Applied Mathematics & Information Sciences **4**(3) (2010), 281–288
– An International Journal

Monogamy and Polygamy of Entanglement in Multipartite Quantum Systems

Barry C. Sanders[1] and Jeong San Kim[2]

[1] Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada
Email Address: sandersb@ucalgary.ca

[2] Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada
Email Address: jekim@ucalgary.ca

Received November 11, 2009; Revised Jamuary 2, 2010

We summarize our recent results about monogamy and polygamy of entanglement in multipartite quantum systems. We also consider convex-roof extended negativity as a strong candidate for bipartite entanglement measure for general monogamy and polygamy relations for multipartite quantum systems.

Keywords: Multipartite quantum entanglement, monogamy, polygamy, convex-roof extended negativity.

1 Introduction

One distinct property of quantum entanglement from other classical correlations is that multipartite entanglements cannot be freely shared among the parties: If two parties in a multi-party system share a maximally entangled state, then they cannot share any entanglement with the rest. This is known as *monogamy of entanglement* (MoE) [1], which is a key ingredient for secure quantum cryptography [2, 3], and it also plays an important role in condensed-matter physics such as the N-representability problem for fermions [4].

Whereas MoE is the restricted sharability of multipartite entanglement, quantifying entanglement itself is about bipartite entanglement among the parties in multipartite systems. Thus, it is important and necessary to have a proper way of quantifying bipartite entanglement for a good description of the multipartite entanglement monogamy. For this reason, certain criteria of bipartite entanglement measure were recently proposed for a good description of the monogamy nature of entanglement in multipartite quantum systems [5]:

1. *Monotonicity*: the property that ensures entanglement cannot be increased under local operations and classical communications.

2. *Separability*: capability of distinguishing entanglement from separability.
3. *Monogamy*: upper bound on a sum of bipartite entanglement measures thereby showing that bipartite sharing of entanglement is bounded.

Using *concurrence* [6] as a bipartite entanglement measure, MoE was first shown to have a mathematical characterization in three-qubit systems as an inequality [1], and it was generalized for arbitrary multi-qubit systems [7]. As a dual concept of MoE, a *polygamy* inequality of multi-qubit entanglement was also established later in terms of *Concurrence of Assistance* (CoA).

However, multi-qubit monogamy inequality using concurrence is known to fail in its generalization for higher-dimensional quantum systems [8, 9], and this exposes the importance of having proper entanglement measure for general MoE in higher-dimensional quantum systems.

2 Concurrence-Based Monogamy and Polygamy Inequalities

For any bipartite pure state $\left|\phi\right\rangle_{AB}$, its concurrence is defined as [6]

$$\mathcal{C}(\left|\phi\right\rangle_{AB}) = \sqrt{2(1-\mathrm{tr}\rho_A^2)}, \tag{2.1}$$

where $\rho_A = \mathrm{tr}_B(\left|\phi\right\rangle_{AB}\left\langle\phi\right|)$, and for any mixed state ρ_{AB}, it is defined as

$$\mathcal{C}(\rho_{AB}) = \min \sum_k p_k \mathcal{C}(\left|\phi_k\right\rangle_{AB}), \tag{2.2}$$

where the minimum is taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k \left|\phi_k\right\rangle_{AB}\left\langle\phi_k\right|$.

As a dual quantity to concurrence, CoA is defined as

$$\mathcal{C}^a(\rho_{AB}) = \max \sum_k p_k \mathcal{C}(\left|\phi_k\right\rangle_{AB}), \tag{2.3}$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB} [10],

Concurrence and CoA are known to have an analytic formula for two-qubit systems [6, 10]: For any two-qubit mixed state ρ_{AB} in $\mathcal{B}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$, its concurrence and CoA are

$$\begin{aligned} \mathcal{C}(\rho_{AB}) &= \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \\ \mathcal{C}^a(\rho_{AB}) &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \end{aligned} \tag{2.4}$$

where λ_i's are the eigenvalues, in decreasing order, of $\sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}$ and $\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y$ with the Pauli operator σ_y.

In three-qubit systems, monogamy inequality in terms of concurrence was first proposed as [1],

$$\mathcal{C}_{A(BC)}^2 \geq \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2, \tag{2.5}$$

for a pure state $|\psi\rangle_{ABC}$, where $\mathcal{C}_{A(BC)} = \mathcal{C}(|\psi\rangle_{A(BC)})$ is the concurrence of a 3-qubit state $|\psi\rangle_{A(BC)}$ with respect to the bipartite cut between A and BC, and $\mathcal{C}_{AB}$ and $\mathcal{C}_{AC}$ are the concurrences of the reduced density matrices onto subsystems AB and AC respectively. Later, Eq. (2.5) was generalized into arbitrary multi-qubit systems as

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} \geq \mathcal{C}^2_{A_1A_2} + \cdots + \mathcal{C}^2_{A_1A_n}, \tag{2.6}$$

for an n-qubit state $\rho_{A_1\cdots A_n}$ [7].

As a dual inequality to Eq. (2.6), a polygamy inequality for multi-qubit systems in terms of CoA was also introduced as,

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} \leq (\mathcal{C}^a_{A_1A_2})^2 + \cdots + (\mathcal{C}^a_{A_1A_n})^2, \tag{2.7}$$

for an n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$ [11].

Although concurrence is a good entanglement measure for multi-qubit systems satisfying the criteria in [5], it is also known that there are some counter examples in higher-dimensional quantum systems violating concurrence-based monogamy inequality in Eq. (2.6) [8, 9].

Let us first consider a pure state $|\psi\rangle$ in $3 \otimes 3 \otimes 3$ quantum systems [8] such that

$$|\psi\rangle_{ABC} = \tfrac{1}{\sqrt{6}} \left(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle\right). \tag{2.8}$$

Then, we can easily check $C^2_{A(BC)} = \frac{4}{3}$, while $C^2_{AB} = C^2_{AC} = 1$, and therefore we have

$$C^2_{AB} + C^2_{AC} = 2 \geq \tfrac{4}{3} = C^2_{A(BC)}, \tag{2.9}$$

which is a violation of the inequality in Eq. (2.6) for higher-dimensional quantum systems.

Now, let us consider a pure state $|\psi\rangle$ in $3 \otimes 2 \otimes 2$ quantum systems such that

$$|\psi\rangle_{ABC} = \tfrac{1}{\sqrt{6}}(\sqrt{2}|010\rangle + \sqrt{2}|101\rangle + |200\rangle + |211\rangle). \tag{2.10}$$

Again, it can be easily seen that $\mathcal{C}^2_{A(BC)} = \frac{12}{9}$ whereas $\mathcal{C}^2_{AB} = \mathcal{C}^2_{AC} = \frac{8}{9}$, which implies the violation of the inequality in Eq. (2.6).

In other words, the concurrence-based monogamy inequality in Eq. (2.6) only holds for multi-qubit systems, and even a tiny extension in any of the subsystems leads to a violation.

3 Convex-Roof Extended Negativity

Besides concurremce, another well-known quantification of bipartite entanglement is the *negativity* [12, 13], which is based on the *positive partial transposition* (PPT) criterion [14, 15]. For a bipartite pure state $|\phi\rangle_{AB}$, its negativity is defined as

$$\mathcal{N}(|\phi\rangle_{AB}) = \left\| |\phi\rangle_{AB}\langle\phi|^{T_B} \right\|_1 - 1, \tag{3.1}$$

where $|\phi\rangle_{AB}\langle\phi|^{T_B}$ is the partial transpose of $|\phi\rangle_{AB}\langle\phi|$, and $\|\cdot\|_1$ is the trace norm.

However, for a mixed state ρ_{AB}, its negativity,

$$\mathcal{N}(\rho_{AB}) = \left\|{\rho_{AB}}^{T_B}\right\|_1 - 1, \tag{3.2}$$

does not even give us a separability criterion because there exist some entangled states with PPT [15, 16].

To overcome this lack of separability criterion, a modified version of negativity was introduced, and it is called *Convex-Roof Extended Negativity* (CREN) [12].

For a bipartite mixed state mixed state ρ_{AB}, CREN is defined as

$$\mathcal{N}_c(\rho) \equiv \min \sum_k p_k \mathcal{N}(|\phi\rangle_k), \tag{3.3}$$

where the minimum is taken over all possible pure state decompositions of $\rho = \sum_k p_k|\phi_k\rangle\langle\phi_k|$. Unlike usual negativity of bipartite mixed states in Eq. (3.2), CREN provides a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum system.

Now, let us consider the relation between CREN and concurrence. For any bipartite pure state $|\phi\rangle_{AB}$ with Schmidt rank 2,

$$|\phi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle_{AB} + \sqrt{\lambda_1}|11\rangle_{AB}, \tag{3.4}$$

we have

$$\mathcal{N}(|\phi\rangle_{AB}) = \left\||\phi\rangle\langle\phi|^{T_B}\right\|_1 - 1 = 2\sqrt{\lambda_0\lambda_1} = \mathcal{C}(|\phi\rangle_{AB}), \tag{3.5}$$

where $\rho_A = \mathrm{tr}_B(|\phi\rangle\langle\phi|)$. In other words, negativity is reduced to concurrence for any pure state with Schmidt rank 2, and consequently, for any 2-qubit mixed state $\rho_{AB} = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, we have

$$\mathcal{N}_c(\rho_{AB}) = \mathcal{C}(\rho_{AB}). \tag{3.6}$$

Similar to the duality between concurrence and CoA, we can also define a dual to CREN by taking the maximum value of average negativity over all possible pure state decomposition, namely *Negativity of Assistance* (NoA). Furthermore, for a two-qubit state ρ_{AB}, we have

$$\mathcal{N}^a(\rho_{AB}) = \max \sum_i p_i \mathcal{N}(|\phi_i\rangle) = \max \sum_i p_i \mathcal{C}(|\phi_i\rangle) = \mathcal{C}^a(\rho_{AB}), \tag{3.7}$$

where $\mathcal{N}^a(\rho_{AB})$ is the NoA of ρ_{AB}, and the maxima are taken over all pure state decompositions of ρ_{AB}. Thus, the monogamy and polygamy inequalities in Eqs. (2.6) and

(2.7) based on concurrence can be rephrased in terms of negativity: For any n-qubit state $\rho_{A_1\cdots A_n}$, we have

$$\mathcal{N}_{cA_1(A_2\cdots A_n)}{}^2 \geq \mathcal{N}_{cA_1A_2}{}^2 + \cdots + \mathcal{N}_{cA_1A_n}{}^2, \tag{3.8}$$

and for any n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$,

$$\mathcal{N}_{cA_1(A_2\cdots A_n)}{}^2 \leq (\mathcal{N}^a_{cA_1A_2})^2 + \cdots + (\mathcal{N}^a_{cA_1A_n})^2, \tag{3.9}$$

where $\mathcal{N}_{cA_1(A_2\cdots A_n)} = \mathcal{N}(|\psi\rangle_{A_1(A_2\cdots A_n)})$, $\mathcal{N}_{cA_1A_i} = \mathcal{N}_c(\rho_{A_1A_i})$ and $\mathcal{N}^a_{cA_1A_i} = \mathcal{N}_c{}^a(\rho_{A_1A_i})$ for $i = 2, \ldots, n$. In other words, multi-qubit monogamy and polygamy relation can be well-characterized in terms of CREN and NoA.

In fact, the states in Eqs. (2.8) and (2.10) are all known counterexamples showing the violation of the concurrence-based monogamy inequality in higher-dimensional quantum systems. However, they still have a monogamy relation in terms of CREN: For the state in Eq. (2.8), it can be directly checked that $\mathcal{N}_{A(BC)} = 2$ and $\mathcal{N}_{cAB} = \mathcal{N}_{cAC} = 1$, and thus

$$\mathcal{N}_{cA(BC)}{}^2 = 4 \geq 1 + 1 = \mathcal{N}_{cAB}{}^2 + \mathcal{N}_{cAC}{}^2. \tag{3.10}$$

Similarly, we can also show $\mathcal{N}_{cA(BC)}{}^2 = 4$ and $\mathcal{N}_{cAB}{}^2 = \mathcal{N}_{cAB}{}^2 = \frac{8}{9}$ for the state in Eq. (2.10) [5].

In other words, the states in Eqs. (2.8) and (2.10) still show the monogamy of entanglement in terms of CREN, although they are counterexamples of the concurrence-based monogamy inequality. Thus, CREN monogamy and polygamy inequalities in Eqs. (3.8) and (3.9) are strong candidates for general monogamy and polygamy inequalities in multipartite higher-dimensional quantum systems without any known counterexample.

4 Monogamy and Polygamy of Entanglement in Higher-Dimensional Quantum Systems

Multipartite entanglement is known to have many inequivalent classes, which are not convertible to each other under *stochastic local operations and classical communications* (SLOCC) [18]. For example, there are two inequivalent classes in three-qubit systems: the Greenberger-Horne-Zeilinger (GHZ) class [17] and the W-class [18]. In terms of monogamy and polygamy relations, W-class states saturate concurrence-based monogamy and polygamy inequalities in Eqs. (2.6) and (2.7), while the differences between terms in the inequalities can assume their largest values for GHZ-class states. In other words, monogamy and polygamy of multipartite entanglement can also be used for an analytical characterization of entanglement in multipartite quantum systems.

As the first step toward general CREN MoE studies in higher-dimensional quantum systems, we consider here a class of quantum states in higher-dimensional quantum systems, which are in partially coherent superposition of a generalized W-class state [9] and the vacuum. We further show that this class of states saturates CREN monogamy and polygamy inequalities in Eqs. (3.8) and (3.9), regardless of the decoherency in the superposition.

A partially coherent superposition of a generalized W-class state and $|0\rangle^{\otimes n}$ is given as

$$\begin{aligned}\rho_{A_1\cdots A_n} =&p\left|W_n^d\right\rangle\left\langle W_n^d\right| + (1-p)|0\rangle^{\otimes n}\langle 0|^{\otimes n}\\ &+\lambda\sqrt{p(1-p)}(\left|\left|W_n^d\right\rangle\langle 0|^{\otimes n} + |0\rangle^{\otimes n}\left\langle W_n^d\right|\right),\end{aligned} \tag{4.1}$$

where $\left|W_n^d\right\rangle$ is the n-qudit W-class state [9],

$$\left|W_n^d\right\rangle_{A_1\cdots A_n} = \sum_{i=1}^{d-1}(a_{1i}|i0\cdots 0\rangle + a_{2i}|0i\cdots 0\rangle + \cdots + a_{ni}|00\cdots 0i\rangle), \tag{4.2}$$

with $\sum_{j=1}^{n}\sum_{i=1}^{d-1}|a_{ji}|^2 = 1$, and λ is the degree of coherence for $0 \le \lambda \le 1$.

By using the method introduced in [5,9], we can directly evaluate the average negativity of the reduced density matrices $\rho_{A_1A_i}$ for $i = 2, \ldots, n$, and we can also show that the average negativity is invariant for any possible choice of pure state decomposition of $\rho_{A_1A_i}$. Furthermore, after a tedious calculation, we obtain the following equalities;

$$\sum_{i=2}^{n}{\mathcal{N}_{cA_1A_i}}^2 = {\mathcal{N}_{cA_1(A_2\cdots A_n)}}^2 = \sum_{i=2}^{n}\left(\mathcal{N}_{cA_1A_i}^{a}\right)^2, \tag{4.3}$$

which is the saturation of CREN monogamy and polygamy inequalities in Eqs. (3.8) and (3.9).

Besides the case of multi-qubit systems and the counterexamples in in Eqs. (2.8) and (2.10), CREN also shows a strong possibility of general monogamy and polygamy relation of entanglement by providing saturated inequalities for a large class of higher-dimensional quantum states.

5 Conclusion

We have shown that CREN is a powerful candidate to characterize general monogamy and polygamy relation of multipartite entanglement in higher-dimensional quantum systems. We have shown that multi-qubit monogamy and polygamy inequalities can be rephrased in terms of CREN, and CREN monogamy inequality is also true even for the counterexamples of concurrence-based inequalities in higher-dimensional quantum systems. We further tested the possibility of CREN monogamy and polygamy inequalities in higher-dimensional quantum systems by showing its saturation for a large class of quantum states in n-qudit systems that are in a partially coherent superpositions of a generalized W-class state and the vacuum. Thus, CREN is a strong candidate for the general monogamy relation of multipartite entanglement in higher-dimensional quantum systems with no obvious counterexamples.

Acknowledgements

We appreciate financial support from *i*CORE, USARO, and MITACS. BCS is a CIFAR Associate.

References

[1] V. Coffman, J. Kundu and W. K. Wootters, Distributed entanglement, *Phys. Rev. A* **61** (2000), 052306.
[2] J. M. Renes and M. Grassl, Generalized decoding, effective channels, and simplified security proofs in quantum key distribution, *Phys. Rev. A* **74** (2006), 022317.
[3] L. Masanes, Universally-composable Privacy Amplification from Causality Constraints, *Phys. Rev. Lett.* **102** (2009), 140501.
[4] A. J. Coleman and V. I. Yukalov, *Reduced Density Matrices: Coulsons Challenge*, in: Lecture Notes in Chemistry, Vol. **72** (Springer-Verlag, Berlin, 2000).
[5] J. S. Kim, A. Das and B. C. Sanders, Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extended negativity, *Phys. Rev. A* **79** (2009), 012329.
[6] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, *Phys. Rev. Lett.* **80** (1998), 2245.
[7] T. Osborne and F. Verstraete, General monogamy inequality for bipartite qubit entanglement, *Phys. Rev. Lett.* **96** (2006), 220503.
[8] Y. Ou, Violation of monogamy inequality for higher-dimensional objects, *Phys. Rev. A* **75** (2007), 034305.
[9] J. S. Kim and B. C. Sanders, Generalized W-class state and its monogamy relation, *J. Phys. A* **41** (2008), 495301.
[10] T. Laustsen, F. Verstraete and S. J. van Enk, Local vs. joint measurements for the entanglement of assistance, *Quantum Inf. Comput.* **3** (2003), 64.
[11] G. Gour, S. Bandyopadhay and B. C. Sanders, Duality for monogamy of entanglement, *J. Math. Phys.* **48** (2007), 012108.
[12] S. Lee, D. P. Chi, S. D. Oh and J. Kim, Convex-roof extended negativity as an entanglement measure for bipartite quantum systems, *Phys, Rev. A* **68** (2003), 62304.
[13] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65** (2002), 032314.
[14] A. Peres, Separability criterion for density matrices, *Phys. Rev. Lett.* **77** (1996), 1413.
[15] M. Horodecki, P. Horodecki and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, *Phys. Lett. A* **223** (1996), 1.
[16] M. Horodecki, P. Horodecki and R. Horodecki, Mixed-state entanglement and distillation: is there a "bound" entanglement in nature?, *Phys. Rev. Lett.* **80** (1998), 5239.
[17] D. M. Greenberger, M. A. Horne and A. Zeilinger, *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.

[18] W. Dür, G. Vidal and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, *Phys. Rev. A* **62** (2000), 062314.

Dr. Barry Sanders is iCORE Chair of Quantum Information Science and Director of the Institute for Quantum Information Science at the University of Calgary. He received his Bachelor of Science degree from the University of Calgary in 1984 and then completed a Diploma of Imperial College. Subsequently he completed a PhD at Imperial College. Following completion of the PhD in 1987, he was a postdoctoral research fellow under the supervision of Professor Gerard Milburn. He joined the Physics Department of Macquarie University in 1991 and was there for 12 years, including 6.5 years as Department Head, before moving to Calgary in 2003. He is especially well known for seminal contributions to theories of quantum-limited measurement, highly nonclassical light, practical quantum cryptography, and optical implementations of quantum information tasks.

Dr. Jeong San Kim is a postdoctoral associate of the Institute for Quantum Information Science at the University of Calgary. He received his Ph.D. in mathematical science from Seoul National University, Korea in 2006, and he was also a postdoctoral research fellow at Seoul National University. In December 2007, he moved to the Institute for Quantum Information Science at the University of Calgary working as a postdoc. under the supervision of Professor Barry C. Sanders. Dr. Kim is currently working on the mathematical aspect of quantum information theory and quantum cryptography, which are mainly about quantum entanglement theory in multipartite quantum systems and the structural property of quantum cryptosystems using bound entanglement.