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The Duality of the Formal Local Homology Together With The Edge Ideal of a Graph

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Abstract: In this paper, is discuted about the formal local homology module and the formal local cohomology module, this last which is the dual of the formal local homology module. For the definition of this two modules are used the definitions of the local cohomology module with respect to an ideal and of the local homology module with respect to an ideal, where this modules are mencioned in the text. Also are put some connection between the formal local cohomology module and the formal local homology module, using some results of the theory of commutative algebra. Moreover, we involve the theory of graphs within of such modules achieving some applications for the edge ideal of a graph.

Keywords: inverse limit, direct limit, formal local cohomology, duality, edge ideal of a graph

1 Introduction

Throughout this paper, R is a commutative ring with non-zero identity. The local cohomology theory of Grothendieck has proved to be an important tool in commutative algebra. The theory of local cohomology if has developed so much six decades after its introduction by Grothendieck. Its theory dual of local homology is also studied by many mathematicians. The theory of local homology was initiated by Matlis [12], in 1974. The study of this theory was continued in Simon [21] and [22]. More after, Greenlees [9] and Alonso Tarrío, Jeremías López and Lipman [2] has initiated a new era in the study of local homology. Moreover, it is worth noting at this moment that there are already articles with relation to generalized local homology and generalized local cohomology modules, according to [15] and [13].

In [18], P. Schenzel has introduced the concept of formal local cohomology module and the i-th I-formal local cohomology module of an R-module M with respect to $\mathfrak{m}$ can be defined by

$$\mathfrak{F}_I^i(M) = \varprojlim_{t\in\mathbb{N}} \mathrm{H}_{\mathfrak{m}}^i\left(M/I^tM\right),$$

where I is an ideal of the local ring $(R,\mathfrak{m})$ and the module M is arbitrary.

Tran Tuan Nam in [14], has introduced the concept of formal local homology with the objective to do the dual to P. Schenzel's concept of formal local cohomology. The notion of the formal local homology module brings the the possibility of solving problems about the structure of this module, such as the module to be Noetherian, Artinian, finitely generated, between others; moreover, as is the dual of the formal local cohomology provides intuitions of duality of this last module. The i-th I-formal local homology module $\mathfrak{F}_{i,J}^I(M)$ of an R-module M with respect to J is defined by

$$\mathfrak{F}_{i,J}^I(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{H}_i^J\left(\left(0 :_M I^t\right)\right).$$

In the case of $J = \mathfrak{m}$, we set $\mathfrak{F}_{i,\mathfrak{m}}^I(M) = \mathfrak{F}_i^I(M)$ and speak simply about the i-th I-formal local homology module.

In [24] we have some results about this R-module of formal local cohomology, such as vanishing results and an application which consist in to prove that the i-th I-formal local cohomology module belongs to a Serre subcategory $\mathfrak{S}$ of the category of R-modules. The importance of the results in [24] consists in the fact that they provide relationships between $\mathfrak{F}_I^i(M)$ and others algebraic structures.

In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involving the edge ideal of a graph G;

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associated to the graph G is a monomial ideal

$$I(G) = (v_i v_j \mid v_i v_j \text{ is an edge of } G), \text{ with } i \neq j,$$

in the polynomial ring $R = K[v_1, v_2, \ldots, v_s]$ over a field K, called the **edge ideal** of G. The preliminaries of the theory of graphs were introduced in this Section 2 together with the concepts suitable for the work.

In the Section 3, we presented some results for the formal local cohomology module, and for this we had to do some results about the formal local homology module.

In the Section 4, we put some results involving the Noetherian dimension of an R-module. The Noetherian dimension can be understood as the dual of the Krull dimension.

Throughout of the paper, we mean by a graph G, a finite simple graph with the vertex set $V(G)$ and with no isolated vertices. Here we use properties of commutative algebra and homological algebra for the development of the results (see [3] and [17]).

2 Some prerequisites and preliminaries of the graphs theory

Let I be an ideal of R, and let M be an R-module. In [4], the i-th local cohomology module $\mathrm{H}^i_I(M)$ of M with respect to I is defined by

$$\mathrm{H}^i_I(M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R\left(R/I^t, M\right),$$

for all $0 \leq i \in \mathbb{Z}$. By [5, Remark 3.5.3(a)], we have $\mathrm{H}^0_I(M) \cong \Gamma_I(M)$, where we have that

$$\Gamma_I(M) := \left\{m \in M \mid I^t m = 0 \text{ for some } t \in \mathbb{N}\right\},$$

is an R-submodule of the R-module M. Moreover, also we have that

$$\left(0 :_M I^t\right) = \left\{m \in M \mid I^t m = 0, \text{ for some } t \in \mathbb{N}\right\},$$

is an R-submodule of the R-module M. It is clear that also we have the definition

$$(0 :_M I) = \{m \in M \mid Im = 0\}.$$

We can also see the definition of local cohomology module of the following form.

Definition 1.([5, Definition 3.5.2]) The local cohomology functors, denoted by $\mathrm{H}^i_I(\bullet)$, are the right derived functors of $\Gamma_I(\bullet)$. In other words, if $\mathbf{I}^\bullet$ is an injective resolution of the R-module M, then $\mathrm{H}^i_I(M) \cong \mathrm{H}^i(\Gamma_I(\mathbf{I}^\bullet_M))$ for all $i \geq 0$, where $\mathbf{I}^\bullet_M$ denotes the deleted injective resolution of M.

We present now the definition of the formal local cohomology module, the object principal of the study of the paper.

Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring. For $\mathfrak{a}$ other ideal of R, we consider the family of local cohomology modules given as it follows by $\left\{\mathrm{H}^i_\mathfrak{m}\left(\frac{M}{\mathfrak{a}^n M}\right)\right\}_{n\in\mathbb{N}}$, for all $i \geq 0$, with M an arbitrary R-module. According to [18], for every $n \in \mathbb{N}$, there exists a natural homomorphism

$$\phi_{n+1,n} : \mathrm{H}^i_\mathfrak{m}\left(\frac{M}{\mathfrak{a}^{n+1}M}\right) \to \mathrm{H}^i_\mathfrak{m}\left(\frac{M}{\mathfrak{a}^n M}\right).$$

These families form an inverse system. Their inverse limit that is given of the following manner by

$$\varprojlim_{n\in\mathbb{N}} \mathrm{H}^i_\mathfrak{m}\left(\frac{M}{\mathfrak{a}^n M}\right),$$

is called, according to [18], the i-th formal local cohomology module of M with respect to $\mathfrak{m}$, and will be denoted by $\mathfrak{F}^i_\mathfrak{a}(M)$.

Now, the definition of local cohomology module suggests the following definition.

Definition 2.([7, Definition 3.1]) Let I be an ideal of R and let M be an R-module. The i-th local homology module $\mathrm{H}^I_i(M)$ of M with respect to I is defined by

$$\mathrm{H}^I_i(M) = \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}^R_i\left(R/I^t, M\right),$$

for all $i \geq 0$.

When $i = 0$, we have that

$$\mathrm{H}^I_0(M) \cong \varprojlim_{t\in\mathbb{N}} \frac{M}{I^t M} = \Lambda_I(M),$$

is the I-adic completion of M. This suggests the following definition, according to [14].

Let I, J be two ideals of R. The i-th I-formal local homology module $\mathfrak{F}^I_{i,J}(M)$ of an R-module M with respect to J is defined by

$$\mathfrak{F}^I_{i,J}(M) := \varinjlim_{t\in\mathbb{N}} \mathrm{H}^J_i\left(\left(0 :_M I^t\right)\right).$$

In the case of $J = \mathfrak{m}$ we set $\mathfrak{F}^I_{i,\mathfrak{m}}(M) = \mathfrak{F}^I_i(M)$ and speak simply about the i-th I-formal local homology module.

We recall also the concept of Matlis dual, which can be found in [4]. Let M be an R-module and $\mathrm{E}(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. The module $\mathrm{D}(M) = \mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{m}))$ is called the Matlis dual of M. We observe that we have

$$\mathrm{D}\left(\mathfrak{F}^I_i(M)\right) = \mathfrak{F}^i_I(M) \text{ and } \mathrm{D}\left(\mathfrak{F}^i_I(M)\right) = \mathfrak{F}^I_i(M), \text{ for all } i \geq 0.$$

The concept of Hausdorff linearly topologized, semi-discrete and linearly compact module, according to [11, 3.1], which will be used in the next sections, is the following: a Hausdorff linearly topologized R-module M is said to be *linearly compact* if $\mathfrak{A}$ is a family of closed cosets, i.e., cosets of closed submodules, in M which has the finite intersection property, then the cosets in $\mathfrak{A}$ have a non-empty intersection. A Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed. Moreover, we have that Artinian R-modules are linearly compact with the discrete topology [8, Theorem 2.1]. So the class of semi-discrete linearly compact modules contains all the Artinian modules. Moreover, if $(R,\mathfrak{m})$ is a complete ring, then the finitely generated R-modules are also linearly compact and semi-discrete.

2.1 Edge ideal of a graph

This subsection is in accordance with [1] and [20].

Let $R = K[v_1,\ldots,v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1,\ldots,z_q\}$ be a finite set of monomials in R. The *monomial subring* spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1,\ldots,z_q] \subset R.$$

In general, it is very difficult to certify whether $K[Z]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1,\ldots,v_s\}$. Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $j \neq i$, in $R = K[v_1,\ldots,v_s]$, such that $\{v_iv_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

Definition 3.A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1,z_1,v_1,\ldots,v_{s-1},z_h,v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

Definition 4.A walk is closed if $v_1 = v_s$. A walk may also be denoted by $\{v_1,\ldots,v_s\}$, the edges being evident by context. A *cycle* of length s is a closed walk, in which the points $v_1,\ldots,v_s$ are distinct.

A *path* is a walk with all the points distinct. A *tree* is a connected graph without cycles and a graph is *bipartite* if all its cycles are even. A vertex of degree one will be called an *end point*.

Definition 5.A subgraph $G' \subseteq G$ is called **induced** if $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G' whenever v_i and v_j are vertices of G' and v_iv_j is an edge of G.

The **complement** of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a **chord** is an edge which is not in the edge set of C_k. A cycle is called **minimal** if it has no a chord.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K.

Definition 6.([1]) According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G).$$

The figure 1 illustrates a connected graph with six vertices and nine edges. Some edge ideals of this graph are given by $I(G) = (v_1v_2, v_2v_3, v_3v_4)$, $I(G) = (v_5v_4, v_4v_6, v_6v_2)$, $I(G) = (v_1v_5, v_5v_6)$.

3 Results for the formal local homology and formal local cohomology modules

We consider again $(R,\mathfrak{m})$ a local Noetherian ring. We recall the concept of Macdonald dual. As before, let M be an R-module and consider $\mathrm{E}(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. According to [11], if M is a Hausdorff linearly topologized R-module, then the Macdonald dual of M is defined by

$$M^* = \mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{m})),$$

i.e., is the set of continuous homomorphisms of R-modules.

The topology on M^* is defined as in [11, 8.1]. Moreover, if M is a semi-discrete R-module, then the topology of M^* coincides with that induced on it as a submodule of $\mathrm{E}(R/\mathfrak{m})^M$, where

$$\mathrm{E}(R/\mathfrak{m})^M = \prod_{x\in M} (\mathrm{E}(R/\mathfrak{m}))^x,$$

with $(\mathrm{E}(R/\mathfrak{m}))^x = \mathrm{E}(R/\mathfrak{m})$ for all $x \in M$ (see [11, 8.6]).

Also, note that $M^* \subseteq \mathrm{D}(M)$ and the equality holds if and only if M is semi-discrete in the following lemma.

Lemma 1.([11, 5.8]) *Let M be a Hausdorff linearly topologized R-module. Then M is semi-discrete if and only if $M^* = \mathrm{D}(M)$.*

Lemma 2.([11, 9.3, 9.12, 9.13]) *Let $(R,\mathfrak{m})$ be a complete local Noetherian ring.*

1. *If M is linearly compact, then M^* is semi-discrete. If M is semi-discrete, then M^* is linearly compact.*
2. *If M is linearly compact, then we have a topological isomorphism $f : M \xrightarrow{\cong} M^{**}$.*

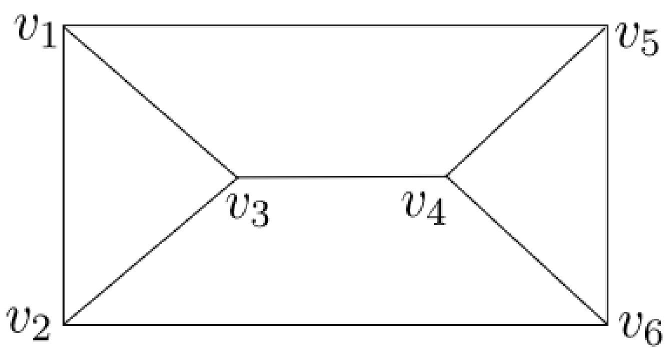

Fig. 1: A graph simple and finite

We have the following dual theorem, as it follows below.

Theorem 1. *Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R = K[v_1,v_2,\ldots,v_s]$, and suppose that R is a complete ring. Let $I(G)$ be the edge ideal, of a graph G, in the polynomial ring R. Suppose that $I(G)$ is a linearly compact R-module. Let J be an ideal of R. Then, we have that:*

1. $\mathfrak{F}_i^J(I(G)^*) \cong \left(\mathfrak{F}_J^i(I(G))\right)^*$,
2. $\mathfrak{F}_J^i(I(G)^*) \cong \left(\mathfrak{F}_i^J(I(G))\right)^*$,

for all $i \geq 0$.

Proof. It should be noted by [6, 6.7] that:

(a) $\frac{I(G)^*}{J^tI(G)^*} \cong \left(0 :_{I(G)} J^t\right)^*$, and
(b) $\left(\frac{I(G)}{J^tI(G)}\right)^* \cong \left(0 :_{I(G)^*} J^t\right)$,

for all $t > 0$ in $\mathbb{Z}$. Then, we have that:

$$\begin{aligned}
\mathfrak{F}_i^J(I(G)^*) &= \varinjlim_{t\in\mathbb{N}} \mathrm{H}_i^{\mathfrak{m}}\left((0 :_{I(G)^*} J^t)\right) \\
&\cong \varinjlim_{t\in\mathbb{N}} \mathrm{H}_i^{\mathfrak{m}}\left(\left(\frac{I(G)}{J^tI(G)}\right)^*\right) \\
&\cong \varinjlim_{t\in\mathbb{N}} \left(\mathrm{H}^i_{\mathfrak{m}}\left(\frac{I(G)}{J^tI(G)}\right)\right)^* \quad \text{(cf. [6, 6.4}(ii)\text{])} \\
&\cong \left(\varprojlim_{t\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{I(G)}{J^tI(G)}\right)\right)^* = \mathfrak{F}_J^i(I(G))^* \quad \text{(cf. [11, 9.14])}
\end{aligned}$$

and that:

$$\begin{aligned}
\mathfrak{F}_J^i(I(G)^*) &= \varprojlim_{t\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{I(G)^*}{J^tI(G)^*}\right) \\
&\cong \varprojlim_{t\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left((0 :_{I(G)} J^t)^*\right) \\
&\cong \varprojlim_{t\in\mathbb{N}} \left(\mathrm{H}_i^{\mathfrak{m}}\left((0 :_{I(G)} J^t)\right)\right)^* \quad \text{(cf. [6, 6.4}(ii)\text{])} \\
&\cong \left(\varinjlim_{t\in\mathbb{N}} \mathrm{H}_i^{\mathfrak{m}}\left((0 :_{I(G)} J^t)\right)\right)^* = \mathfrak{F}_i^J(I(G))^* \quad \text{(cf. [11, 2.6])}.
\end{aligned}$$

Thus, the proof is complete.

In this section, we presented some results about the formal local cohomology and formal local homology modules which involve the theory of graphs together with the edge ideal of a graph G. For the concept of linearly compact modules, which are we going to use, see [11, Definition 3.1], according to the which was placed in the Section 2.

Here, we take K a fixed field and we consider $K[v_1,\ldots,v_s]$ the polynomial ring over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1,\ldots,v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis). Moreover, $K[v_1,\ldots,v_s]$ is a local ring because K is a field.

Corollary Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R = K[v_1,v_2,\ldots,v_s]$, and suppose that R is a complete ring. Let $I(G)$ be the edge ideal, of a graph G, in the polynomial ring R. Suppose that $I(G)$ is a linearly compact R-module. Let J be an ideal of R. Then:

1. $\mathfrak{F}_J^i(I(G)^*)^* \cong \mathfrak{F}_i^J(I(G))$, and
2. $\mathfrak{F}_i^J(I(G)^*)^* \cong \mathfrak{F}_J^i(I(G))$,

for all $i \geq 0$.

Proof. Combining the Lemma 2 (2) with the Theorem 1 we have that:

1. $\mathfrak{F}_i^J(I(G)) \cong \mathfrak{F}_i^J(I(G)^{**}) \cong \mathfrak{F}_J^i(I(G)^*)^*$, and
2. $\mathfrak{F}_J^i(I(G)) \cong \mathfrak{F}_J^i(I(G)^{**}) \cong \mathfrak{F}_i^J(I(G)^*)^*$,

as required.

In order to state the next theorem about the long exact sequence of formal local cohomology modules we need of the following lemmas.

Lemma 3. *Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R = K[v_1,v_2,\ldots,v_s]$, and suppose that R is a complete ring. Let $I(G)$ be the edge ideal, of a graph G, in the polynomial ring R. Note that $I(G)$ is an R-module.*

1.*If $I(G)$ is a finitely generated R-module, then $I(G)^*$ is Artinian;*
2.*If $I(G)$ is an Artinian R-module, then $I(G)^*$ is finitely generated.*

*Proof.*According to the hypothesis, we have that $(R,\mathfrak{m})$ is a complete local Noetherian ring.

1.As $I(G)$ is a finitely generated R-module over the complete local Noetherian ring $(R,\mathfrak{m})$, it follows from [11, 7.3] that $I(G)$ is linearly compact and semi-discrete. Then, by Lemma 1, we have that $I(G)^* = \mathrm{D}(I(G))$. Now, the conclusion it follows from [23, 3.4.11], as required.
2.It should be noted that an Artinian R-module is a linearly compact R-module with the discrete topology (see [8, 2.1]). Then, by Lemma 1, it follows that $I(G)^* = \mathrm{D}(I(G))$. Finally, the conclusion it follows from [23, 3.4.12], as required.

Lemma 4.*Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R = K[v_1, v_2, \ldots, v_s]$. Let $I(G)$ be the edge ideal, of a graph G, in the polynomial ring R. Suppose that $I(G)$ is a semi-discrete linearly compact R-module. Let J be an ideal of R. Then the formal local cohomology modules $\mathfrak{F}^i_J(I(G))$ are linearly compact R-modules, for all $i \geq 0$.*

*Proof.*By [6, 7.9], the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}\left(\frac{I(G)}{J^tI(G)}\right)$ are Artinian R-modules, for all $i \geq 0$ and for all $t \in \mathbb{N}$. Note that, by definition, we have

$$\mathfrak{F}^i_J(I(G)) = \varprojlim_{t\in\mathbb{N}} \mathrm{H}^i_{\mathfrak{m}}\left(\frac{I(G)}{J^tI(G)}\right),$$

for all $i \geq 0$. Therefore, the conclusion it follows from [11, 3.7, 3.10], as required.

Theorem 2.*Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R = K[v_1, v_2, \ldots, v_s]$. Moreover, suppose that R is a complete ring. Let $I(G)$, $I(G)'$ and $I(G)''$ be edge ideals, of a graph G, in the polynomial ring R, and suppose that they are Artinian R-modules. Let J be an ideal of R. Also, suppose that*

$$0 \to I(G)' \to I(G) \to I(G)'' \to 0$$

is a short exact sequence. Then there exists a long exact sequence of I-formal local cohomology modules

$$\ldots \to \mathfrak{F}^i_J(I(G)) \to \mathfrak{F}^i_J(I(G)'') \to \mathfrak{F}^{i+1}_J(I(G)') \to \mathfrak{F}^{i+1}_J(I(G)) \to \ldots.$$

Moreover, the modules of this sequence are all linearly compact R-modules.

*Proof.*It should be noted by [19, 1.11] that an Artinian R-module, over a local Noetherian ring $(R,\mathfrak{m})$, has a natural structure of Artinian module over $\hat{R} = \varprojlim_{t\in\mathbb{N}} \frac{R}{\mathfrak{m}^t}$, and a subset of $I(G)$ is an R-submodule if and only if it is an $\hat{R}$-submodule. Thus, we now consider the short exact sequence of Artinian R-modules

$$0 \to I(G)' \xrightarrow{f} I(G) \xrightarrow{g} I(G)'' \to 0.$$

Note that the Artinian R-modules are linearly compact and semi-discrete, then the homomorphisms f, g are continuous. Combining [6, 6.5] with the Lemma 3 we have the following short exact sequence of finitely generated R-modules

$$0 \to (I(G)'')^* \xrightarrow{f^*} I(G)^* \xrightarrow{g^*} (I(G)')^* \to 0,$$

where the induced homomorphisms f^*, g^* are continuous. It gives rise by [18, 3.11] a long exact sequence of I-formal local cohomology modules

$$\ldots \to \mathfrak{F}^{i-1}_J((I(G)')^*) \to \mathfrak{F}^i_J((I(G)'')^*) \xrightarrow{g_i} \mathfrak{F}^i_J(I(G)^*) \xrightarrow{f_i} \mathfrak{F}^i_J((I(G)')^*) \to \ldots.$$

It should be mentioned from Lemma 4 that the I-formal local cohomology modules $\mathfrak{F}^i_J(I(G)^*)$, $\mathfrak{F}^i_J((I(G)')^*)$, $\mathfrak{F}^i_J((I(G)'')^*)$ are linearly compact and the homomorphisms of the long exact sequence are inverse limits of the homomorphisms of Artinian modules. Note that the homomorphisms of Artinian modules are continuous, because the topologies on the Artinian modules are semi-discrete. Moreover, the inverse limits of the continuous homomorphisms are also continuous. Then the homomorphisms of the long exact sequence are continuous. Therefore the long exact sequence induces an exact sequence by [6, 6.5], of the following form

$$\ldots \to (\mathfrak{F}^i_J(I(G)^*))^* \to \left(\mathfrak{F}^i_J((I(G)'')^*)\right)^* \to \left(\mathfrak{F}^{i-1}_J((I(G)')^*)\right)^* \to \ldots.$$

Now, it follows from Corollary 3 (1) that we have $\left(\mathfrak{F}^i_J(I(G)^*)\right)^* \cong \mathfrak{F}^J_i(I(G))$, for all $i \geq 0$. Therefore, we obtain the long exact sequence for formal local homology modules. The conclusion now it follows applying the Matlis dual module to long exact sequence of formal local homology modules.

By the Lemma 4, we have that the modules of this sequence are all linearly compact R-modules.

Remark In the Theorem 2 we can consider, for example, that $R = K[v_1, v_2, v_3]$, and in this case the edge ideal $I(G)$ can be of the form, for example, $I(G) = (v_1v_2, v_2v_3)$, $I(G) = (v_1v_2)$, etc. The graph G, in this case, or is a cycle of length $s = 3$, or is a graph of the type tree.

4 Results involving Noetherian dimension

We now recall the concept of *Noetherian dimension* of an R-module M denoted by $\mathrm{Ndim}(M)$. Note that the notion of Noetherian dimension was introduced first by R.N. Roberts in [16] by the name Krull dimension. Later, D. Kirby in [10] changed this terminology of Roberts and refereed to *Noetherian dimension* for to avoid confusion with well-know Krull-dimension of finitely generated modules. Let M be an R-module. When $M=0$ we put $\mathrm{Ndim}(M)=-1$. Then by induction, for any ordinal α, we put $\mathrm{Ndim}(M)=\alpha$ when

1. $\mathrm{Ndim}(M)<\alpha$ is false, and
2. for every ascending chain $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of M, there exists a positive integer m_0 such that $\mathrm{Ndim}(M_{m+1}/M_m)<\alpha$ for all $m \geq m_0$.

Thus, M is non-zero and finitely generated if and only if $\mathrm{Ndim}(M)=0$. If $0 \to M'' \to M \to M' \to 0$ is a short exact sequence of R-modules, then $\mathrm{Ndim}(M)=\max\left\{\mathrm{Ndim}(M''),\mathrm{Ndim}(M')\right\}$.

Proposition Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R=K[v_1,v_2,\ldots,v_s]$. Let $I(G)$ be the edge ideal, of a graph G, in the polynomial ring R. Suppose that $I(G)$ is an Artinian R-module. Let J be an ideal of R. If $\mathrm{Ndim}\left((0:_{I(G)}J)\right)=0$, then

1. $\mathfrak{F}^i_J(I(G)) \cong 0$, if $i \neq 0$;
2. $\mathfrak{F}^i_J(I(G)) \cong \mathrm{D}(I(G))$, if $i=0$.

Proof. Since $\mathrm{Ndim}\left((0:_{I(G)}J)\right)=0$, we have that $\left(0:_{I(G)}J\right)$ has finite length (cf. [16, p. 269]) and then $\left(0:_{I(G)}J\right)$ is $\mathfrak{m}$-separated. It follows from [6, 3.8] that

a. $\mathrm{H}^{\mathfrak{m}}_i\left((0:_{I(G)}J^t)\right) \cong 0$, if $i \neq 0$;
b. $\mathrm{H}^{\mathfrak{m}}_i\left((0:_{I(G)}J^t)\right) \cong \left(0:_{I(G)}J^t\right)$, if $i=0$,

for all $0<t \in \mathbb{N}$. By passing to direct limits we have

1. $\mathfrak{F}^J_i(I(G)) \cong 0$, if $i \neq 0$;
2. $\mathfrak{F}^J_i(I(G)) \cong \Gamma_J(I(G))$, if $i=0$.

Now, it follows from [19, 1.4], that as $I(G)$ is an Artinian R-module over the local ring $(R,\mathfrak{m})$, we have $\Gamma_J(I(G))=I(G)$. By passing to Matlis dual module, we have that

1. $\mathfrak{F}^i_J(I(G)) \cong \mathrm{D}(0)=0$, if $i \neq 0$;
2. $\mathfrak{F}^i_J(I(G)) \cong \mathrm{D}(I(G))$, if $i=0$,

as required.

Remember that the (Krull) dimension $\dim_R(M)$ of a non-zero R-module M is the supremum of lengths of chains of primes ideals in the support of M if this supremum there exists, and ∞ otherwise. For convenience, we set $\dim_R(M)=-1$, if $M=0$. Note that if M is non-zero and Artinian, then $\dim_R(M)=0$. If M is a finitely generated R-module, then we have that

$$\dim_R(M)=\max\left\{\dim_R\left(\frac{R}{\mathfrak{p}}\right) \mid \mathfrak{p} \in \mathrm{Ass}_R(M)\right\}.$$

In [6, 4.10], if M is a non-zero semi-discrete linearly compact R-module, then

1. $\mathrm{Ndim}(\Gamma_{\mathfrak{m}}(M))=\max\{i \mid \mathrm{H}^{\mathfrak{m}}_i(M) \neq 0\}$, if $\Gamma_{\mathfrak{m}}(M) \neq 0$;
2. $\mathrm{Ndim}(M)=\max\{i \mid \mathrm{H}^{\mathfrak{m}}_i(M) \neq 0\}$, if $\mathrm{Ndim}(M) \neq 1$.

We have the non-vanishing theorem of formal local homology modules.

Theorem 3. *Let $(R,\mathfrak{m})$ be a local Noetherian ring, where we have that the ring $R=K[v_1,v_2,\ldots,v_s]$. Let $I(G)$ be the edge ideal, of a graph G, in the polynomial ring R. Suppose that $I(G)$ is a semi-discrete linearly compact R-module. Let J be an ideal of R. Then,*

1. $\mathrm{Ndim}\left((0:_{I(G)}J)\right)=\max\left\{i \mid \mathfrak{F}^J_i(I(G)) \neq 0\right\}$, *if we have that* $0 \leq \mathrm{Ndim}\left((0:_{I(G)}J)\right) \neq 1$;
2. $\mathrm{Ndim}\left((0:_{\Gamma_{\mathfrak{m}}(I(G))}J)\right)=\max\left\{i \mid \mathfrak{F}^J_i(I(G)) \neq 0\right\}$, *if we have the following* $\mathrm{Ndim}\left((0:_{\Gamma_{\mathfrak{m}}(I(G))}J)\right) \neq 0$.

Proof. (1). We begin by proving that

$$\mathrm{Ndim}\left((0:_{I(G)}J^t)\right)=\mathrm{Ndim}\left((0:_{I(G)}J)\right),$$

for all $0<t \in \mathbb{Z}$. As $I(G)$ is a semi-discrete linearly compact R-module, it should be noted by [6, 7.1, 7.2] that $I(G)$ has a natural structure of semi-discrete linearly compact module over the ring $\hat{R}$ and $\mathrm{Ndim}_R(I(G))=\mathrm{Ndim}_{\hat{R}}(I(G))$. Thus, we may assume that $(R,\mathfrak{m})$ is a complete ring. At first, we prove in the special case when $I(G)$ is an Artinian R-module. Then, in this case, $\mathrm{D}(I(G))$ is a finitely generated R-module, by Matlis dual. We have then that

$$\dim\left(\frac{\mathrm{D}(I(G))}{J^t\mathrm{D}(I(G))}\right)=\dim\left(\frac{\mathrm{D}(I(G))}{J\mathrm{D}(I(G))}\right).$$

Combining Lemma 1 with [6, 7.4] yields

$$\begin{aligned}\mathrm{Ndim}\left((0:_{I(G)}J^t)\right) &= \dim\left(\mathrm{D}\left((0:_{I(G)}J^t)\right)\right)\\ &= \dim\left(\frac{\mathrm{D}(I(G))}{J^t\mathrm{D}(I(G))}\right)\\ &= \dim\left(\frac{\mathrm{D}(I(G))}{J\mathrm{D}(I(G))}\right)\\ &= \dim\left(\mathrm{D}\left((0:_{I(G)}J)\right)\right)=\mathrm{Ndim}\left((0:_{I(G)}J)\right).\end{aligned}$$

We now assume that $I(G)$ is a semi-discrete linearly compact R-module. By [25] there exists a short exact sequence

$$0 \to N \to I(G) \to A \to 0,$$

where N is a finitely generated R-module and A is an Artinian R-module. It induces an exact sequence

$$0 \to \left(0:_N J^t\right) \xrightarrow{f} \left(0:_{I(G)}J^t\right) \xrightarrow{g} \left(0:_A J^t\right) \xrightarrow{\delta} \mathrm{Ext}^1_R\left(R/J^t,N\right).$$

Then we have two short exact sequences

$$0 \to \left(0:_N J^t\right) \xrightarrow{f} \left(0:_{I(G)}J^t\right) \to \mathrm{Im}(g) \to 0,$$

$$0 \to \mathrm{Im}(g) \to \left(0:_A J^t\right) \to \mathrm{Im}(\delta) \to 0.$$

Since $(0:_N J^t)$ and $\mathrm{Im}(\delta)$ are finitely generated R-modules, we get $\mathrm{Ndim}(0:_N J^t) = \mathrm{Ndim}(\mathrm{Im}(\delta)) = 0$. It follows that

$$\begin{aligned}\mathrm{Ndim}\left((0:_{I(G)} J^t)\right) &= \mathrm{Ndim}(\mathrm{Im}(g))\\ &= \mathrm{Ndim}\left((0:_A J^t)\right)\\ &= \mathrm{Ndim}\left((0:_A J)\right)\\ &= \mathrm{Ndim}\left((0:_{I(G)} J)\right).\end{aligned}$$

Now, it follows from [6, 4.10(ii)] that

$$d = \mathrm{Ndim}\left((0:_{I(G)} J)\right) = \mathrm{Ndim}\left((0:_{I(G)} J^t)\right) = \max\left\{i \mid \mathrm{H}^{\mathfrak{m}}_i\left((0:_{I(G)} J^t)\right) \neq 0\right\}$$

for all $t > 0$. Then $\mathrm{H}^{\mathfrak{m}}_d\left((0:_{I(G)} J^t)\right) \neq 0$ and $\mathrm{H}^{\mathfrak{m}}_i\left((0:_{I(G)} J^t)\right) = 0$, for all $i > d$ and $t > 0$. The short exact sequences for all $t > 0$

$$0 \to (0:_{I(G)} J^t) \to (0:_{I(G)} J^{t+1}) \to \left(0:_{I(G)} J^{t+1}/(0:_{I(G)} J^t)\right) \to 0$$

induces exact sequences

$$\ldots \to \mathrm{H}^{\mathfrak{m}}_{d+1}\left((0:_{I(G)} J^{t+1}/(0:_{I(G)} J^t))\right) \to \mathrm{H}^{\mathfrak{m}}_d\left((0:_{I(G)} J^t)\right) \to \mathrm{H}^{\mathfrak{m}}_d\left((0:_{I(G)} J^{t+1})\right) \to \ldots,$$

for all $t > 0$, by [6, 3.7]. As $\mathrm{Ndim}\left((0:_{I(G)} J^{t+1}/(0:_{I(G)} J^t))\right) \leq d$, [6, 4.8] shows that $\mathrm{H}^{\mathfrak{m}}_{d+1}\left((0:_{I(G)} J^{t+1}/(0:_{I(G)} J^t))\right) = 0$. Then the homomorphisms

$$\mathrm{H}^{\mathfrak{m}}_d\left((0:_{I(G)} J^t)\right) \longrightarrow \mathrm{H}^{\mathfrak{m}}_d\left((0:_{I(G)} J^{t+1})\right)$$

are injective for all $t > 0$. Therefore, it follows that

(a)$\varinjlim_{t\in\mathbb{N}} \mathrm{H}^{\mathfrak{m}}_d\left((0:_{I(G)} J^t)\right) \neq 0$,
(b)$\varinjlim_{t\in\mathbb{N}} \mathrm{H}^{\mathfrak{m}}_i\left((0:_{I(G)} J^t)\right) = 0$,

for all $i > d$. Hence, (1) is proved.
(2). It should be noted that $\left(0:_{\Gamma_{\mathfrak{m}}(I(G))} J\right) = \Gamma_{\mathfrak{m}}\left((0:_{I(G)} J)\right)$. Set

$$d = \mathrm{Ndim}\left((0:_{\Gamma_{\mathfrak{m}}(I(G))} J)\right) = \mathrm{Ndim}\left((0:_{\Gamma_{\mathfrak{m}}(I(G))} J^t)\right).$$

From [6, 4.10(i)] we have that

$$\begin{aligned}d &= \mathrm{Ndim}\left((0:_{\Gamma_{\mathfrak{m}}(I(G))} J^t)\right)\\ &= \mathrm{Ndim}\left(\Gamma_{\mathfrak{m}}\left((0:_{I(G)} J^t)\right)\right)\\ &= \max\left\{i \mid \mathrm{H}^{\mathfrak{m}}_i\left((0:_{I(G)} J^t)\right) \neq 0\right\}.\end{aligned}$$

The rest of the proof is analogous to that was made in the proof of (1).

We have the following corollary.

Corollary In the same previous context, let $I(G)$ be an Artinian R-module, with $(R,\mathfrak{m})$ a local Noetherian ring, such that $\left(0:_{I(G)} J\right) \neq 0$. Then

$$\mathrm{Ndim}\left((0:_{I(G)} J)\right) = \max\left\{i \mid \mathfrak{F}^J_i(I(G)) \neq 0\right\}.$$

*Proof.*It should be noted that an Artinian R-module is semi-discrete and linearly compact (see [8]). Moreover, $\Gamma_{\mathfrak{m}}(I(G)) = I(G)$. Therefore, the conclusion it follows from Theorem 3 (2), as required.

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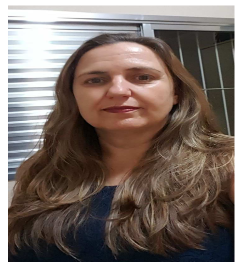

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