices adjacent to the 1-vertices $v$ and $w$ respectively. Let $x_1$ and $x_2$ be the other 3-neighbors of $v_1$ and $v_2$ respectively. (By claim 3, both are 3-vertices). For $i = 1, 2$, let $N(x_i) = \{z_i, z_i', v_1\}$ and (let $d(x_3) = 3$) $N(x_3) = \{z_3, z_3', u\}$.

Consider $H = G \setminus \{v_1v_2\}$, which by minimality has a star edge coloring $\varphi$ with six colors. If $x_1 = x_2$, then by counting the number of maximum possible forbidden colors it is easy to see that we have at least one color available for $v_1v_2$. So, we assume that $x_1$ is distinct from $x_2$. Suppose there is no color available for $v_1v_2$. Then, $|\varphi(u) \cap \varphi(x_1)| = 0$ and $\varphi(v_1x_1) \in \varphi(z_1) \cap \varphi(z_1')$ and $\varphi(v_1u) \in \varphi(z_1) \cap \varphi(v_2)$ making bi-colored paths of length three.

Without loss of generality, we can assume a coloring that depicts this situation. Let $\varphi(v_1x_1) = 1$, $\varphi(x_1z_1) = 2$, $\varphi(x_1z_1') = 3$, $\varphi(v_1u) = 4$, $\varphi(uw) = 5$ and $\varphi(v_1w) = 6$.

When $\varphi(v_1u) = \varphi(v_2w) = 4$, if there is an available color for $v_2w$ other than $4$, we are done. Otherwise, we can assume that $\varphi(v_2x_2) \subseteq \{1, 2, 3\}$, $\varphi(v_2z_2) \in \varphi(z_2) \cap \varphi(z_2')$ and $\varphi(x_2w) = 6 \in \varphi(x_3)$ forming bi-colored paths of length three. This situation is shown in Figure 1. Remove the colors of the edges $v_1w$ and $v_2w$, then recolor the edge $v_2x_2$ using the Observation. After recoloring, let the color of $v_2x_2$ be $c$. Clearly, $c \in \{4, 5, 6\}$. If $c \in \{5, 6\}$, then we first color $uv_2$ with an available color from $\{1, 2, 3\}$ then, we color the edges $v_1x_1$ and $v_2x_2$ in order. Otherwise, we color the edge $v_1x_1$, then color $uv_2$ with an available color from $\{1, 2, 3\}$. Finally, we color $v_1u, v_1v$ and $v_2w$ in order. At each step there is a color available for each of the edge.

![Figure 1](image_url)

When $\varphi(v_1u) = \varphi(v_2x_2) = 4$, we remove the colors of the edges $uv_2$ and $v_2w$ and recolor the edge $v_2x_2$ using the Observation. After recoloring, let the color of $v_2x_2 = c$. In the worst case, let $c \neq \varphi(z_2)$. If $c = \varphi(v_2x_2) = 4$, we uncolor the edge $v_1u$, this gives at least two colors available for $uv_2$. We choose a color other than $6$ for $uv_2$. Then color $v_1v$ and $v_2x_2$ in order. Otherwise, (when $c \neq 4$) we color $uv_2, v_1v$ and $v_2w$ in order. At each step there is at least one color available for each of the edge.

From the above, we can also see that a 3-vertex is not adjacent to a light 3-vertex and a 2-vertex.

Claim 10. $G$ does not contain a 3-vertex adjacent to either (i) a 3-1-vertex or (ii) a 3-vertex which is adjacent to a light 3-vertex.

Suppose to the contrary that $G$ contains a 3-vertex $u$ adjacent to such a 3-vertex $v_1$. Let $N(v_1) = \{u, v_2, x_3\}$. Let $v_2$ be a 2-vertex. Let $u_1$ and $u_2$ be neighbors of $u$. By claim 7, $v_2 \neq u_1$ or $u_2$. By claim 5, $v_2$ is not adjacent to $u_1$ or $u_2$. Let $w_i$ be the other neighbor of $u_i$ that is distinct from $u$ and $x_3$ be a 3-neighbor of $v_2$ which is distinct from $v_1$. By claim 5, $x_2$ and $w_i, i = 1, 2$ are three vertices. Let $N(u_i) = \{z_i, z_i', w_i\}$ and $N(x_2) = \{y_i, y_i', v_1\}$. If $v_2$ is a light 3-vertex, let the adjacent 1-vertex be $v$. Since, $G'$ is a minimum counterexample, $H = (G \setminus \{uu_1\}, i = 1, 2)$ has a star edge coloring $\varphi$ with six colors. To extend $\varphi$ to $G'$, we remove the colors of the edges $uu_1$ and $v_1v_2$. (When $v_2$
is a light 3-vertex then, we remove the color of the edge $v_2v'$ also.) By claim 7, $x_1 \neq x_2$. By using the Observation for the edges $v_ix_i$, $i = 1, 2$ we can assume that $\varphi(v_ix_i) \in \varphi(y_i)$.

When $\varphi(v_1x_1) = \varphi(v_2x_2)$, there are at most five forbidden colors for the edge $v_1v_2$ so, we color it with the available color. (Then we color the edge $v_2v'$, when $v_2$ is a light 3-vertex, with the color $\varphi(x_1y_1)$, if it is available. Otherwise, $\varphi(x_1y_1) = \varphi(x_2y_2)$. So, we set the color of $v_2v'$ = $\varphi(x_2y_2)$). This makes at most three colors forbidden for the edge $uv_1$. Therefore, $|A(\alpha_{uv_1})| \geq 3$.

When $\varphi(v_1x_1) \neq \varphi(v_2x_2)$, there are at least two colors available for the edge $v_1v_2$. We choose a color that appears on $\varphi(x_1)$ for the edge $v_1v_2$, if it is available. Otherwise, $|\varphi(x_1) \cap \varphi(x_2)| \geq 1$, so we get a color which does not appear on $\varphi(x_1)$, for both $i = 1, 2$. We set this color for $v_1v_2$. (Then we color the edge $v_2v'$ as above when $v_2$ is a light 3-vertex.) Therefore, again there are at most three forbidden colors for the edge $uv_1$, $|A(\alpha_{uv_1})| \geq 3$.

In both the cases above we can observe that there is at least one color in $A(\alpha_{uv_1})$ which is not present in $\varphi(x_1)$ and $\varphi(v_2)$. Let $\alpha_{uv_1} = \{\alpha_1, \alpha_2, \alpha_3\}$.

$$N(\alpha_{uv_1}) = \{y_1, y_2, y_3\}.$$ Consider $H = G \setminus \{u_1v_1, u_2v_2\}$. By minimality of $G'$, $H$ has a star edge coloring $\varphi$ using six colors. If $v_1 = v_2$ and we cannot extend $\varphi$ to the edges of $G'$. We recolor the edges $v_1'v_1'$ and $u_1v_1$ using the Observation. In the worst case, we assume that $\varphi(u_1v_1) \in \varphi(u_2v_2)$ or $\varphi(u_1v_1) \in \varphi(u_3v_1)$ and $\varphi(v_1v_1') \in \varphi(y_1)$. If $\varphi(u_1v_1) = \varphi(v_1v_1')$, then we color the edges $u_1v_1$ and $u_1v_1'$ in this order. There exists at least one color for each of them. Otherwise, $|\alpha_{uv_1}| \geq 2$, so, we set the color of $u_1v_1 = \alpha \in \alpha_{uv_1}$, such that $\alpha \in \varphi(u)$ and $\alpha \notin \varphi(v_1')$, if such $\alpha$ exists. Else, we get a color $\alpha \notin \{\varphi(u), \varphi(v_1')\}$. So, we set the color of $u_1v_1 = \alpha$. This gives $|A(\alpha_{uv_1})| \geq 1$ so, $\varphi$ can be extended to $G'$. Similarly, we can extend $\varphi$ to $G'$ when $v_1 = v_2$. Therefore, in $G'$, $v_1$ is distinct from $v_2$ and $v_1'$ is distinct from $v_2'$.

Also, it can be observed that for $i = 1, 2$, $v_i$ is not adjacent to $v_j$ (by claim 5) and $w_1$ is distinct from $w_j$ (by claim 8).

Now, suppose $\varphi$ cannot be extended to the edges of $G'$. We recolor the edges $v_1w_1$ and $v_1v_1'$ using the Observation. In the worst case we can assume that the color of $v_1w_1 = (say)t_1 \in \varphi(x_1)$ and the color of $v_1v_1' = (say)t_2 \in \varphi(y_1)$. If we get colors available for the edges $u_1v_1$ and $u_1v_1'$, we are done. Otherwise, we recolor the edge $uv_1$. After recoloring, let the color of $uu_1 = \alpha$. We consider two cases.

**Case 1:** Suppose $\alpha \notin \{\varphi(u_1) \cup \varphi(u_2)\}$.

When $\alpha \neq t_1 \neq t_2$ or $\alpha \neq t_1 = t_2$ we can easily get available colors for the edges of $G'$. When $\alpha = t_1 \neq t_2$ (ref $t_2 \neq t_1$ respectively), first we color $u_1v_1$ ($u_1v_1'$ respectively), then we have $A(\alpha_{u_1v_1}) \geq 2$. But $A(\alpha_{u_1v_1'}) \geq 2$ respectively.

When $\alpha = t_1 = t_2$ and there is no color available for one of the edges in $G'$. We extend this coloring to $G'$ in the following way. Before recoloring the edge $uu_1$ with $\alpha$ -

**Case 1.1:** If $\varphi(uu_1)$ is different from $\alpha$ then, $\varphi(uu_1) \neq t_1 = t_2$. As $\varphi$ could not be extended, $|\{\varphi(\alpha_{uu_1}) \cap \varphi(v_1v_1')\}| = 3$ and all the three colors incident with the vertex $u$ must be forbidden. So, $\varphi(uu_1)$ must appear on one of the edges incident to $u_2$ and also on $u_3$.

So, the coloring can be assumed as follows. $\varphi(u_1v_1) = 1$, $\varphi(w_3x_1) = 2$. $\varphi(w_1x_1) = 3$. $\varphi(u_2v_1) = 1$. $\varphi(w_1v_1') = 2$. $\varphi(w_2v_1') = 3$ and $\varphi(uu_1) = 4$, $\varphi(uu_2) = 5$, $\varphi(uu_3) = 6$.

Let $\varphi(u_1v_1') = 4$ and $\varphi(u_2v_1') = 4$. After recoloring, now we get the case that the color of $u_1v_1 = \alpha \notin \{\varphi(uu_2) \cup \varphi(u_3)\}$ and $\alpha = t_1 = t_2 = 1$. If there is any color available for the edge $uu_1$ other than 1 and 4, we are done. Otherwise, we can assume that $\varphi(u_2v_2) = 5$ and 6 $\in \varphi(v_1)$. If $\varphi(v_1v_1') \neq 1$ (or 1 is an available color for $uu_1$) we can swap the colors of the edges $uu_1$ and $uu_2$.

Then it is easy to see that, there is at least one color available for each of the two edges $u_1v_1$ and $u_1v_1'$ of $G'$ and we are done. Therefore, we assume that $\varphi(v_1v_1') = 1$.

Now, we remove the colors of the edges $u_2v_2, u_2v_2, uu_2$.

![Fig. 2: Configuration of claim 10](image-url)
and \( uu_1 \). We set \( \Phi(u_1v_2') = \alpha_1 \) and \( \Phi(u_2v_3) = \alpha_2 \), such that \( \alpha_1 \notin \{ \Phi(w_2), \Phi(u_1v_2), \Phi(u_2v_3) \} \). (Clearly \( \alpha_1 \notin \{2, 3, 4\} \) and \( \alpha_2 \notin \{ \Phi(w_2), \Phi(u_1v_2), \Phi(u_2v_3) \} \). For \( \alpha_2 \neq 1 \), we set \( \Phi(uu_2) = 1 \) and \( \Phi(uu_1) = 5 \). For \( \alpha_2 = 1 \) and \( \alpha_1 = 2 \) or 3, we set \( \Phi(uu_2) = 4 \) and \( \Phi(uu_1) = 5 \). For \( \alpha_2 = 1 \) and \( \alpha_1 = 4 \), we set \( \Phi(uu_2) = 5 \) and \( \Phi(uu_1) \in \{2, 3\} \). Therefore, in each case, the color of \( uu_1 \neq t_1 = t_2 \). So, \( \Phi \) can be easily extended to \( G' \).

**Case 1.2:** If \( \Phi(uu_1) \) is the same as \( \alpha \) then \([\{\Phi(uu_2) \cap \Phi(uu_3)\}] = 1 \). If we get a color other than \( \alpha \) for \( uu_1 \) we can easily extend the coloring to \( G' \). Otherwise, we can assume the coloring as in (i) except that the colors of the three edges out of \( u_2v_2, u_2v_3, u_3v_3 \) and \( uu_1 \) are from \( \{2, 3, 4\} \) and \( \alpha_1 = t_1 = t_2 = 1 \). If \( \{\Phi(u_2v_2), \Phi(u_2v_3)\} \in \{2, 3, 4\} \), then we can swap the colors of the edges \( uu_2 \) and \( uu_1 \) and hence we are done. If \( \{\Phi(u_3v_3), \Phi(u_3v_1)\} \in \{2, 3, 4\} \), then we can again swap the colors of \( uu_2 \) and \( uu_1 \), whenever \( \Phi(v_2w_2') \neq 1 \) or 1 is available for \( uu_2 \). Otherwise, when \( \phi(v_2w_2') = 1 \), we remove the colors of the edges \( uu_2 \) and \( uu_1 \) and then recolor the edge \( uu_1 \) using the Observation. After recoloring \( \Phi(uu_1) \in \{1, 4, 5\} \). (Observe that before recoloring \( \Phi(uu_1) \in \{\Phi(v_1) \cap \Phi(v_2)\} \). If \( \Phi(uu_2) = 1 \), retain \( \Phi(uu_2) = 5 \) and set \( \Phi(uu_1) = 6 \). If \( \Phi(uu_1) = \{4, 5\} \) and \( \Phi(uu_2) \neq 6 \), then set \( \Phi(uu_2) = 6 \) and \( \Phi(uu_1) \in \{4, 5\} \). When \( \Phi(uu_2) = 6 \), then \( 6 \in \Phi(w_2) \) (as 1 is not available for \( uu_2 \)). So, we remove the color of the edge \( uu_2 \) and recolor it with a color which is not in \( \{\Phi(u) \cup \Phi(v_2) \cup \Phi(w_2)\} \). There is at least one such color. Observe that after recoloring \( \phi(u_2v_2) \in \{2, 3\} \), so we set \( \Phi(uu_1) = 1 \) and \( \Phi(uu_2) = 6 \). Therefore, in each case, the color of \( uu_1 \neq t_1 = t_2 \). So, \( \Phi \) can be easily extended to \( G' \).

**Case 2:** Suppose \( \alpha \in \Phi(uu_2) \) or \( \alpha \notin \Phi(uu_3) \).

**Case 2.1:** When \( \alpha \in \Phi(uu_3) \). (and there is no color available for one of the edges).

From the Observation \([\{\Phi(uu_2) \cap \Phi(uu_3)\}] = 0 \) so, without loss of generality we can assume that \( \alpha \notin \Phi(u_2v_2) \), \( \Phi(uu_3) = 2 \), \( \Phi(uu_1v_1) = 3 \), \( \Phi(uu_2) = 4 \), \( \Phi(u_2v_3) = 5 \), \( \Phi(u_2v_2') = 6 \). In order to extend \( \Phi \) to the edges of \( G' \), we try to get a color for the edge \( uu_1 \) other than 1. If such color is available, we are done. Otherwise, we can assume that \( \Phi(v_2w_2) = \Phi(v_2w_2') = 4 \). So, we swap the colors of the edges \( uu_1 \) and \( uu_2 \) and set the color of \( uu_2 = 1 \) and the color of \( uu_1 = 4 \). This situation is shown in Figure 3.

**Case 2.2:** When \( \alpha \in \Phi(uu_2) \) and there is no color available for one of the edges.

Again from the Observation \([\{\Phi(uu_2) \cap \Phi(uu_3)\}] = 0 \). Let \( \alpha = \phi(u_2v_2) = 1 \), \( \phi(uu_3) = 2 \), \( \phi(u_1v_3) = 3 \), \( \phi(uu_2) = 4 \), \( \phi(u_2v_3) = 5 \), \( \phi(u_2v_2') = 6 \). If there is an available color for the edge \( uu_1 \) other than 1 then we are done. Otherwise, we can assume that \( 2 \in \{\Phi(v_2) \cap \Phi(v_2')\} \) and \( \phi(v_2w_2') = 4 \). This situation is depicted in Figure 4. So, we remove the colors of the edges \( u_2v_2, uu_1 \) and \( uu_2 \). Then, we recolor the edge \( uu_1 \) using the observation. Clearly, after recoloring \( \Phi(uu_1) \in \{1, 4, 6\} \).

When \( \Phi(uu_1) = 4 \), we retain the color of \( uu_2 \) and recolor the edge \( uu_2 \) with an available color from \{3, 5\} and set \( \Phi(uu_1) = 2 \).

When \( \Phi(uu_1) = 6 \), we set \( \Phi(uu_1) = 2 \) and color \( uu_2 \) with its available color. There is at least one available color for it, as \( \phi \notin \Phi(w_2) \). If \( \phi(u_2v_2) \neq 2 \), we set \( \Phi(uu_1) = 2 \). Otherwise, we set \( \Phi(uu_1) = 4 \).

When \( \Phi(uu_1) = 4 \), \( \phi(v_2w_2) \neq 4 \), we retain the colors of the edges \( uu_2 \) and \( uu_1 \) and set \( \Phi(uu_1) = 2 \).

![Fig. 3: Configuration of claim 11, case 2.1](image)

![Fig. 4: Configuration of claim 11, case 2.2](image)

Now, in cases 2.1 and 2.2 above, when \( \alpha = 1 = t_1 = t_2 \), after recoloring, \( \Phi(uu_1) \neq t_1 = t_2 \), so, there are at least two colors available for \( uu_1v_1 \) and at least one for \( uu_1v_2 \). When \( \alpha = 1 \neq t_1 \neq t_2 \), after recoloring, if \( \Phi(uu_1) \neq t_1 \neq t_2 \), there are at least three colors available for \( uu_2v_2 \) and at least one for \( uu_1v_1 \). Otherwise, if \( \Phi(uu_1) = t_1 = t_2 \) (resp. \( t_2 \neq t_1 \) ), we color the edges \( uu_1v_1 \) (resp. \( uu_2v_2 \) ) and \( uu_1v_1 \) (resp. \( uu_2v_2 \) ) in order. At each step there is a color available for each of the edges. Hence, \( \Phi \) can be extended to \( G' \), a contradiction.

Now, we delete all the vertices of degree one from \( G' \) and obtain a graph \( H' = G' \setminus \{v \in V(G') \mid d(v) = 1\} \). Clearly, it is connected and has maximum average degree < \( \frac{4}{3} \). Using all the above claims we get the following reducible configurations in \( H' \).

C1) The minimum degree of \( H' \geq 2 \). (from claims 1 & 2)
C2) \( H' \) does not contain -
C2.1) two adjacent 2-vertices. (from claims 3, 4 & 5)
C2.2) a 3-vertex adjacent to three 2-vertices. (from claims 6 & 9)
C2.3) a 31-vertex adjacent to a 32-vertex. (from claims 7, 8, 9 & 10)
C2.4) a 3-vertex adjacent to two 32-vertices. (from claims 8, 9 & 11)
Next, we set a weight function $w : V(H') \to \mathbb{R}$ with $w(v) = d(v) - \delta$, $\forall v \in V(H')$. It can be easily observed that as $\text{mad}(H') < \frac{\Delta}{3}$, the total sum of weights of all the vertices in the graph $H'$ is strictly negative. That is, $\sum_{v \in V(H')} w(v) < 0$.

Then, we redistribute the weights among the vertices according to the discharging rules described below, to obtain the weight function $w'$. During the discharging process, the sums of the values of $w'$ and $w$, counting over all the vertices remain same.

Let $v \in V(H')$ be a $k$-vertex, $k = 2, 3$. The initial weight assigned to the 2-vertices is $\frac{1}{4}$ units and to the 3-vertices is $\frac{1}{3}$ units. We move the positive weights from 3-vertices to 2-vertices using the following discharging rules:

A 3-vertex sends-

R1 $\frac{1}{4}$ units to the adjacent 3-vertex.
R2 $\frac{1}{3}$ units to each of the adjacent 2-vertices.

**Case 1:** Let $v \in V(H')$ be a 2-vertex. From C2.1 and C2.2, a 2-vertex $v$ is adjacent to 3-vertices only. Therefore, using R2, the 3-vertices send $\frac{1}{3}$ units to $v$. So, $w'(v) = \frac{1}{3} + 2 \times \frac{1}{3} - \frac{1}{4} = 0$.

**Case 2:** Let $v \in V(H')$ be a 3-vertex. From C2.2, a 3-vertex $v$ can be adjacent to at most two 2-vertices. When $v$ is a 3₂-vertex, by C2.3 and C2.4, $v$ is adjacent to at least one 3₀-vertex. So, by R1, it receives $\frac{1}{4}$ units from the 3₀-vertex, then by R2 it sends $\frac{1}{3}$ units to each of the adjacent 2-vertices. Therefore, $w'(v) = \frac{1}{3} + \frac{1}{3} - 2 \times \frac{1}{3} = 0$.

When $v$ is a 1-vertex, it sends $\frac{1}{3}$ units to the adjacent 2-vertex. Therefore, by using R2, $w'(v) = \frac{1}{3} - \frac{1}{3} = 0$.

When $v$ is a 3₀-vertex, by C2.4 it can be adjacent to at most one 3₂-vertex. So, it sends $\frac{1}{4}$ units to it. Therefore, by using R1, $w'(v) = \frac{1}{4} - \frac{1}{3} = 0$. If $v$ is not adjacent to any 3₂-vertex, then its weight remains unchanged. So, $w'(v) = w(v) = \frac{1}{3}$.

Therefore, $w'(v) \geq 0$ for every $v \in V(H')$. This leads us to the contradiction $0 \leq \sum_{v \in V(H')} w'(v) = \sum_{v \in V(H')} w(v) < 0$. Therefore, $H'$ cannot exist. Hence, the minimal counter example $G$ also does not exist. This completes the proof.

2 Conclusion

We have determined the star chromatic index of graphs with maximum degree $\Delta \leq 3$ and maximum average degree less than $\frac{\Delta}{2}$. We have shown that $\chi_s(G) \leq 6$ for such graphs.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References


V Vijayalakshmi

is an Assistant Professor of Mathematics in Anna University, India. Her research interests include Graph Theory and Combinatorics. She has published research articles in reputed international journals.

Kavita Pradeep

is a research scholar in the department of Mathematics, Anna University, India. She pursued her Master of Science in Mathematics at MIT Campus, Anna University and Master of Philosophy in Mathematics at Ethiraj College, affiliated to University of Madras. Her area of research is Graph Theory.