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Standard Map Spatial Dynamics in a Ring-Phase Conjugated Resonator

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Abstract: The spatial dynamics of an optical ring phase-conjugated resonator with Standard chaos is presented. It is shown that this behavior takes place when a specific chaos-generating element is introduced on the resonator. Assuming ray optics inside the cavity with parameters p and θ for the effective distance to the optical axis and the angle to the same axis (*beam's divergence*) respectively. The matrix of a standard map generating device is found in terms of the specific map parameters, the state variables and the resonator parameters. One interesting feature of these kind of systems is that allow to model different bigger and uncontrollable systems, e.g. ocean dynamics, weather, social and economical systems, among others; This feature is possible thanks to the parameters control facility.

Keywords: Spatial chaos, optical mapping, laser resonator, dynamical systems

1 Introduction

The standard map is the name of a specific two-dimensional dynamic map that is useful for studying the basic features of chaotic motion and it has been played an important role in classical and quantum chaos. This particular map can be derived directly from a periodically kicked system (*the kicked rotor*) [1,2] The map describes a situation when nonlinear resonances are equidistant in phase space that corresponds to a local description of dynamical chaos. Due to this property various dynamical systems and maps can be locally reduced to the standard map. Thus, this map describes a universal behavior with divided phase space when integrable islands of stability are surrounded by a chaotic component. As in most dynamical maps, the effect of the map represents the evolution of a real physical system in time [2]. The physical system represented by the standard map is the so-called kicked rotor. A rotor can be imagined as a bead sliding in a frictionless manner around on a circular wire. The kicked part comes from the fact that the rotor is subjected to a driving force that comes at exactly even intervals in time. In physical terms, the kick might be delivered by an electromagnetic pulse, for example. To make things as simple as possible, the driving force (*that is, the kick*) always arrives from the same direction in space, and with exactly the same intensity. The driving force either speeds up or slows down the bead, depending on where the bead happens to be at the moment the kick arrives. In other words, even though the intensity of the kick is always the same, and it always comes from the same direction in space, the amount of the kick that is absorbed by the bead in the form of a change of speed depends on the position of the bead around the circle at that instant. The relationship is expressed mathematically through the map equations [1]. The most interesting aspect of the standard map is not simply that it shows chaos, but that it shows the most basic conditions under which chaos can occur [3,4]. One of the most interesting features of this kind of systems is that they can be applied to optical resonators and, by these means; the system can be easily studied due to the simply way that the parameters can be changed in an optical cavity [5,6,7,8,9]. This paper is organized as follows. After this introduction, Section 2 presents some basic features of the standard map when it is treated by the ABCD formalism, Section 3 presents the theoretical problem and shows the main characteristics of the map generation matrix and the standard beams (*i.e. beams that behave accordingly to a standard map*), Section 4 presents the numerical results

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and the discussion on the observed spatial dynamics at the resonator's output and Section 5 presents the conclusions.

2 The standard map as an optical mapping

The standard map is a non-linear, two-dimensional map that obeys the following two difference equations:

$$p_{n+1} = p_n + k\sin(\theta_n), \qquad (1)$$
$$\theta_{n+1} = \theta_n + p_{n+1},$$

where p_n and θ_n are taken *modulo*2π. The variables p_n and θ_n determine the angular position of the stick and its angular momentum after the n-th kick respectively. The constant k measures the intensity of the kicks on the kicked rotator [1]. In the proposed ring phase-conjugated resonator p_n and θ_n means the vertical distance to the optical axis and the angle to the same axis respectively. Using matrix optics, the following relation can then be used for calculating how these parameters are modified by an optical element where the iterated quantities refer to the beam after passing the optical component. From the following general expression [10] holds:

$$\begin{bmatrix} p_{n+1} \\ \theta_{n+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} p_n \\ \theta_n \end{bmatrix} \qquad (2)$$

Substituting Eq. (1) into Eq. (2) the obtained *ABCD* standard map matrix system after some algebra becomes:

$$\begin{bmatrix} p_{n+1} \\ \theta_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{k\sin(\theta_n)}{\theta_n} \\ 1 & \frac{k\sin(\theta_n)}{\theta_n} + 1 \end{bmatrix} \times \begin{bmatrix} p_n \\ \theta_n \end{bmatrix} \qquad (3)$$

This system will be utilized for the generation of standard beams in the setup proposed in Fig. 1.

The ring phase conjugated resonator setup shown is composed by two ideal plain mirrors (M), by a phase-conjugated mirror (PM), as well as by a chaos-generating element separated by a distance $d/2$ between the two mirrors [11, 12, 13, 14].

The matrices involved and used to describe the propagation of a light beam in the optical resonator are: for M $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, for PM $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, for a translational distance d $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$, and, finally, for an unknown Standard chaos generating element matrix $\begin{bmatrix} a & b \\ c & e \end{bmatrix}$ [11]. Therefore, the total transformation matrix *ABCD* for a complete round trip is calculated as follows [12]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & e \end{bmatrix} \cdots \qquad (4)$$
$$\cdots \begin{bmatrix} 1 & d/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

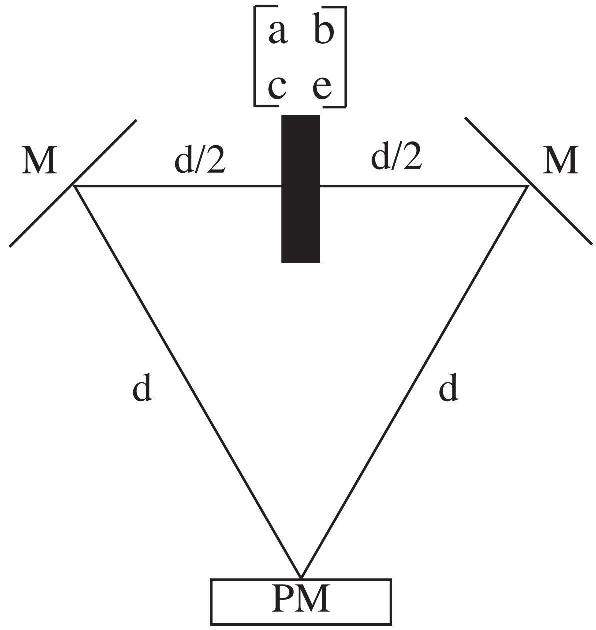

Fig. 1: Ring phase-conjugated laser resonator setup with an intracavity chaos-generating element

Solving Eq. (4) the resulting matrix is:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a + \frac{3cd}{2} & b + \frac{3d}{4}(2a + 3cd + 2e) \\ -c & -\frac{3cd}{2} - e \end{bmatrix} \qquad (5)$$

If one does want the standard map dynamics reproduced by a ray in a ring laser cavity, then, each round trip a ray described by (p_n, θ_n) undergoes in the laser cavity has to be considered as an iteration of the standard map equations [13]. Then, when the matrix transformation, Eq. (2), and the coupled equations, Eq. (1), that generates the standard map, are both taken into consideration one obtains by inspection the following equalities:

$$A = 1, \qquad (6)$$
$$B = \frac{k\sin(\theta_n)}{\theta_n}, \qquad (7)$$
$$C = 1, \qquad (8)$$
$$D = \frac{k\sin(\theta_n)}{\theta_n} + 1. \qquad (9)$$

Then, the *abce* matrix of the standard chaos generating element able to produce standard beams consistent with the standard map are found by substituting the *ABCD* elements of Eq. (5) in the above equalities (6) to (9), obtaining:

$$a + \frac{3cd}{2} = 1, \tag{10}$$

$$b + \frac{3d}{4}(2a + 3cd + 2e) = \frac{k\sin(\theta_n)}{\theta_n}, \tag{11}$$

$$c = -1, \tag{12}$$

$$-\frac{3cd}{2} - e = \frac{k\sin(\theta_n)}{\theta_n} + 1. \tag{13}$$

The solutions for the standard chaos matrix elements *abce*, able to produce standard beams in terms of the standard map are the following:

$$a = 1 + \frac{3d}{2}, \tag{14}$$

$$b = \frac{k\sin(\theta_n)}{\theta_n} + d\left[\frac{3k\sin(\theta_n)}{2\theta_n} - \frac{9d}{4}\right], \tag{15}$$

$$c = -1, \tag{16}$$

$$e = \frac{3d}{2} - \frac{k\sin(\theta_n)}{\theta_n} - 1. \tag{17}$$

Taking the solutions (14) to (17) for the *abce* matrix elements into Eq. (4) the total transformation *ABCD* matrix for a round trip now has the elements given by (6) to (9). Substituting in Eq. (2), we obtain the equations for p_{n+1} and θ_{n+1} as the standard map, Eq. (1), making the problem reversible. As can be seen the matrix elements of the chaos generating matrix depend on the standard map parameter k, on the resonator main parameter d and on the state variable θ_n.

It has to be noted that the used approximation is only valid for an element thickness that tends to zero.

3 Standard beams: general case

The results given by equations (14) – (17) are only valid for $b \approx 0$. For a general case, the thickness of the chaos-generating element has to be taken into account, so, Eq. (4) becomes:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{d-b}{2} \\ 0 & 1 \end{bmatrix}\cdots \tag{18}$$

$$\cdots\begin{bmatrix} a & b \\ c & e \end{bmatrix}\begin{bmatrix} 1 & \frac{d-b}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}.$$

Therefore the round trip total transformation matrix yields:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a - \frac{c}{2}(b - 3d) & \frac{1}{4}\left[2\alpha b + 3\beta d + b^2 c\right] \\ -c & \frac{1}{2}(bc - 3cd - 2e) \end{bmatrix}, \tag{19}$$

where $\alpha = (2 - a - 3cd - e)$ and $\beta = (2a + 3cd + 2e)$.

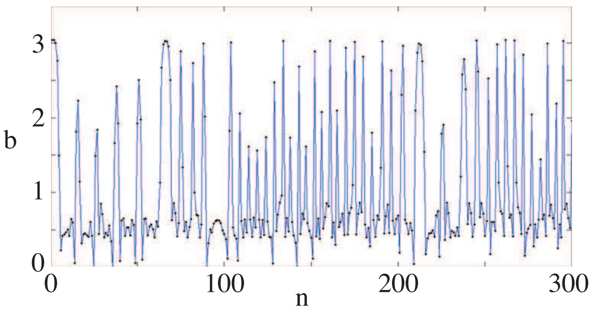

Fig. 2: Estimation of the magnitude of matrix element b of the standard map generating device for a resonator with $d = 1$ and a standard parameter $k = 1$ for the first 300 round trips.

Using Eq. (18) and Eqs. (6)–(9) the general standard map chaos-generating matrix elements, *abce*, can be calculated through the following expressions:

$$a = 2 + \frac{1}{2}\left(\frac{k\sin(\theta_n)}{\theta_n} \pm \gamma\right), \tag{20}$$

$$b = \frac{k\sin(\theta_n)}{\theta_n} - 3d + 2 \pm \gamma, \tag{21}$$

$$c = -1, \tag{22}$$

$$e = \frac{1}{2}\left(2 - \frac{k\sin(\theta_n)}{\theta_n} \pm \gamma\right), \tag{23}$$

where $\gamma = \sqrt{\frac{k\sin(\theta_n)}{\theta_n}\left(\frac{k\sin(\theta_n)}{4\theta_n} + 2\right) - 3d + 1}$.

Eqs. (20)–(23) are written taking into account the thickness of the chaos-generating element by mean of the translational distance between the element and the mirrors, this distance is represented by $(d - b)/2$ in the proposed system.

4 Numerical results

As it has been shown in the previous section, only the c matrix element is constant, being a, b and e dependent on the state variable θ_n, on the standard map parameter k and on the resonator main parameter d. It has to be noted that all parameters are dimensionless.

Figure 2 shows the simulation results for the first 300 round trips of matrix element b of the standard map generating device for a resonator of unitary length ($d = 1$) and map parameter $k = 1$, these parameters were found using brute force calculations and they were selected due to the matrix-element b behavior. In a similar way, Fig. 3 shows the simulation results for the first 300 round trips of matrix element b of the standard map generating device for a resonator of length ($d = 0.5$) and a map parameter $k = 2$. It should be noted, that several

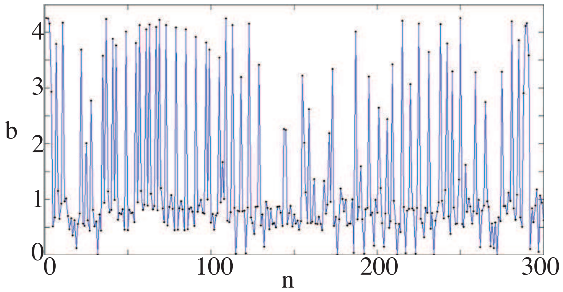

Fig. 3: Numerical results of the magnitude of matrix element b of the standard map generating device for a resonator with $d = 0.5$ and a standard parameter $k = 2$ for the first 300 round trips.

simulations were performed by varying the map parameter k and the state variables obtaining qualitatively the same complex behavior.

5 Conclusions

In this article it is shown how the introduction of a particular map-generating device in a ring optical phase-conjugated resonator can generate beams with the behavior of a specific two-dimensional map. In particular it is explicitly shown how the difference equations of standard map can be used to describe the spatial dynamical behavior of what it is called standard beams i.e. beams that behave accordingly to the standard map dynamics. The matrix elements a, b, c and e of a map-generating device are found in terms of the standard map variable k, the state variable θ_n and the resonator parameter d. To our knowledge this is the first time that the mathematical characteristics of an intracavity optical device inside a resonator are stated so that the standard map dynamics is obtained as the spatial dynamics for the output ray beams.

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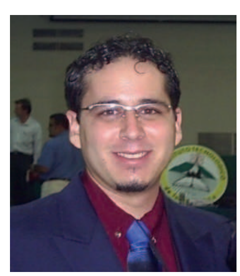

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