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Dynamical Behavior of Fractional-Order Rumor Model Based on the Activity of Spreaders

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Abstract: According to the similarity between the infectious disease transmission and the rumor spreading, we introduce this manuscript. In this work, the dynamical behavior of the fractional-order rumor model (FOM) is investigated in details. Also, we determine all the equilibrium fixed points of model. Nevertheless, the stability at this equilibrium points is studied. The basic reproduction number of FOM is obtained. Some valuable and essential definitions about the Caputo fractional derivative are introduced. Various methods use to solve this model such as Generalized Mittag-Leffler Function method (GMLFM) is an approximate solution and Predictor-Corrector method (PCM) as a numerical solution. Numerical simulations are performed to confirm our analytical results and elucidates the effect of various parameters on the rumor spreading.

Keywords: Rumor model, equilibrium points, stability analysis, basic reproduction number, Mittag-Leffler function, predictor-corrector.

1 Introduction

Despite the development and increasing use of social media platforms for information and news gatherings however its leads to spread and emergence rumors. At the same time, the openness of using social media platforms provide opportunities to investigate how users discuss and share rumors. Rumors play an effective and important role in social topics such that their spreading has obvious effect on people's life. Recently, online social networks such as Facebook, Wechat, Twitter and Instagram become an innovative and effective channel for spreading information and news between millions of people's. There are many definitions of rumors that differ from one to another [1,2,3] but the common definition of rumors are the spreading of public interested things, some events or disinformation of problems which usually spread widely through different channels. Many scholars have focused in their studies on the spreading of rumors [4,5,6,7,8,9,10,11,12], also some scholar have identified rumor as subset of sociology, propaganda and psychology [13]. The rapidly spread of rumors or disinformation has a negative impact on many fields of your life such political consequence, economic damage and reputation damage [14,15,16]. At is well known, the disease spreading from person to another implies physical contacts but the spreading of rumor thought online connection is widest in spreading from the traditional face to face communications.
In 1965, the rumor spreading named DK was introduced by Daley and Kendall [17]. In [18], both Maki and Thomson focused in studying on developing the DK model and interested to the analysis of the rumor spreading based on mathematical theory. A lot of authors investigated the spreading of rumor and dynamical behaviors of model depends on the theory of mathematical and related them to topological properties of social networks [19,20,21,22,23,24,25], Zenette determined the spreading of rumor model on small-world networks and establish the rumor spread threshold [26, 27], Zhao et al [28] explained the rumor spreading processes by forgetting mechanism in SIR model on small world networks. Furthermore, many of researcher have been investigated various kinds of rumor spreading by using different ways [29,30,31,32,33,34,35,36]. In this work, we focus our studying on the fractional-order rumor model such that

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individuals divide into four species:
A is called ignorants (individuals who never heard the rumors), B tends to the low rate of active spreaders (individuals who spread rumors with low probability), C stand for the high rate of active spreaders (individuals who spread rumors with high probability) and G refer to stiflers (individuals who know the rumor but never spread it).
The fractional-order of rumor spreading is described as follows:

$$\begin{aligned} {}_0^CD_t^\alpha A &= p^\alpha - \eta^\alpha \sigma^\alpha AB - \xi^\alpha \sigma^\alpha AC - q^\alpha A, \\ {}_0^CD_t^\alpha B &= \eta^\alpha \sigma^\alpha AB + \gamma^\alpha C - {\mu_1}^\alpha \sigma^\alpha BG - q^\alpha B - \theta^\alpha B, \\ {}_0^CD_t^\alpha C &= \xi^\alpha \sigma^\alpha AC - \gamma^\alpha C - {\mu_2}^\alpha \sigma^\alpha CG - q^\alpha C + \theta^\alpha B, \\ {}_0^CD_t^\alpha G &= {\mu_1}^\alpha \sigma^\alpha BG + {\mu_2}^\alpha \sigma^\alpha CG - q^\alpha G, \end{aligned} \quad (1)$$

where both p, q denote to the coming rate of ignorant (or inflow rate) and the leaving rate of the population (or outflow rate) respectively such that $p = q$, η is the probability of the ignorants who become the low rate of active spreader, ξ refer to the transmission probability from ignorants to the high rate of active spreaders and we suppose that $\xi > \eta$, μ_1 is called the stifling rate such that the first rumor spreader may become a stifler, μ_2 stand for the probability of the high rate of active spreader becomes removed, γ is called the decay rate, assume that $\mu_1 = \mu_2 = \mu$, θ tends to the probability of the high active spreaders lead to the low active spreaders also the probability of the low rate of active spreader lead to the high rate of active spreader where we assume that it equal to zero i.e $\theta = 0$, σ is the average degree of freedom, and both $A(t), B(t), C(t), G(t)$ express to the density of ignorants, the low rate of active spreaders, the high rate of active spreaders, and stiflers at time t, respectively and satisfy the normalization condition $A(t) + B(t) + C(t) + G(t) = 1$.
According to the relation between the given parameters, the system (1) is transformed to

$$\begin{aligned} {}_0^CD_t^\alpha A &= q^\alpha - \eta^\alpha \sigma^\alpha AB - \xi^\alpha \sigma^\alpha AC - q^\alpha A, \\ {}_0^CD_t^\alpha B &= \eta^\alpha \sigma^\alpha AB + \gamma^\alpha C - \mu^\alpha \sigma^\alpha BG - q^\alpha B, \\ {}_0^CD_t^\alpha C &= \xi^\alpha \sigma^\alpha AC - \gamma^\alpha C - \mu^\alpha \sigma^\alpha CG - q^\alpha C, \\ {}_0^CD_t^\alpha G &= \mu^\alpha \sigma^\alpha BG + \mu^\alpha \sigma^\alpha CG - q^\alpha G. \end{aligned} \quad (2)$$

The main aim of this work is to formulate a rumor model using fractional-order derivatives which have an advantage over their corresponding integer-order counterparts because of their memory effect property. Also, solve this model by two methods GMLFM as an analytical solution and PCM is numerical solution

The rest of this paper is organized as follows. Many various definitions concerned to fractional-order differential equations (FODEs) are introduced in Section 2. In section 3, we illustrate the non-negative solution of the rumor model. Also, the stability and the equilibrium fixed points of model are analyzed in details in Section 4. in Section 5, we explain the used methods and its applications. Numerical simulation are presented to confirm the obtained theoretical results in Section 6. Finally, we offer our conclusion in Section 7.

2 Preliminaries

In this section, we introduce some important definitions concerned to fractional calculus (see [37]):

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by*

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \qquad \alpha > 0, \qquad t > a,$$
$$ {}_aI_t^0 f(t) = f(t),$$

where $\Gamma(\alpha)$ is Eular Gamma function is defined as follows

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \qquad \Re(z) > 0.$$

Definition 2. *The Riemann-Liouville fractional derivatives is defined by*

$$ {}_aD_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha-n)} \frac{d^n}{dt^n} \int_a^t \frac{f(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi, & n-1 < \alpha < n, n \in N, \\ \frac{d^n f(t)}{dt^n}, & \alpha = n, n \in N. \end{cases}$$

Definition 3. *The Caputo fractional derivative of order $\alpha > 0$ is defined as follows*

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, & n-1 < \alpha < n, n \in N, \\ \frac{d^n f(x)}{dt^n}, & \alpha = n, n \in N. \end{cases} $$

Also, we present some properties of fractional calculus their detailedly explained in references [38,39]

$$ {}_{a}^{C}D_x^{\alpha} \, {}_{a}I_x^{\alpha} f(x) = f(x), $$

$$ {}_{a}I_x^{\alpha} \, {}_{a}^{C}D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(x-a)^k}{\Gamma(k+1)}. $$

Definition 4. *In (1902-1905), the Mittag-Leffler functions E_α and $E_{\alpha,\beta}$ defined by the power series as*

$$ E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha+1)}, \qquad E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha+\beta)}, \qquad \alpha, \beta > 0, \tag{3} $$

the Caputo fractional derivative of GMLFM is decomposed by an infinite series of components [40,41,42,43] is given by

$$ {}_{0}^{C}D_t^{\alpha} E_\alpha(ax^\alpha) = \sum_{n=1}^{\infty} a^n \frac{x^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}. \tag{4} $$

3 Non-negative solution

Assume that $R_+^4 = \{X \in R^4 : X \geq 0\}$, and $X(t) = (A, B, C, G)^T$. To prove the non-negative solution, we investigate the following theorem and corollary.

Theorem 1. *[44](Generalized mean value theorem). Suppose that $f(x) \in C(0,a]$ and $D^\alpha f(x) \in C(0,a]$, for $0 < \alpha \leq 1$. Then we have*

$$ f(x) = f(0) + \frac{1}{\Gamma(\alpha)} (D^\alpha f)(\zeta)(x)^\alpha $$

with $0 \leq \zeta \leq x$, $\forall x \in (0,a]$,

Corollary 1. *Let $f(x) \in C[0,a]$ and $D^\alpha f(x) \in C(0,a]$, for $0 < \alpha \leq 1$.. It is clear that from Theorem 1 that if $D^\alpha f(x) \geq 0$, $\forall x \in (0,a]$, then $f(x)$ is non-decreasing and if $D^\alpha f(x) \leq 0$, $\forall x \in (0,a]$, then $f(x)$ is non-increasing $\forall x \in [0,a]$.*

Proof. In [45], the proof is explained in theorem (1).

Theorem 2. *There is a unique solution $X(t) = (A, B, C, G)^T$ for (2) at $t \geq 0$ and the solution will remain in R_+^4.*

Proof. From [46], we can get the existence and uniqueness of the solution of the initial value problem (2) in $(0, \infty)$. Now, we will clear that R_+^4 is positively invariant domain.
Then

$$ \begin{aligned} {}_{0}^{C}D_t^{\alpha} A \mid_{A=0} &= p^\alpha \geq 0, \\ {}_{0}^{C}D_t^{\alpha} B \mid_{B=0} &= \gamma^\alpha C \geq 0, \\ {}_{0}^{C}D_t^{\alpha} C \mid_{C=0} &= 0, \\ {}_{0}^{C}D_t^{\alpha} G \mid_{G=0} &= 0. \end{aligned} $$

According to corollary 1, we will deduce that the solution remain in R_+^4.

4 Stability and equilibrium points

To determine the equilibrium points of the fractional-order model (2), Assume that

$$ {}_0^CD_t^{\alpha}A = 0, $$
$$ {}_0^CD_t^{\alpha}B = 0, $$
$$ {}_0^CD_t^{\alpha}C = 0, $$
$$ {}_0^CD_t^{\alpha}G = 0. $$

By simple calculating, the system (2) has five equilibrium points as:

– The axial equilibrium point (rumor-free equilibrium point) $E_0 = (A_0, B_0, C_0, G_0) = (1, 0, 0, 0)$.

– The boundary equilibrium fixed point $E_1 = (A_1, B_1, C_1, G_1) = (\frac{q^{\alpha}}{\eta^{\alpha}\sigma^{\alpha}}, \frac{\eta^{\alpha}\sigma^{\alpha}-q^{\alpha}}{\eta^{\alpha}\sigma^{\alpha}}, 0, 0)$,

– The boundary equilibrium fixed point $E_2 = (A_2, B_2, C_2, G_2) = (\frac{\mu^{\alpha}}{\mu^{\alpha}+\eta^{\alpha}}, \frac{q^{\alpha}}{\mu^{\alpha}\sigma^{\alpha}}, 0, \frac{\eta^{\alpha}\mu^{\alpha}\sigma^{\alpha}-q^{\alpha}\eta^{\alpha}-q^{\alpha}\mu^{\alpha}}{\mu^{\alpha}\sigma^{\alpha}(\mu^{\alpha}+\eta^{\alpha})})$,

– The equilibrium fixed point E_3 where

$$ E_3 = (A_3, B_3, C_3, G_3) = (\frac{q^{\alpha}+\gamma^{\alpha}}{\xi^{\alpha}\sigma^{\alpha}}, \frac{\gamma^{\alpha}(\xi^{\alpha}\sigma^{\alpha}-q^{\alpha}-\gamma^{\alpha})}{\sigma^{\alpha}(q^{\alpha}+\gamma^{\alpha})(\xi^{\alpha}-\eta^{\alpha})}, \frac{(q^{\alpha}+\gamma^{\alpha}-\xi^{\alpha}\sigma^{\alpha})(q^{\alpha}\eta^{\alpha}+\gamma^{\alpha}\eta^{\alpha}-q^{\alpha}\xi^{\alpha})}{\xi^{\alpha}\sigma^{\alpha}(q^{\alpha}+\gamma^{\alpha})(\xi^{\alpha}-\eta^{\alpha})}, 0), $$

– The positive equilibrium fixed point (the interior point) $E_* = (A_*, B_*, C_*, G_*)$, where

$$ A_* = \frac{\gamma^{\alpha} + \frac{\mu^{\alpha}q^{\alpha}R_0}{\eta^{\alpha}}}{\sigma^{\alpha}(\mu^{\alpha}+\xi^{\alpha})}, $$

$$ B_* = \frac{\frac{\eta^{\alpha}\sigma^{\alpha}}{R_0}(\gamma^{\alpha}\mu^{\alpha}+\gamma^{\alpha}\xi^{\alpha})}{\mu^{\alpha}\sigma^{\alpha}(\gamma^{\alpha}+\mu^{\alpha}\sigma^{\alpha})(\xi^{\alpha}-\eta^{\alpha})}, $$

$$ C_* = \frac{\eta^{\alpha}(\gamma^{\alpha}\eta^{\alpha}+\gamma^{\alpha}\mu^{\alpha}+\eta^{\alpha}\mu^{\alpha}\sigma^{\alpha}-\mu^{\alpha}\xi^{\alpha}\sigma^{\alpha})}{\mu^{\alpha}R_0(\gamma^{\alpha}+\mu^{\alpha}\sigma^{\alpha})(\eta^{\alpha}-\xi^{\alpha})}, $$

$$ G_* = \frac{\frac{\mu^{\alpha}\xi^{\alpha}q^{\alpha}R_0}{\eta^{\alpha}} - q^{\alpha}\mu^{\alpha} - \gamma^{\alpha}\mu^{\alpha} - q^{\alpha}\xi^{\alpha}}{\mu^{\alpha}\sigma^{\alpha}(\mu^{\alpha}+\xi^{\alpha})}, $$

where R_0 refer to the basic reproduction number [47] is given by the following relation:

$$ R_0 = \rho(FV^{-1}) = \frac{\eta^{\alpha}\sigma^{\alpha}}{q^{\alpha}}, $$

where

$$ F = \begin{pmatrix} \eta^{\alpha}\sigma^{\alpha} & 0 \\ 0 & 0 \end{pmatrix}, $$

$$ V = \begin{pmatrix} q^{\alpha} & -\gamma^{\alpha} \\ 0 & -\xi^{\alpha}\sigma^{\alpha}+\gamma^{\alpha}+q^{\alpha} \end{pmatrix}. $$

Stability analysis of the rumor-free equilibrium point E_0

To study the asymptotic stability of the rumor-free equilibrium point $E_0 = (1, 0, 0, 0)$, we calculate the Jacobian matrix of the model (2) at E_0 as follows

$$ J(E_0) = \begin{pmatrix} -q^{\alpha} & -\eta^{\alpha}\sigma^{\alpha} & -\xi^{\alpha}\sigma^{\alpha} & 0 \\ 0 & \eta^{\alpha}\sigma^{\alpha}-q^{\alpha} & \gamma^{\alpha} & 0 \\ 0 & 0 & \xi^{\alpha}\sigma^{\alpha}-\gamma^{\alpha}-q^{\alpha} & 0 \\ 0 & 0 & 0 & -q^{\alpha} \end{pmatrix}, \quad (5) $$

the characteristic equation of matrix $J(E_0)$ is given by

$$|J(E_0)-\lambda I| = \begin{pmatrix} -q^\alpha-\lambda & -\eta^\alpha\sigma^\alpha & -\xi^\alpha\sigma^\alpha & 0 \\ 0 & \eta^\alpha\sigma^\alpha-q^\alpha-\lambda & \gamma^\alpha & 0 \\ 0 & 0 & \xi^\alpha\sigma^\alpha-\gamma^\alpha-q^\alpha-\lambda & 0 \\ 0 & 0 & 0 & -q^\alpha-\lambda \end{pmatrix}, \tag{6}$$

$$= (-q^\alpha-\lambda)(\eta^\alpha\sigma^\alpha-q^\alpha-\lambda)(\xi^\alpha\sigma^\alpha-\gamma^\alpha-q^\alpha-\lambda)(-q^\alpha-\lambda),$$

has four eigenvalues are $\lambda_{01}=-q^\alpha$, $\lambda_{02}=\eta^\alpha\sigma^\alpha-q^\alpha$, $\lambda_{03}=\xi^\alpha\sigma^\alpha-\gamma^\alpha-q^\alpha$, $\lambda_{04}=-q^\alpha$. The equilibrium point E_0 is locally asymptotically stable if $\frac{q^\alpha}{\xi^\alpha\sigma^\alpha}<1$, $\frac{q^\alpha+\gamma^\alpha}{\xi^\alpha\sigma^\alpha}<1$.

Stability analysis of the boundary equilibrium point E_1

In the following, we estimate the stability of the boundary equilibrium fixed point $E_1=(\frac{q^\alpha}{\eta^\alpha\sigma^\alpha},\frac{\eta^\alpha\sigma^\alpha-q^\alpha}{\eta^\alpha\sigma^\alpha},0,0)$. The Jacobian matrix at E_1 take the form

$$J(E_1) = \begin{pmatrix} -\eta^\alpha\sigma^\alpha & -q^\alpha & \frac{-\xi^\alpha q^\alpha}{\eta^\alpha} & 0 \\ \eta^\alpha\sigma^\alpha-q^\alpha & 0 & \gamma^\alpha & \frac{-\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}{\eta^\alpha} \\ 0 & 0 & \frac{\xi^\alpha q^\alpha}{\eta^\alpha}-\gamma^\alpha-q^\alpha & 0 \\ 0 & 0 & 0 & \frac{\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}{\eta^\alpha}-q^\alpha \end{pmatrix}, \tag{7}$$

the characteristic equation of E_1 is described as

$$|J(E_1)-\lambda I| = \begin{pmatrix} -\eta^\alpha\sigma^\alpha-\lambda & -q^\alpha & \frac{-\xi^\alpha q^\alpha}{\eta^\alpha} & 0 \\ \eta^\alpha\sigma^\alpha-q^\alpha & 0-\lambda & \gamma^\alpha & \frac{-\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}{\eta^\alpha} \\ 0 & 0 & \frac{\xi^\alpha q^\alpha}{\eta^\alpha}-\gamma^\alpha-q^\alpha-\lambda & 0 \\ 0 & 0 & 0 & \frac{\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}{\eta^\alpha}-q^\alpha-\lambda \end{pmatrix}, \tag{8}$$

$$= (-q-\lambda)(-\eta^\alpha\sigma^\alpha-q-\lambda)(\frac{-\xi^\alpha q^\alpha}{\eta^\alpha}-\gamma^\alpha-q^\alpha-\lambda)(\frac{\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}{\eta^\alpha}-q^\alpha-\lambda),$$

has the following eigenvalues
$\lambda_{11}=-q^\alpha$, $\lambda_{12}=q^\alpha-\eta^\alpha\sigma^\alpha$, $\lambda_{13}=\frac{\xi^\alpha q^\alpha}{\eta^\alpha}-\gamma^\alpha-q^\alpha$, $\lambda_{14}=\frac{\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}{\eta^\alpha}-q^\alpha$. The equilibrium point E_1 is locally asymptotically stable if $\frac{q^\alpha}{\eta^\alpha\sigma^\alpha}<1$, $\frac{\eta^\alpha(q^\alpha+\gamma^\alpha)}{\xi^\alpha q^\alpha}<1$, $\frac{\eta^\alpha q^\alpha}{\mu^\alpha(\eta^\alpha\sigma^\alpha-q^\alpha)}<1$.

Stability analysis of the boundary equilibrium point E_2

In this subsection, we investigate the asymptotic stability of the model (2) at the equilibrium point $E_2=(\frac{\mu^\alpha}{\eta^\alpha+\mu^\alpha},\frac{q^\alpha}{\mu^\alpha\sigma^\alpha},0,\frac{\eta^\alpha\mu^\alpha\sigma^\alpha-q^\alpha\eta^\alpha-q^\alpha\mu^\alpha}{\mu^\alpha\sigma^\alpha(\eta^\alpha+\mu^\alpha)})$, the Jacobian matrix is given by

$$J(E_2) = \begin{pmatrix} \frac{-\eta^\alpha q^\alpha}{\mu^\alpha}-q^\alpha & \frac{-\eta^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & \frac{-\xi^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & 0 \\ \frac{\eta^\alpha q^\alpha}{\mu^\alpha} & \frac{q^\alpha\eta^\alpha+q^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha}-q^\alpha & \gamma^\alpha & -q^\alpha \\ 0 & 0 & \frac{\xi^\alpha\sigma^\alpha\mu^\alpha-\eta^\alpha\sigma^\alpha\mu^\alpha+q^\alpha\eta^\alpha+q^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha}-\gamma^\alpha-q^\alpha & 0 \\ 0 & \frac{\eta^\alpha\sigma^\alpha\mu^\alpha-q^\alpha\eta^\alpha-q^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & \frac{\eta^\alpha\sigma^\alpha\mu^\alpha-q^\alpha\eta^\alpha-q^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & 0 \end{pmatrix}, \tag{9}$$

the characteristic equation of matrix $J(E_2)$ is given by

$$|J(E_2)-\lambda I| = \begin{pmatrix} \frac{-\eta^\alpha q^\alpha}{\mu^\alpha}-q^\alpha-\lambda & \frac{-\eta^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & \frac{-\xi^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & 0 \\ \frac{\eta^\alpha q^\alpha}{\mu^\alpha} & -\lambda & \gamma^\alpha & -q^\alpha \\ 0 & 0 & \frac{\xi^\alpha\sigma^\alpha\mu^\alpha-\eta^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha}-\gamma^\alpha-\lambda & 0 \\ 0 & \frac{\eta^\alpha\sigma^\alpha\mu^\alpha-q^\alpha\eta^\alpha-q^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & \frac{\eta^\alpha\sigma^\alpha\mu^\alpha-q^\alpha\eta^\alpha-q^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} & -\lambda \end{pmatrix}, \tag{10}$$

$$= (\frac{\xi^\alpha\sigma^\alpha\mu^\alpha-\eta^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha}-\gamma^\alpha-\lambda)(\lambda^3+\Delta_2\lambda^2+\Delta_1\lambda+\Delta_0).$$

From the previous equation, we deduce that $\lambda_{21} = \frac{\xi^\alpha\sigma^\alpha\mu^\alpha - \eta^\alpha\sigma^\alpha\mu^\alpha}{\eta^\alpha+\mu^\alpha} - \gamma^\alpha$, and

$$\lambda^3 + \Delta_2\lambda^2 + \Delta_1\lambda + \Delta_0 = 0, \tag{11}$$

where
$\Delta_2 = \frac{\eta^\alpha q^\alpha}{\mu^\alpha} + q^\alpha$, $\Delta_1 = \eta^\alpha\sigma^\alpha q^\alpha - q^{\alpha 2}$, $\Delta_0 = \frac{q^{\alpha 2}(\eta^\alpha\sigma^\alpha\mu^\alpha - q^\alpha(\eta^\alpha+\mu^\alpha))}{\mu^\alpha}$, we can deduce that both $\Delta_2 > 0$, $\Delta_1 > 0$, and $\Delta_1\Delta_2 - \Delta_0 = \frac{q^{\alpha 2}\eta^{\alpha 2}\sigma^\alpha}{\mu^\alpha} > 1$. The equilibrium point E_2 is locally asymptotically stable if $\frac{q^\alpha}{\eta^\alpha\sigma^\alpha} < 1$ and $\frac{\xi^\alpha\sigma^\alpha\mu^\alpha - \eta^\alpha\sigma^\alpha\mu^\alpha + q^\alpha\eta^\alpha + q^\alpha\mu^\alpha}{(\eta^\alpha+\mu^\alpha)(q^\alpha+\sigma^\alpha)} < 1$.

Stability analysis of the boundary equilibrium point E_3

Similarly, we compute the Jacobin matrix of the model (2) at the equilibrium $E_3 = (\frac{q^\alpha+\gamma^\alpha}{\xi^\alpha\sigma^\alpha}, \frac{\gamma^\alpha(\xi^\alpha\sigma^\alpha - q^\alpha - \gamma^\alpha)}{\sigma^\alpha(q^\alpha+\gamma^\alpha)(\xi^\alpha-\eta^\alpha)}, \frac{(q^\alpha+\gamma^\alpha-\xi^\alpha\sigma^\alpha)(q^\alpha\eta^\alpha+\gamma^\alpha\eta^\alpha-q^\alpha\xi^\alpha)}{\xi^\alpha\sigma^\alpha(q^\alpha+\gamma^\alpha)(\xi^\alpha-\eta^\alpha)}, 0)$ and evaluate its the eigenvalues as follows

$$J(E_3) = \begin{pmatrix} -\eta^\alpha u_1 - u_2 - q^\alpha & \frac{-\eta^\alpha(q^\alpha+\gamma^\alpha)}{\xi^\alpha} & -\gamma^\alpha - q^\alpha & 0 \\ \eta^\alpha u_1 & \frac{\eta^\alpha(q^\alpha+\gamma^\alpha)}{\xi^\alpha} - q^\alpha & \gamma^\alpha & -\mu^\alpha u_1 \\ u_2 & 0 & 0 & \frac{-\mu^\alpha u_2}{\xi^\alpha} \\ 0 & 0 & 0 & \frac{-\gamma^\alpha\mu^\alpha + \mu^\alpha\sigma^\alpha\xi^\alpha - q^\alpha(\mu^\alpha+\eta^\alpha)}{\xi^\alpha} \end{pmatrix}, \tag{12}$$

where
$u_1 = \frac{\gamma^\alpha(\xi^\alpha\sigma^\alpha - q^\alpha - \gamma^\alpha)}{(q^\alpha+\gamma^\alpha)(\xi^\alpha-\eta^\alpha)}$, $u_2 = \frac{(q^\alpha+\gamma^\alpha-\xi^\alpha\sigma^\alpha)(q^\alpha\eta^\alpha+\gamma^\alpha\eta^\alpha-q^\alpha\xi^\alpha)}{(q^\alpha+\gamma^\alpha)(\xi^\alpha-\eta^\alpha)}$. The characteristic equation of the Jacobian matrix $J(E_3)$ is given by

$$|J(E_3) - \lambda I| = \begin{pmatrix} -\eta^\alpha u_1 - u_2 - q^\alpha - \lambda & \frac{-\eta^\alpha(q^\alpha+\gamma^\alpha)}{\xi^\alpha} & -\gamma^\alpha - q^\alpha & 0 \\ \eta^\alpha u_1 & \frac{\eta^\alpha(q^\alpha+\gamma^\alpha)}{\xi^\alpha} - q^\alpha - \lambda & \gamma^\alpha & -\mu^\alpha u_1 \\ u_2 & 0 & -\lambda & \frac{-\mu^\alpha u_2}{\xi^\alpha} \\ 0 & 0 & 0 & \frac{-\gamma^\alpha\mu^\alpha + \mu^\alpha\sigma^\alpha\xi^\alpha - q^\alpha(\mu^\alpha+\eta^\alpha)}{\xi^\alpha} - \lambda \end{pmatrix}, \tag{13}$$

$$= \left(\frac{-\gamma^\alpha\mu^\alpha + \mu^\alpha\sigma^\alpha\xi^\alpha - q^\alpha(\mu^\alpha+\eta^\alpha)}{\xi^\alpha} - \lambda\right)(\lambda^3 + \Phi_2\lambda^2 + \Phi_1\lambda + \Phi_0) = 0.$$

We can obtain one eigenvalue is $\lambda_{31} = \frac{\mu^\alpha\sigma^\alpha\xi^\alpha - \gamma^\alpha\mu^\alpha - q^\alpha(\mu^\alpha+\eta^\alpha)}{\xi^\alpha}$, and the following equation

$$\lambda^3 + \Phi_2\lambda^2 + \Phi_1\lambda + \Phi_0 = 0, \tag{14}$$

from the equation (14) both Φ_2, Φ_1 and Φ_0 are given by

$\Phi_2 = \frac{q^\alpha(\sigma^\alpha\xi^{\alpha 2} + \gamma^\alpha(\xi^\alpha - 2\eta^\alpha)) + q^{\alpha 2}(\xi^\alpha - \eta^\alpha) - \gamma^{\alpha 2}\eta^\alpha}{\xi^\alpha(q^\alpha+\gamma^\alpha)}$, $\Phi_1 = \frac{(\gamma^\alpha - \sigma^\alpha\xi^\alpha)(\gamma^{\alpha 2}\eta^\alpha + q^{\alpha 2}(\eta^\alpha - 2\xi^\alpha) + q^\alpha\gamma^\alpha(2\eta^\alpha - \xi^\alpha)) - q^{\alpha 3}\xi^\alpha}{\xi^\alpha(q^\alpha+\gamma^\alpha)}$,
$\Phi_0 = \frac{q^\alpha(\gamma^\alpha\eta^\alpha + q^\alpha(\eta^\alpha - \xi^\alpha)(q^\alpha + \gamma^\alpha - \sigma^\alpha\xi^\alpha))}{\xi^\alpha}$, we note that $\Phi_2 > 0, \Phi_1 > 0$, and $\Phi_1\Phi_2 - \Phi_0 > 0$. The equilibrium point E_3 is locally asymptotically stable at $\frac{\mu^\alpha\sigma^\alpha\xi^\alpha}{\gamma^\alpha\mu^\alpha + q^\alpha(\mu^\alpha+\eta^\alpha)} < 1$.

Stability analysis of the positive equilibrium point E_*

The Jacobian matrix at the interior equilibrium point $E_* = (A_*, B_*, C_*, G_*)$ of model (2) is given by the form

$$J(E_*) = \begin{pmatrix} v_1 & -\eta^\alpha\sigma^\alpha A_* & -\xi^\alpha\sigma^\alpha & 0 \\ \eta^\alpha\sigma^\alpha B_* & v_2 & \gamma^\alpha & -\mu^\alpha\sigma^\alpha B_* \\ \xi^\alpha\sigma^\alpha C_* & 0 & v_3 & -\mu^\alpha\sigma^\alpha C_* \\ 0 & \mu^\alpha\sigma^\alpha G_* & \mu^\alpha\sigma^\alpha G_* & v_4 \end{pmatrix}, \tag{15}$$

where $v_1 = -\sigma^\alpha\eta^\alpha B_* - \sigma^\alpha\xi^\alpha C_* - q^\alpha$, $v_2 = \sigma^\alpha\eta^\alpha A_* - \sigma^\alpha\mu^\alpha G_* - q^\alpha$, $v_3 = \sigma^\alpha\xi^\alpha A_* - \sigma^\alpha\mu^\alpha G_* - \gamma^\alpha - q^\alpha$, $v_4 = \mu^\alpha\sigma^\alpha B_* + \mu^\alpha\sigma^\alpha C_* - q^\alpha$.

The characteristic equation of matrix $J(E_*)$ is described as

$$|J(E_*)-\lambda I| = \begin{pmatrix} v_1-\lambda & -\eta^\alpha\sigma^\alpha A_* & -\xi^\alpha\sigma^\alpha & 0 \\ \eta^\alpha\sigma^\alpha B_* & v_2-\lambda & \gamma^\alpha & -\mu^\alpha\sigma^\alpha B_* \\ \xi^\alpha\sigma^\alpha C_* & 0 & v_3-\lambda & -\mu^\alpha\sigma^\alpha C_* \\ 0 & \mu^\alpha\sigma^\alpha G_* & \mu^\alpha\sigma^\alpha G_* & v_4-\lambda \end{pmatrix}, \quad (16)$$

$$= \lambda^4+\Psi_3\lambda^3+\Psi_2\lambda^2+\Psi_1\lambda+\Psi_0=0,$$

where

$$\begin{cases} \Psi_3 = \eta^\alpha\sigma^\alpha(B_*-A_*)+\xi^\alpha\sigma^\alpha(C_*-A_*)+\mu^\alpha\sigma^\alpha(2G_*-B_*-C_*)+\gamma^\alpha+4q^\alpha, \\ \Psi_2 = \Big(-\eta^\alpha\sigma^\alpha B_*-\xi^\alpha\sigma^\alpha C_*-q^\alpha\Big)\Big(\eta^\alpha\sigma^\alpha A_*+\xi^\alpha\sigma^\alpha A_*+\mu^\alpha\sigma^\alpha(B_*+C_*-2G_*)-\gamma^\alpha-3q^\alpha\Big) \\ \quad +\Big(\eta^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-q^\alpha\Big)\Big(\xi^\alpha\sigma^\alpha A_*+\mu^\alpha\sigma^\alpha(B_*+C_*-G_*)-\gamma^\alpha-2q^\alpha\Big) \\ \quad +\Big(\xi^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-\gamma^\alpha-q^\alpha\Big)\Big(\mu^\alpha\sigma^\alpha B_*-\mu^\alpha\sigma^\alpha C_*-q^\alpha\Big)+\mu^{\alpha 2}\sigma^{\alpha 2}C_*(B_*+G_*) \\ \quad +\eta^{\alpha 2}\sigma^{\alpha 2}A_*B_*+\xi^{\alpha 2}\sigma^{\alpha 2}A_*C_*, \\ \Psi_1 = \Big(\eta^\alpha\sigma^\alpha B_*+\xi^\alpha\sigma^\alpha C_*+q^\alpha\Big)\Big(\eta^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-q^\alpha\Big)\Big(\xi^\alpha\sigma^\alpha A_*+\mu^\alpha\sigma^\alpha(B_*+C_*-G_*)-\gamma^\alpha-2q^\alpha\Big) \\ \quad -\Big(\xi^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-\gamma^\alpha-q^\alpha\Big)\Big(\mu^\alpha\sigma^\alpha(B_*+C_*)-q^\alpha\Big)\Big(\sigma^\alpha(-\eta^\alpha B_*-\xi^\alpha C_*+\eta^\alpha A_*-\mu^\alpha G_*)-2q^\alpha\Big) \\ \quad +\mu^{\alpha 2}\sigma^{\alpha 2}C_*(G_*+B_*)\Big(\eta^\alpha\sigma^\alpha B_*+\xi^\alpha\sigma^\alpha C_*+q^\alpha\Big)-\sigma^{\alpha 2}C_*\Big(\mu^{\alpha 2}G_*+\xi^{\alpha 2}A_*\Big)\Big(\eta^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-q^\alpha\Big) \\ \quad -\sigma^{\alpha 2}B_*\Big(\mu^{\alpha 2}C_*+\eta^{\alpha 2}A_*\Big)\Big(\xi^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-\gamma^\alpha-q^\alpha\Big)-\sigma^{\alpha 2}\mu^{\alpha 2}\gamma^\alpha G_*-\eta^\alpha\xi^\alpha\sigma^{\alpha 2}\gamma^\alpha A_*C_* \\ \quad -\sigma^{\alpha 2}A_*\Big(\eta^{\alpha 2}B_*+\xi^{\alpha 2}C_*\Big)\Big(\mu^\alpha\sigma^\alpha B_*+\mu^\alpha\sigma^\alpha C_*-q^\alpha\Big), \\ \Psi_0 = \Big[\Big(-\eta^\alpha\sigma^\alpha B_*-\xi^\alpha\sigma^\alpha C_*-q^\alpha\Big)\Big(\eta^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-q^\alpha\Big)\Big(\xi^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-\gamma^\alpha-q^\alpha\Big) \\ \quad \Big(\mu^\alpha\sigma^\alpha B_*+\mu^\alpha\sigma^\alpha C_*-q^\alpha\Big)\Big]+\sigma^{\alpha 2}\mu^{\alpha 2}C_*G_*\Big(-\eta^\alpha\sigma^\alpha B_*-\xi^\alpha\sigma^\alpha C_*-q^\alpha\Big)\Big(\eta^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-q^\alpha\Big) \\ \quad -\sigma^{\alpha 2}\mu^{\alpha 2}\gamma^\alpha C_*G_*\Big(-\eta^\alpha\sigma^\alpha B_*-\xi^\alpha\sigma^\alpha C_*-q^\alpha\Big)+\sigma^{\alpha 2}\mu^{\alpha 2}B_*C_*\Big[\Big(-\eta^\alpha\sigma^\alpha B_*-\xi^\alpha\sigma^\alpha C_*-q^\alpha\Big) \\ \quad \Big(\xi^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-\gamma^\alpha-q^\alpha\Big)\Big]+\sigma^{\alpha 2}\eta^{\alpha 2}A_*B_*\Big[\Big(\xi^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-\gamma^\alpha-q^\alpha\Big) \\ \quad \Big(\mu^\alpha\sigma^\alpha B_*+\mu^\alpha\sigma^\alpha C_*-q^\alpha\Big)\Big]-\eta^\alpha\xi^\alpha\sigma^{\alpha 2}\gamma^\alpha A_*C_*\Big(\mu^\alpha\sigma^\alpha B_*+\mu^\alpha\sigma^\alpha C_*-q^\alpha\Big)+\eta^{\alpha 2}\sigma^{\alpha 2}\mu^{\alpha 2}A_*B_*C_*G_* \\ \quad +\xi^{\alpha 2}\sigma^{\alpha 2}\mu^{\alpha 2}A_*C_*\Big(\eta^\alpha\sigma^\alpha A_*-\mu^\alpha\sigma^\alpha G_*-q^\alpha\Big)\Big(\mu^\alpha\sigma^\alpha B_*+\mu^\alpha\sigma^\alpha C_*-q^\alpha\Big)+\xi^{\alpha 2}\sigma^{\alpha 2}\mu^{\alpha 2}A_*B_*C_*G_*. \end{cases}$$

This clear that the interior equilibrium point E_* of model (2) is global asymptotic stable when $R_0 < 1$.

Sensitivity analysis of the basic reproduction number R_0

In this subsection, we study the sensitivity of R_0 at all its parameters

$$\frac{\partial R_0}{\partial \eta^\alpha} = \frac{\sigma^\alpha}{q^\alpha} > 0,$$
$$\frac{\partial R_0}{\partial \sigma^\alpha} = \frac{\eta^\alpha}{q^\alpha} > 0,$$
$$\frac{\partial R_0}{\partial q^\alpha} = \frac{-\eta^\alpha\sigma^\alpha}{q^{\alpha 2}} < 0.$$

According to previous results deduce that R_0 is increasing at η^α and σ^α , decreasing at q^α.

5 Applications and results

In this section, we investigate the used methods for solving the nonlinear FOM (2) in details.

Generalized Mittag-Leffler function method

No doubt, we know that the efficiency and performance of the generalized Mittag-Leffler function method for solving the fractional-order differential equations (see [21,22,23]) for the reason will use it for solving our model (2) as an approximate solution.
Assume that

$$\begin{aligned} A &= \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \qquad B = \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ C &= \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \qquad G = \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{17}$$

by using Caputo fractional derivative we get

$$\begin{aligned} {}_0^C D_t^{\alpha} A &= \sum_{n=1}^{\infty} a^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} a^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ {}_0^C D_t^{\alpha} B &= \sum_{n=1}^{\infty} b^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} b^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ {}_0^C D_t^{\alpha} C &= \sum_{n=1}^{\infty} c^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} c^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ {}_0^C D_t^{\alpha} G &= \sum_{n=1}^{\infty} d^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} d^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{18}$$

by substituting from equations (17) and (18) in model (2) we obtain

$$\begin{cases} \sum_{n=0}^{\infty} a^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = q^{\alpha} - \eta^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - \xi^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ \qquad - q^{\alpha} \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ \sum_{n=0}^{\infty} b^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \eta^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \gamma^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ \qquad - \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - q^{\alpha} \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ \sum_{n=0}^{\infty} c^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \xi^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - \gamma^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ \qquad - \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - q^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ \sum_{n=0}^{\infty} d^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ \qquad - q^{\alpha} \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{cases} \tag{19}$$

after make some calculation the equation (19) transformed to

$$\begin{cases} \sum_{n=0}^{\infty} a^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = q^{\alpha} - \eta^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} l^n t^{n\alpha} - \xi^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} m^n t^{n\alpha} - q^{\alpha} \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ \sum_{n=0}^{\infty} b^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \eta^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} l^n t^{n\alpha} + \gamma^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} k^n t^{n\alpha} - q^{\alpha} \sum_{n=0}^{\infty} b^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ \sum_{n=0}^{\infty} c^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \xi^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} m^n t^{n\alpha} - \gamma^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} g^n t^{n\alpha} - q^{\alpha} \sum_{n=0}^{\infty} c^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ \sum_{n=0}^{\infty} d^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} k^n t^{n\alpha} + \mu^{\alpha}\sigma^{\alpha} \sum_{n=0}^{\infty} g^n t^{n\alpha} - q^{\alpha} \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{cases} \tag{20}$$

where

$$\begin{aligned} l^n &= \sum_{k=0}^{n} \frac{a^k b^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}, \qquad m^n = \sum_{k=0}^{n} \frac{a^k c^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}, \\ k^n &= \sum_{k=0}^{n} \frac{b^k d^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}, \qquad g^n = \sum_{k=0}^{n} \frac{c^k d^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}. \end{aligned}$$

Additionally, after sum the similar quantities in equation (20) we have

$$\begin{cases} \sum_{n=0}^{\infty}\left[\frac{a^{n+1}}{\Gamma(n\alpha+1)}+\eta^{\alpha}\sigma^{\alpha}l^{n}+\xi^{\alpha}\sigma^{\alpha}m^{n}+q^{\alpha}\frac{a^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=q^{\alpha}, \\ \sum_{n=0}^{\infty}\left[\frac{b^{n+1}}{\Gamma(n\alpha+1)}-\eta^{\alpha}\sigma^{\alpha}l^{n}-\gamma^{\alpha}\frac{c^{n}}{\Gamma(n\alpha+1)}+\mu^{\alpha}\sigma^{\alpha}k^{n}+q^{\alpha}\frac{b^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0, \\ \sum_{n=0}^{\infty}\left[\frac{c^{n+1}}{\Gamma(n\alpha+1)}-\xi^{\alpha}\sigma^{\alpha}m^{n}+\gamma^{\alpha}\frac{c^{n}}{\Gamma(n\alpha+1)}+\mu^{\alpha}\sigma^{\alpha}g^{n}+q^{\alpha}\frac{c^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0, \\ \sum_{n=0}^{\infty}\left[\frac{d^{n+1}}{\Gamma(n\alpha+1)}-\mu^{\alpha}\sigma^{\alpha}k^{n}-\mu^{\alpha}\sigma^{\alpha}g^{n}+q^{\alpha}\frac{d^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0. \end{cases} \tag{21}$$

By taking the first limit of the equation (21) we deduce that

$$\begin{aligned} a^{1}&=q^{\alpha}-\eta^{\alpha}\sigma^{\alpha}a^{0}b^{0}-\xi^{\alpha}\sigma^{\alpha}a^{0}c^{0}-q^{\alpha}a^{0}, \\ b^{1}&=\eta^{\alpha}\sigma^{\alpha}a^{0}b^{0}+\gamma^{\alpha}c^{0}-\mu^{\alpha}\sigma^{\alpha}b^{0}d^{0}-q^{\alpha}b^{0}, \\ c^{1}&=\xi^{\alpha}\sigma^{\alpha}a^{0}c^{0}-\gamma^{\alpha}c^{0}-\mu^{\alpha}\sigma^{\alpha}c^{0}d^{0}-q^{\alpha}c^{0}, \\ d^{1}&=\mu^{\alpha}\sigma^{\alpha}b^{0}d^{0}+\mu^{\alpha}\sigma^{\alpha}c^{0}d^{0}-q^{\alpha}d^{0}. \end{aligned} \tag{22}$$

According to equation (22), equation (21) turned into

$$\begin{cases} \sum_{n=1}^{\infty}\left[\frac{a^{n+1}}{\Gamma(n\alpha+1)}+\eta^{\alpha}\sigma^{\alpha}l^{n}+\xi^{\alpha}\sigma^{\alpha}m^{n}+q^{\alpha}\frac{a^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0, \\ \sum_{n=1}^{\infty}\left[\frac{b^{n+1}}{\Gamma(n\alpha+1)}-\eta^{\alpha}\sigma^{\alpha}l^{n}-\gamma^{\alpha}\frac{c^{n}}{\Gamma(n\alpha+1)}+\mu^{\alpha}\sigma^{\alpha}k^{n}+q^{\alpha}\frac{b^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0, \\ \sum_{n=1}^{\infty}\left[\frac{c^{n+1}}{\Gamma(n\alpha+1)}-\xi^{\alpha}\sigma^{\alpha}m^{n}+\gamma^{\alpha}\frac{c^{n}}{\Gamma(n\alpha+1)}+\mu^{\alpha}\sigma^{\alpha}g^{n}+q^{\alpha}\frac{c^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0, \\ \sum_{n=1}^{\infty}\left[\frac{d^{n+1}}{\Gamma(n\alpha+1)}-\mu^{\alpha}\sigma^{\alpha}k^{n}-\mu^{\alpha}\sigma^{\alpha}g^{n}+q^{\alpha}\frac{d^{n}}{\Gamma(n\alpha+1)}\right]t^{n\alpha}=0. \end{cases} \tag{23}$$

In equation (23) $t^{n\alpha}$ is not equal zero then the rest of coefficients equal to zero therefore we deduce that the following relations that are called recurrence relations and we can get the values of constants a^n, b^n, c^n, d^n, where $n = 1,2,3,4,\cdots\infty$.

$$\begin{aligned} a^{n+1}&=-\eta^{\alpha}\sigma^{\alpha}l^{n}\Gamma(n\alpha+1)-\xi^{\alpha}\sigma^{\alpha}m^{n}\Gamma(n\alpha+1)-q^{\alpha}a^{n}, \\ b^{n+1}&=\eta^{\alpha}\sigma^{\alpha}l^{n}\Gamma(n\alpha+1)+\gamma^{\alpha}c^{n}-\mu^{\alpha}\sigma^{\alpha}k^{n}\Gamma(n\alpha+1)-q^{\alpha}b^{n}, \\ c^{n+1}&=\xi^{\alpha}\sigma^{\alpha}m^{n}\Gamma(n\alpha+1)-\gamma^{\alpha}c^{n}-\mu^{\alpha}\sigma^{\alpha}g^{n}\Gamma(n\alpha+1)-q^{\alpha}c^{n}, \\ d^{n+1}&=\mu^{\alpha}\sigma^{\alpha}k^{n}\Gamma(n\alpha+1)+\mu^{\alpha}\sigma^{\alpha}g^{n}\Gamma(n\alpha+1)-q^{\alpha}d^{n}. \end{aligned} \tag{24}$$

At $n = 1$, we obtain

$$\begin{aligned} a^{2}&=(-\eta^{\alpha}\sigma^{\alpha}b^{0}-\xi^{\alpha}\sigma^{\alpha}c^{0}-q^{\alpha})a^{1}-\eta^{\alpha}\sigma^{\alpha}a^{0}b^{1}-\xi^{\alpha}\sigma^{\alpha}a^{0}c^{1}, \\ b^{2}&=\eta^{\alpha}\sigma^{\alpha}b^{0}a^{1}+(\eta^{\alpha}\sigma^{\alpha}a^{0}-\mu_{1}{}^{\alpha}\sigma^{\alpha}d^{0}-q^{\alpha})b^{1}+\gamma^{\alpha}c^{1}-\mu^{\alpha}\sigma^{\alpha}b^{0}d^{1}, \\ c^{2}&=\xi^{\alpha}\sigma^{\alpha}c^{0}a^{1}+(\xi^{\alpha}\sigma^{\alpha}a^{0}-\gamma^{\alpha}-\mu^{\alpha}\sigma^{\alpha}d^{0}-q^{\alpha})c^{1}-\mu_{2}{}^{\alpha}\sigma^{\alpha}c^{0}d^{1}, \\ d^{2}&=\mu^{\alpha}\sigma^{\alpha}d^{0}b^{1}+\mu^{\alpha}\sigma^{\alpha}d^{0}c^{1}+(\mu^{\alpha}\sigma^{\alpha}b^{0}+\mu^{\alpha}\sigma^{\alpha}c^{0}-q^{\alpha})d^{1}. \end{aligned} \tag{25}$$

Also when $n=2$, we get

$$
\begin{aligned}
a^3 &= -(\eta^\alpha\sigma^\alpha b^0+\xi^\alpha\sigma^\alpha c^0+q^\alpha)a^2-\eta^\alpha\sigma^\alpha a^0 b^2-\xi^\alpha\sigma^\alpha a^0 c^2-\frac{(\eta^\alpha\sigma^\alpha a^1 b^1+\xi^\alpha\sigma^\alpha a^1 c^1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2},\\
b^3 &= \eta^\alpha\sigma^\alpha b^0 a^2+(\eta^\alpha\sigma^\alpha a^0-\mu^\alpha\sigma^\alpha d^0-q^\alpha)b^2+\gamma^\alpha c^2-\mu^\alpha\sigma^\alpha b^0 d^2+\frac{(\eta^\alpha\sigma^\alpha a^1 b^1-\mu^\alpha\sigma^\alpha b^1 d^1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2},\\
c^3 &= \xi^\alpha\sigma^\alpha c^0 a^2+(\xi^\alpha\sigma^\alpha a^0-\mu^\alpha\sigma^\alpha d^0-\gamma^\alpha-q^\alpha)c^2-\mu^\alpha\sigma^\alpha c^0 d^2+\frac{(\xi^\alpha\sigma^\alpha a^1 c^1-\mu^\alpha\sigma^\alpha c^1 d^1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2},\\
d^3 &= \mu^\alpha\sigma^\alpha d^0 b^2+\mu^\alpha\sigma^\alpha d^0 c^2+(\mu^\alpha\sigma^\alpha b^0+\mu^\alpha\sigma^\alpha c^0-q^\alpha)d^2+\frac{(\mu^\alpha\sigma^\alpha b^1 d^1+\mu^\alpha\sigma^\alpha c^1 d^1)\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}.
\end{aligned}
\tag{26}
$$

In additionally to at $n=3$ we obtain the following relations as follows

$$
\begin{aligned}
a^4 &= -(\eta^\alpha\sigma^\alpha b^0+\xi^\alpha\sigma^\alpha c^0+q^\alpha)a^3-\eta^\alpha\sigma^\alpha a^0 b^3-\xi^\alpha\sigma^\alpha a^0 c^3\\
&\quad -\frac{\Big(\eta^\alpha\sigma^\alpha(a^1b^2+a^2b^1)+\xi^\alpha\sigma^\alpha(a^1c^2+a^2c^1)\Big)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)},\\
b^4 &= \eta^\alpha\sigma^\alpha b^0 a^3+(\eta^\alpha\sigma^\alpha a^0-\mu^\alpha\sigma^\alpha d^0-q^\alpha)b^3+\gamma^\alpha c^3-\mu^\alpha\sigma^\alpha b^0 d^3\\
&\quad +\frac{\Big(\eta^\alpha\sigma^\alpha(a^1b^2+a^2b^1)-\mu^\alpha\sigma^\alpha(b^1d^2+b^2d^1)\Big)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)},\\
c^4 &= \xi^\alpha\sigma^\alpha c^0 a^3+(\xi^\alpha\sigma^\alpha a^0-\mu^\alpha\sigma^\alpha d^0-\gamma^\alpha-q^\alpha)c^3-\mu^\alpha\sigma^\alpha c^0 d^2\\
&\quad +\frac{\Big(\xi^\alpha\sigma^\alpha(a^1c^2+a^2c^1)-\mu^\alpha\sigma^\alpha(c^1d^2+c^2d^1)\Big)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)},\\
d^4 &= \mu^\alpha\sigma^\alpha d^0 b^3+\mu^\alpha\sigma^\alpha d^0 c^3+(\mu^\alpha\sigma^\alpha b^0+\mu^\alpha\sigma^\alpha c^0-q^\alpha)d^3\\
&\quad +\frac{\Big(\mu^\alpha\sigma^\alpha(b^1d^2+b^2d^1)+\mu^\alpha\sigma^\alpha(c^1d^2+c^2d^1)\Big)\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)},
\end{aligned}
\tag{27}
$$

similarly by using the recurrence relations (24) and substituting the value of $n=4,5,6,7,\cdots\infty$ we can obtain the another values of required constants $a^5,b^5,c^5,d^5,a^6,b^6,c^6,d^6,\cdots$.
By substituting from equations (25), (26) and (27) in (17) we get the following relations in the infinite series form as follows:

$$
\begin{aligned}
A &= a^0+a^1\frac{t^\alpha}{\Gamma(\alpha+1)}+a^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+a^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+a^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+a^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}+\cdots,\\
B &= b^0+b^1\frac{t^\alpha}{\Gamma(\alpha+1)}+b^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+b^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+b^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+b^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}+\cdots,\\
C &= c^0+c^1\frac{t^\alpha}{\Gamma(\alpha+1)}+c^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+c^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+c^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+c^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}+\cdots,\\
G &= d^0+d^1\frac{t^\alpha}{\Gamma(\alpha+1)}+d^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+d^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+d^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+d^5\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}+\cdots.
\end{aligned}
$$

Predictor-corrector method

Here, we illustrate our method is called the Predictor-Corrector method (PCM) [48] for solving the fractional-order rumor model (2) to confirm our theoretical results. We study the approximate solution of FOM by given this algorithm in the following approach.
The considered differential equation

$$D_t^{\alpha} y(t) = f(t, y(t)), \qquad 0 \leq t \leq T, \tag{28}$$
$$y^{(k)}(0) = y_0^{(k)}, \qquad k = 0, 1, 2, \cdots [\alpha] - 1,$$

is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{[\alpha]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(y(\tau)) d\tau. \tag{29}$$

Let $h = \frac{T}{N}$, $tn = nh$, and $n = 0, 1, \cdots, N \in Z^+$. Then (29) transformed to

$$y_n(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(y_n^P(t_{n+1})) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(y_n(t_j)), \tag{30}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & if\, j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & if 0 \leq j \leq n, \\ 1, & if\, j = n+1. \end{cases} \tag{31}$$

Also, the predicted value $y_n^P(t_{n+1})$ is given by

$$y_n^P(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(y_n(t_j)), \tag{32}$$

where

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n-j+1)^{\alpha} - (n-j)^{\alpha}). \tag{33}$$

This is clear that the Predictor-Corrector method is an approximation for the fractional-order integration. By applying the above method, system (2) turned into

$$\begin{aligned}
A_{n+1} &= A_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left[q^{\alpha} - \eta^{\alpha}\sigma^{\alpha} A_{n+1}^P B_{n+1}^P - \xi^{\alpha}\sigma^{\alpha} A_{n+1}^P C_{n+1}^P - q^{\alpha} A_{n+1}^P \right] \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} \left[q^{\alpha} - \eta^{\alpha}\sigma^{\alpha} A_j B_j - \xi^{\alpha}\sigma^{\alpha} A_j C_j - q^{\alpha} A_j \right], \\
B_{n+1} &= B_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left[\eta^{\alpha}\sigma^{\alpha} A_{n+1}^P B_{n+1}^P + \gamma^{\alpha} C_{n+1}^P - \mu^{\alpha}\sigma^{\alpha} B_{n+1}^P G_{n+1}^P - q^{\alpha} B_{n+1}^P \right] \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} \left[\eta^{\alpha}\sigma^{\alpha} A_j B_j + \gamma^{\alpha} C_j - \mu^{\alpha}\sigma^{\alpha} B_j G_j - q^{\alpha} B_j \right], \\
C_{n+1} &= C_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left[\xi^{\alpha}\sigma^{\alpha} A_{n+1}^P C_{n+1}^P - \gamma^{\alpha} C_{n+1}^P - \mu^{\alpha}\sigma^{\alpha} C_{n+1}^P G_{n+1}^P - q^{\alpha} C_{n+1}^P \right] \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} \left[\xi^{\alpha}\sigma^{\alpha} A_j C_j - \gamma^{\alpha} C_j - \mu^{\alpha}\sigma^{\alpha} C_j G_j - q^{\alpha} C_j \right], \\
G_{n+1} &= G_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left[\mu^{\alpha}\sigma^{\alpha} B_{n+1}^P G_{n+1}^P + \mu^{\alpha}\sigma^{\alpha} C_{n+1}^P G_{n+1}^P - q^{\alpha} G_{n+1}^P \right] \\
&\quad + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} \left[\mu^{\alpha}\sigma^{\alpha} B_j G_j + \mu^{\alpha}\sigma^{\alpha} C_j G_j - q^{\alpha} G_j \right].
\end{aligned} \tag{34}$$

6 Numerical simulations

In this section, we introduce many numerical simulations to confirm the analysis results. In system (2), we take the following parameters of all simulations as follows:
$q = 0.1,\ \eta = 0.03,\ \xi = 0.001,\ \gamma = 0.8,\ \sigma = 10,\ \mu = 0.001,\ \alpha = 1$, with the initial condition equal to the values $(1, 0.000001, 0, 0)$ and $t = 15weeks$, where t is denote the time of rumor spreading.
Fig. 1 illustrate the density of ignorant A start the peak and descends until reaches to the balance, B start the origin point and increase even reaches to the balance but both C and G are stable.
Figs. 2 and 3 describes the changing effect in the value of α about the four densities. Figs. 4 and 5 depicts the influence of the probability of the ignorants who become the low rate of active spreader η on the individuals who never heard the rumors A and the individuals who spread rumors with low probability B.

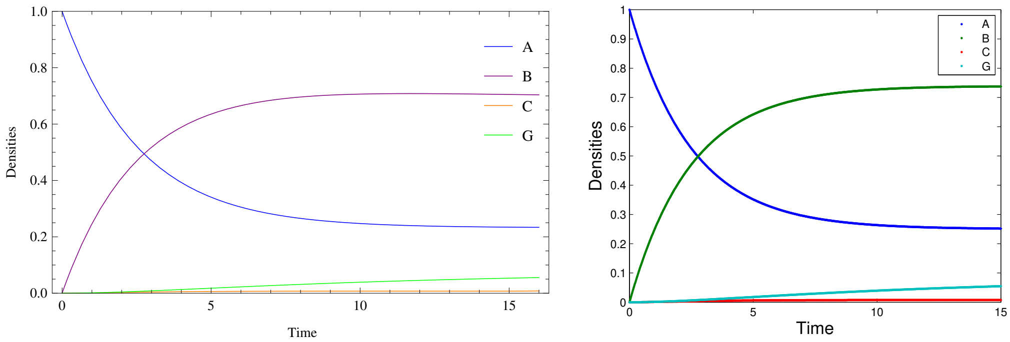

Fig. 1: The density of four groups over time at $\alpha = 1$ by using 1(a) GMLFM and 2(a) PCM.

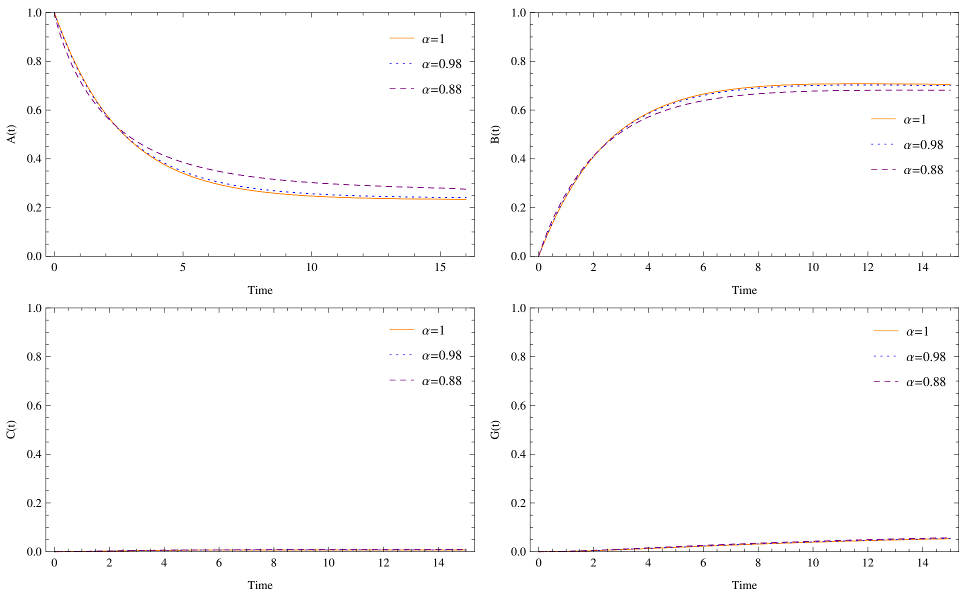

Fig. 2: Simulation result of model (2) using previous parameter values at different densities A, B, C, G for different values of $\alpha = 1, 0.98, 0.88$ and t by using GMLFM.

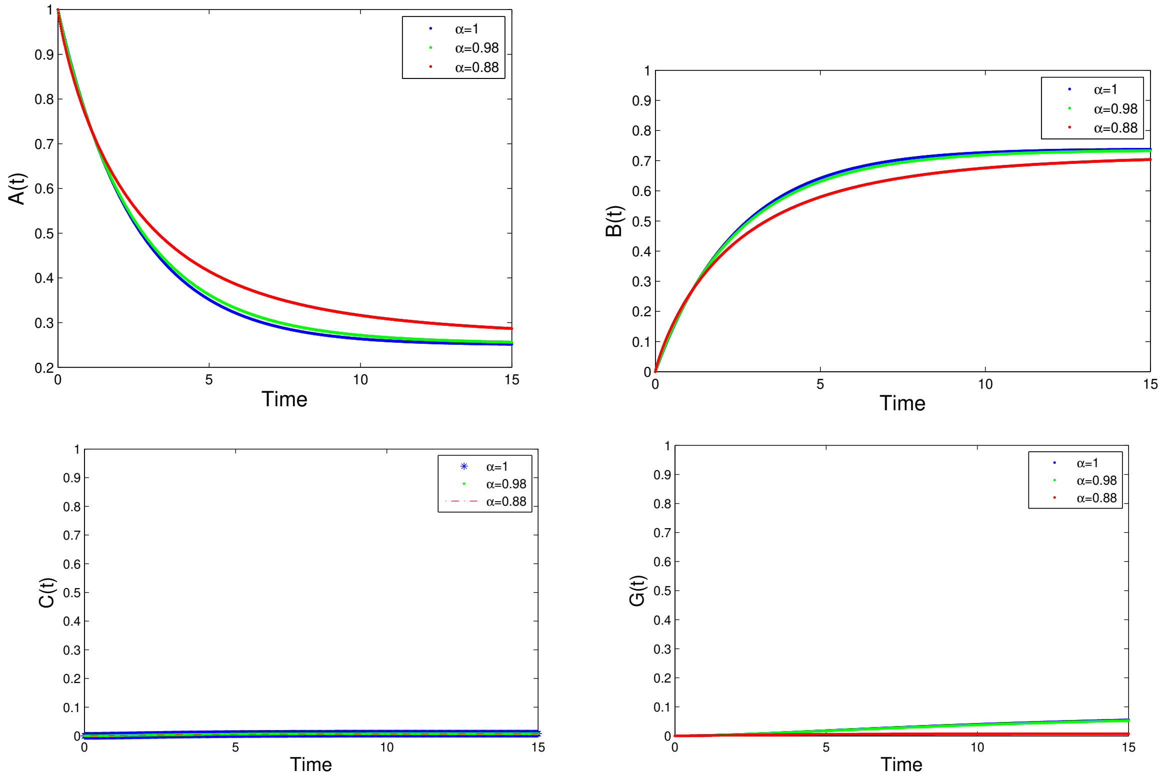

Fig. 3: Simulation result of model (2) using previous parameter values at different densities A,B,C,G for different values of $\alpha = 1, 0.98, 0.88$ and t by using PCM.

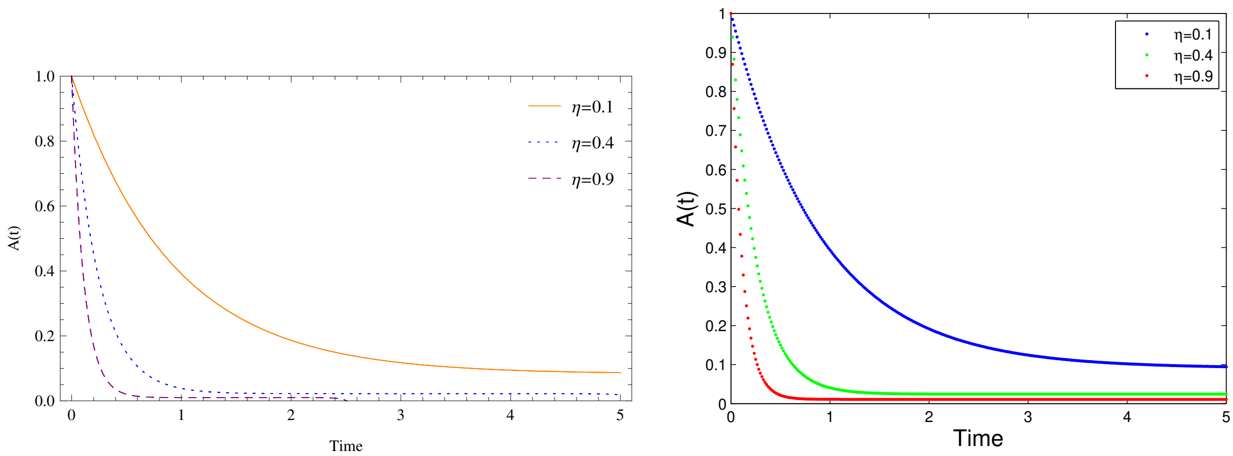

Fig. 4: Simulation result of model (2) using previous parameter values at the ignorant individuals A for different values of $\eta = 0.1, 0.4, 0.9$ and $t = 5weeks$ by using GMLFM and PCM.

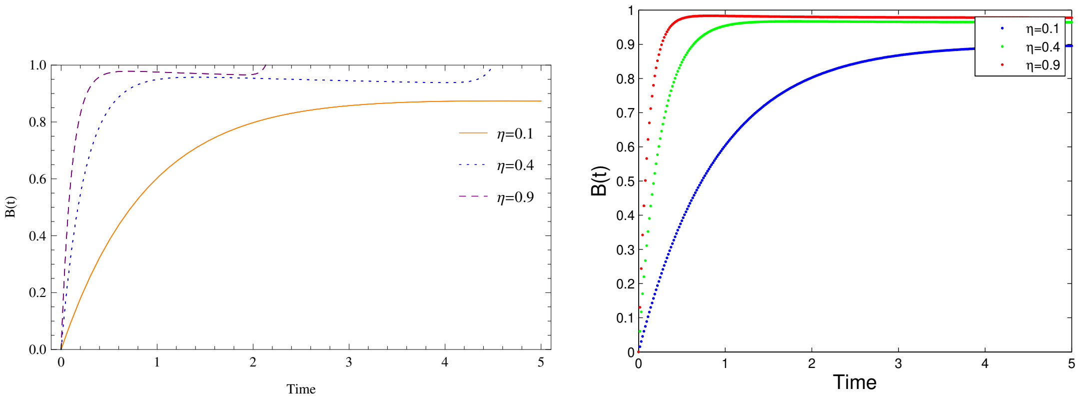

Fig. 5: Simulation result of model (2) using previous parameter values at the density B for different values of $\eta = 0.1, 0.4, 0.9$ and t by using GMLFM and PCM.

7 Conclusion

In this paper, we investigate the dynamics of the rumor spreading model (2). Also, we have analyzed all equilibrium points of the system in details. Furthermore, we have concluded that the importance of the basic reproduction number and its impact on the stability. In addition to, we have explained the fractional-order rumor spreading model by using an approximate method (Generalized Mittag-Leffler Function) and the numerical method (Predictor-Corrector Method). In addition to, we have deduced that the effect some parameters on the density of four groups such that the different values of α, the transmission probability η from ignorants to the low rate of active spreaders. Finally, The various numerical simulations support our obtained theoretical results. These results show that the fractional-order rumor spreading model (2) have rich dynamics.

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