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Applications of N-Structures to Ideal Theory of LA-Semigroup

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Abstract: In this paper, the concept of $([e],[e]\vee[c])$-ideals in an LA-semigroup is introduced and proved fundamental results to determine the relation between these notions and ideals of a LA-semigroup. Moreover, we introduce the concept of $([e],[e]\vee[c])$-sub LA-semigroup and investigated related properties. To obtain more general form $[(\alpha),(\beta)]$-ideals and $[(\alpha),(\beta)]$-sub LA-semigroup are introduced and further characterization is discussed.

Keywords: LA-semigroup, N-structure, Ideal, ([e],[e]v[c])-ideal

1 Introduction

A left almost semigroup, abbreviated as an LA-semigroup, is an algebraic structure betwixt between a groupoid and a commutative semigroup. An LA-semigroup is a non-commutative and non-associative algebraic structure. This structure was introduced by M. A. Kazim and M. Naseeruddin [4] in 1972. It has been defined in [1] and [12] that a groupoid G with left invertive law, that is: $(ab)c=(cb)a\ \forall a,b,c\in G$ is called an LA-semigroup. Naseeruddin has investigated some basic characteristics of this structure in his thesis. He has generalized some important results of semigroup theory. Moreover, he has established the relationships between LA-semigroups and quasi groups, semigroups, loops, monoids and groups. Kazim and Naseeruddin in their paper on almost semigroup [4] have shown that G is medial. That is, $(ab)(cd)=(ac)(bd)$. Later, this structure has further investigated by Q.Mushtaq and others and many useful results have been added to theory of LA-semigroups [7,8,9].

In 1965, L. A. Zadeh generalized the Crisp Set Theory and introduced the fundamental concept of a fuzzy set [13]. Since then this concept has been applied to various algebraic structures. That is why; there is a rapid increase in research and literature on Fuzzy Set Theory and its applications. Many areas, for example, artificial intelligence, computer science, control engineering, expert, robotics, automat theory, finite state machine and graph theory etc deals with the study of Fuzzy Set Theory [13]. Ideals of LA-semigroups were defined by Mushtaq and Khan in his paper [10]. Further, Khan and Ahmad characterized the ideals of LA-semigroup in [5]. In 2010, Khan and Khan introduced fuzzy ideals in LA-semigroups [6] and proved some important results.

So far, no negative information was involved. All the research relied on positive information only. To cope with this problem, in 2009, Jun et al [1] introduced a new function which is called a negative valued function and then introduced N-structure and discussed N-algebras and N-ideals in BCK/BCI algebras. Later on, Y. B. Jun and M. S. Kang [2] introduced the notion of N-ideal of BE-algebra introduced by, Kim and Kim [3] and investigated several characterizations of N-ideals. For further generalization of N-ideals, they proposed a definition of a point N-structure which is employed or conditionally employed in an N-structure. Using these notions, they introduced the concept of $([e],[e]\vee[c])$-ideals and investigated these ideals on BE-algebras.

The aim of this paper is to make development in fuzzy technology and to promote research in the specified field. Our goal is to explain new mathematical techniques and development in the field of fuzzy algebra and LA-semigroup, which would be of great importance in

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future. This paper can be a bridge passing from the theory of N-fuzzy algebra to the theory of LA-semigroup. In this paper, we have used the idea of Y. B. Jun and M. S. Kang [2] and introduced $([e],[e]\vee[c])$-ideals in LA-semigroup. We have proved some fundamental results that determine the relation between the newly introduced notions and ideals of an LA-semigroup. Moreover, we have introduced the notion of $([e],[e]\vee[c])$-sub LA-semigroup and investigated their relationship with sub LA-semigroup. For the generalization of $([e],[e]\vee[c])$-ideals and $([e],[e]\vee[c])$-sub LA-semigroup, we have introduced the concept of $([\alpha],[\beta])$-ideals and $([\alpha],[\beta])$-sub LA-semigroup and proved some fundamental results.

2 Preliminaries

Definition 1.*Let I be a non-empty subset of S. Then I is called a left (resp. right) ideal in LA-semigroup if for $x\in I$, $y\in S \Longrightarrow xy\in I$ (resp. $yx\in I$).*

Definition 2.*Let I be a non-empty subset of an LA-semigroup S. Then I is called a sub-LA-semigroup if for all $x,y\in I \Longrightarrow xy\in I$.*

Definition 3.*An N-structure (S,f) is called a left (resp. right) N-ideal of S if for all $x,y\in S$, $f(xy)\leq f(y)$ (resp. $f(xy)\leq f(x)$).*

Definition 4.*Let (X,f) be an N-structure. Then, a point N-structure be defined as following*

$$f(y)=\{.0 \quad \text{if } y\neq x\theta \quad \text{if } y=x \tag{1}$$

where $\theta\in[-1,0)$. In this case, f is denoted by $\frac{x}{\theta}$ and we call $(X,\frac{x}{\theta})$ a point N-structure. We say that a point N-structure $(X,\frac{x}{\theta})$ is employed in an N-structure (X,f), denoted by $(X,\frac{x}{\theta})[e](X,f)$ (or briefly $\frac{x}{\theta}[e]f$), if $f(x)\leq\theta$. A point N-structure $(X,\frac{x}{\theta})$ is said to be conditionally employed in an N-structure (X,f), denoted by $(X,\frac{x}{\theta})[c](X,f)$ (or briefly $\frac{x}{\theta}[c]f$), if $f(x)+\theta+1<0$.

To say that $(X,\frac{x}{\theta})[e]\vee[c](X,f)$ (or briefly $\frac{x}{\theta}[e]\vee[c]f$), we mean $(X,\frac{x}{\theta})[e](X,f)$ or $(X,\frac{x}{\theta})[c](X,f)$ (or briefly, $\frac{x}{\theta}[e]f$ or $\frac{x}{\theta}[c]f$). To say that $\frac{x}{\theta}\overline{\alpha}f$, we mean $\frac{x}{\theta}\alpha f$ does not hold for $\alpha\in\{[e],[e]\vee[c]\}$.

3 $([e],[e]\vee[c])$-ideals

Definition 5.*An N-structure (S,f) is called an $([e],[e]\vee[c])$-ideal in LA-semigroup if it satisfies:*

1. $\frac{x}{t}[e]f \Longrightarrow \frac{xy}{t}[e]\vee[c]f$
2. $\frac{x}{t}[e]f \Longrightarrow \frac{yx}{t}[e]\vee[c]$, for all $x,y\in S$ and $t\in[-1,0)$.

*Example 1.*Let $S=\{1,2,3\}$ be an LA-semigroup defined by following Cayle's table

.	1	2	3
1	2	2	2
2	2	2	2
3	1	1	2

An N-structure (S,f) is defined by

$$f(x)=\{.-0.7 \quad \text{if } x=1,2-0.3 \quad \text{if } x=3$$

Thus, by routine (S,f) is an $([e],[e]\vee[c])$-ideal in LA-semigroup S.

*Example 2.*Let $S=\{1,2,3,4\}$ be an LA-semigroup defined by following Cayle's table

.	1	2	3	4
1	4	4	2	2
2	4	4	4	4
3	4	4	2	4
4	4	4	4	4

N-structure (S,f) is defined by

$$f(x)=\{.\begin{array}{llll} -0.2 & \text{if } x=1 & -0.3 & \text{if } x=3 \\ -0.7 & \text{if } x=2 & -0.8 & \text{if } x=4 \end{array}$$

Thus, by routine calculation (S,f) is an $([e],[e]\vee[c])$-ideal in LA-semigroup.

Theorem 1.*For any N-structure (S,f), the following conditions are equivalent.*

1. (S,f) is an $([e],[e]\vee[c])$-ideal.
2. $(\forall\, x,y\in S)\,(f(xy)\leq\max\{f(y),-0.5\}$ and $f(xy)\leq\max\{f(x),-0.5\})$

*Proof.*Let (S,f) is an $([e],[e]\vee[c])$-ideal of S. For all $x,y\in S$ let $f(xy)>\max\{f(y),-0.5\}=t_y$. If $f(y)\leq t_y \Longrightarrow \frac{y}{t_y}[e]f$ and $\frac{xy}{t_y}\overline{[e]}f$. But if $t_y>-0.5$ then $f(xy)+t_y+1>-0.5-0.5+1=0 \Longrightarrow \frac{xy}{t_y}\overline{[c]}f \Longrightarrow \frac{xy}{t_y}\overline{[e]\vee[c]}f$ so $\frac{y}{t_y}[e]f \Longrightarrow \frac{xy}{t_y}\overline{[e]\vee[c]}f$. This is a contradiction. Hence our supposition is wrong. So $f(xy)\leq\max\{f(y),-0.5\}$. Similarly, we can show this for $f(xy)\leq\max\{f(x),-0.5\}$.

Conversely, suppose that (S,f) satisfies the condition $f(xy)\leq\max\{f(y),-0.5\}$. Let $x,y\in S$ and $t\in[-1,0)$ such that $\frac{y}{t}[e]f$. Then $f(y)\leq t$. Suppose that $\frac{xy}{t}\overline{[e]}f$ i.e. $f(xy)>t$. Now either $f(y)>-0.5$ or $f(y)\leq-0.5$. If $f(y)>-0.5$ then $f(xy)\leq\max\{f(y),-0.5\}=f(y)\leq t$ which is not true. If $f(y)\leq-0.5 \Longrightarrow f(xy)+t+1< 2f(xy)+1\leq 2\max\{f(y),-0.5\}+1=0$ i.e. $\frac{xy}{t}[c]f$. Thus $\frac{xy}{t}[e]\vee[c]f$. Similarly, we can show this for $\frac{yx}{t}[e]\vee[c]f$. Thus (S,f) is an $([e],[e]\vee[c])$-ideal of S.

Theorem 2.*For any N-structure (S,f), the following conditions are equivalent.*

1.(S,f) is an $([e],[e]\vee[c])$-ideal.
2.$C(f,t)$ is an ideal of S.

*Proof.*Let (S,f) is an $([e],[e]\vee[c])$-ideal and let $t\in[-0.5,0)$ be such that $C(f,t)\neq\Phi$. By Theorem 1 $f(xy)\leq\max\{f(y),-0.5\}$ and $f(xy)\leq\max\{f(x),-0.5\}$ for any $x,y\in C(f,t)$. It follows that $f(xy)\leq\max\{t,-0.5\}=t\Longrightarrow xy\in C(f,t)$ and $f(yx)\leq\max\{t,-0.5\}=t\Longrightarrow yx\in C(f,t)$. Therefore, $C(f,t)$ is an ideal of S.

Conversely, suppose that $C(f,t)$ is an ideal of S. If $\exists x,y\in S$ such that $f(xy)>\max\{t,-0.5\}$ then $f(xy)>t_y\geq\max\{t,-0.5\}$ for some $t_y\in[-0.5,0)$. Thus $y\in C(f,t)$ but $xy\notin C(f,t)$ which is a contradiction. Thus $f(xy)\leq\max\{f(y),-0.5\}$ $\forall x,y\in S$. Similarly we can show this for $f(yx)$. By previous theorem, we can say that (S,f) is an $([e],[e]\vee[c])$-ideal of S.

Theorem 3.*Let S be an LA-semigroup. If (S,f) is an $([e],[e]\vee[c])$-ideal of S such that $f(x)>-0.5\forall x\in S$ then (S,f) is an N-ideal of S.*

*Proof.*Assume that (S,f) is an $([e],[e]\vee[c])$-ideal of S such that $f(x)>-0.5$ $\forall x\in S$. By Theorem 1 $f(xy)\leq\max\{f(x),-0.5\}$. Since $f(x)>-0.5$ so $f(xy)\leq f(x)$. Similarly, we can show this for $f(yx)\leq f(x)$. Therefore, (S,f) is an N-ideal of S.

Theorem 4.*If (S,f) is an $([e],[e]\vee[c])$-ideal of an LA-semigroup, then $Q(f,t)$ is an ideal of S. Where $Q(f,t):=\{x\in X|\frac{x}{t}[c]f\}$ $\forall t\in[-1,-0.5]$.*

*Proof.*Let (S,f) is an $([e],[e]\vee[c])$-ideal of an LA-semigroup S. Suppose that $Q(f,t)\neq\Phi$ for all $t\in[-1,-0.5]$ then there exist $x\in Q(f,t)$ such that $f(xy)+t+1<0\Longrightarrow f(x)<-t-1$. By theorem 1 $f(xy)\leq\max\{f(x),-0.5\}<\max\{-t-1,-0.5\}<-t-1\Longrightarrow f(xy)+t+1<0\Longrightarrow xy\in Q(f,t)$. Similarly, we can show for $yx\in Q(f,t)$. Therefore, $Q(f,t)$ is an ideal of S.

Theorem 5.*Let S be an LA-semigroup. Then an N-structure (S,f) is an $([e],[e]\vee[c])$-ideal of S if and only if $[f]_t$ is an ideal of S where $[f]_t:=C(f,t)\cup Q(f,t)$ $\forall t\in[-1,0)$.*

*Proof.*Assume that (S,f) is an $([e],[e]\vee[c])$-ideal of S and let $t\in[-1,0)$ be such that $[f]_t\neq\Phi$. Then there exist $x\in[f]_t$ such that $f(x)\leq t$ or $f(x)+t+1<0$. If $f(x)\leq t$ then $f(xy)\leq\max\{f(x),-0.5\}\leq\max\{t,-0.5\}=t\Longrightarrow xy\in[f]_t$ and if $f(x)+t+1<0\Longrightarrow f(x)<-t-1$ then $f(xy)\leq\max\{f(x),-0.5\}\leq\max\{-t-1,-0.5\}=-t-1\Longrightarrow xy\in[f]_t$. So $xy\in[f]_t$. Similarly we can show for $yx\in[f]_t$. Thus $[f]_t$ is an ideal of S.

Conversely, let (S,f) be an ideal of S. Suppose that $f(xy)>\max\{f(x),-0.5\}$ for some $x,y\in S$. Taking $t:=\max\{f(x),-0.5\}$. Since $[f]_t$ is an ideal of S, so $xy\in[f]_t\Longrightarrow f(xy)\leq t$ or $f(xy)<-t-1$. But the inequality 3 induces that $xy\notin C(f,t)$. So our supposition is wrong and $f(xy)\leq\max\{f(x),-0.5\}$. Similarly, it can be shown for $f(xy)\leq\max\{f(y),-0.5\}$. Thus (S,f) is an $([e],[e]\vee[c])$-ideal.

4 $([e],[e]\vee[c])$-sub LA-semigroup

Definition 6.*Let (S,f) be an N-structure. Then (S,f) is called $([e],[e]\vee[c])$-sub LA-semigroup if the following condition holds*

1.$\frac{x}{t_1}[e]f,\frac{y}{t_2}[e]f\Longrightarrow\frac{xy}{\max\{t_1,t_2\}}[e]\vee[c]f$

for all $x,y\in S$ and $t_1,t_2\in[-1,0)$.

Theorem 6.*For any N-stature (S,f),the following are equivalent*

1.(S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup.
2.$(\forall x,y\in S)$ $(f(xy)\leq\max\{f(x),f(y),-0.5\})$.

*Proof.*Assume that (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup. Let $f(xy)>\max\{f(x),f(y),-0.5\}$ $\forall x,y\in S$ be such that $t_y:=\max\{f(x),f(y),-0.5\}$. Now either $\max\{f(x),f(y)\}>-0.5$ or $\max\{f(x),f(y)\}\leq-0.5$. If $\max\{f(x),f(y)\}>-0.5\Longrightarrow f(x)\leq t_y$ and $f(y)\leq t_y\Longrightarrow\frac{x}{t_y}[e]f$ and $\frac{y}{t_y}[e]f$ since $f(xy)>t_y\Longrightarrow\frac{xy}{t_y}\overline{[e]}f$. And if $\max\{f(x),f(y)\}\leq-0.5\Longrightarrow f(xy)+t_y+1>-0.5-0.5+1=0\Longrightarrow\frac{xy}{t_y}\overline{[c]}f$ so $\frac{xy}{t_y}\overline{[e]\vee[c]}f$ which is a contradiction. Hence $f(xy)\leq\max\{f(x),f(y),-0.5\}$.

Conversely, Suppose that N-structure (S,f) satisfies the given condition $f(xy)\leq\max\{f(x),f(y),-0.5\}$. Let $x,y\in S$ and $t\in[-1,0)$ such that $\frac{x}{t_1}[e]f$, $\frac{y}{t_2}[e]f$ or $f(x)\leq t_1$, $f(y)\leq t_2$. Then $f(xy)\leq\max\{f(x),f(y),-0.5\}\leq\max\{t_1,t_2,-0.5\}$. Suppose that $f(xy)>\max\{t_1,t_2\}$ or $\frac{xy}{\max\{t_1,t_2\}}\overline{[e]}f$. If $\max\{t_1,t_2\}>-0.5$ then $f(xy)\leq\max\{f(x),f(y)\}$ which is contradiction so this is not possible. But if $\max\{t_1,t_2\}\leq-0.5$ then $f(xy)+\max\{t_1,t_2\}+1<f(xy)+f(xy)+1=2f(xy)+1<2\max\{f(x),f(y),-0.5\}+1<2(-0.5)+1=0\Longrightarrow f(xy)+\max\{t_1,t_2\}+1<0$ i.e. $\frac{xy}{\max\{t_1,t_2\}}[c]f$. Thus $\frac{xy}{\max\{t_1,t_2\}}[e]\vee[c]f$ and therefore (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup.

Theorem 7.*For any N-structure (S,f) the following are equivalent*

1.(S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup.
2.$C(f,t)$ is a sub LA-semigroup.

*Proof.*Assume that (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup and $C(f,t)\neq 0$. Let $x,y\in C(f,t)$, by Theorem 6 $f(xy)\leq\max\{f(x),f(y),-0.5\}<\max\{t,t,-0.5\}=t\Longrightarrow f(xy)\leq t$ and $t\in[-0.5,0)\Longrightarrow x,y\in C(f,t)$. Thus $C(f,t)$ is a sub LA-semigroup.

Conversely, suppose that $C(f,t)$ is a sub LA-semigroup of S. Let $f(xy)>\max\{f(x),f(y),-0.5\}$

for $x,y \in S$ then $f(xy) > t_y \geq \max\{f(x), f(y), -0.5\}$ for some $t_y \in [-0.5, 0)$. Thus $x,y \in C(f,t)$ but $xy \notin C(f,t)$ which is a contradiction. Thus $f(xy) \leq \max\{f(x), f(y), -0.5\}$. Thus (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup.

Theorem 8. *If (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup, then $Q(f,t)$ is a sub LA-semigroup, where $Q(f,t) := \{x \in X \mid \frac{x}{t}[c]f\}$ $\forall t \in [-1,-0.5]$.*

Proof. Let (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup. Suppose $Q(f,t) \neq 0$ for all $t \in [-1,-0.5]$. Then there exist $x,y \in Q(f,t) \Longrightarrow f(x)+t+1 < 0$ and $f(y)+t+1 < 0$ or $f(x) < -t-1$ and $f(y) < -t-1$. Then by Theorem 6 $f(xy) \leq \max\{f(x), f(y), -0.5\} < \max\{-t-1, -t-1, -0.5\} = -t-1 \Longrightarrow f(xy)+t+1 < 0 \Longrightarrow xy \in Q(f,t)$. Thus $Q(f,t)$ is a sub LA-semigroup.

Theorem 9. *Let S be a sub LA-semigroup. Then an N-structure (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup if and only if $[f]_t$ is a sub LA-semigroup for all $t \in [-1,0)$ where $[f]_t := C(f,t) \cup Q(f,t)$.*

Proof. Assume that (S,f) is an $([e],[e]\vee[c])$-sub LA-semigroup. Then by Theorems 7 and 8, $C(f,t)$ and $Q(f,t)$ are sub LA-semigroups. Therefore, $[f]_t$ is a sub LA-semigroup.

Conversely, suppose that $[f]_t$ is a sub LA-semigroup. Suppose that $f(xy) > \max\{f(x), f(y), -0.5\}$ for some $x,y \in S$. Taking $t := \max\{f(x), f(y), -0.5\}$. Since $[f_t]$ is a sub LA-semigroup. So for $x,y \in [f_t] \Longrightarrow xy \in [f_t] \Longrightarrow f(xy) \leq t$ or $f(xy) < -t-1$. But $xy \notin C(f,t)$. Thus $xy \notin [f_t]$ which is a contradiction. Hence our supposition is wrong. Therefore $f(xy) \leq \max\{f(x), f(y), -0.5\}$. Thus $[f_t]$ is an $([e],[e]\vee[c])$-sub LA-semigroup.

5 $([\alpha],[\beta])$-ideal and $([\alpha],[\beta])$-sub LA-semigroup

Definition 7. *Let (S,f) be an N-structure. Then (S,f) is called a right (resp. left) $([\alpha],[\beta])$-ideal in an LA-semigroup if following conditions hold:*

1. $\frac{x}{t}[\alpha]f \Longrightarrow \frac{xy}{t}[\beta]f$ (resp. $\frac{x}{t}[\alpha]f \Longrightarrow \frac{yx}{t}[\beta]f$) where $f : S \longrightarrow [-1,0]$ and for all $x,y \in S$, $\forall t \in [-1,0)$

Definition 8. *An N-structure (S,f) is called an $([\alpha],[\beta])$-sub LA-semigroup if it satisfies following implication*

1. $\frac{x}{t_1}[\alpha]f, \frac{y}{t_2}[\alpha]f \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[\beta]f$ for all $x,y \in S$ and $\forall t_1,t_2 \in [-1,0)$

Theorem 10. *Let (S,f) be a non-empty $([\alpha],[\beta])$-sub LA-semigroup. Then $(S,f)^* = \{x \in S : f(x) < 0\}$ is a sub LA-semigroup.*

Proof. Let (S,f) be a non-empty $([\alpha],[\beta])$-sub LA-semigroup. We have to show that $(S,f)^*$ is a sub LA-semigroup. i.e. for all $x,y \in (S,f)^* \Longrightarrow xy \in (S,f)^*$. Let $x,y \in (S,f)^*$ i.e. $f(x) < 0$ and $f(y) < 0$. Suppose that $f(xy) = 0$ and $\alpha \in \{[e],[c],[e]\vee[c]\}$.

(i) For $\alpha \in \{[e],[e]\vee[c]\}$ we have $\frac{x}{t_1}[\alpha]f$, $\frac{y}{t_2}[\alpha]f$. As $f(xy) = 0 > \max\{t_1,t_2\} \Longrightarrow f(xy) > \max\{t_1,t_2\} \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}\overline{[e]}f$. So $f(xy)+\max\{t_1,t_2\}+1 \geq 0$ (clearly)$\Longrightarrow \frac{xy}{\max\{t_1,t_2\}}\overline{[c]}f$. So $\frac{xy}{\max\{t_1,t_2\}}\overline{[\beta]}f$ for every $\beta \in \{[e],[c],[e]\vee[c],[e]\wedge[c]\}$.

(ii) For $\alpha = [c]$ we have $\frac{x}{-1}[\alpha]f, \frac{y}{-1}[\alpha]f \Longrightarrow f(x)+(-1)+1 < 0, f(y)+(-1)+1 < 0$. As $f(xy) = 0 > -1$ so $f(xy)+(-1)+1 = 0+0 = 0 \Longrightarrow \frac{xy}{-1}\overline{[\beta]}f$. Thus $\frac{xy}{\max\{t_1,t_2\}}\overline{[\beta]}f$ for every $\beta \in \{[e],[c],[e]\vee[c],[e]\wedge[c]\}$. Hence our supposition is wrong. So $f(xy) < 0 \Longrightarrow xy \in (S,f)^*$. Thus (S,f) is a sub LA-semigroup.

Theorem 11. *Let (S,f) be a non-empty $([\alpha],[\beta])$-left ideal of S. Then $(S,f)^* = \{x \in S : f(x) < 0\}$ is a left ideal of S.*

Proof. Let (S,f) be a non-empty $([\alpha],[\beta])$-left ideal. We have to show that $(S,f)^*$ is a left ideal. i.e. for $y \in (S,f)^* \Longrightarrow xy \in (S,f)^*$. Let $y \in (S,f)^*$ i.e. $f(y) < 0$. Suppose that $f(xy) = 0$ and $\alpha \in \{[e],[c],[e]\vee[c]\}$.

(i) For $\alpha \in \{[e],[e]\vee[c]\}$ we have $\frac{y}{t_2}[\alpha]f$. As $f(xy) = 0 > t \Longrightarrow f(xy) > t \Longrightarrow \frac{xy}{t}\overline{[e]}f$. Thus $f(xy)+t+1 \geq 0 \Longrightarrow \frac{xy}{t}\overline{[c]}f \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}\overline{[\beta]}f$ for every $\beta \in \{[e],[c],[e]\vee[c],[e]\wedge[c]\}$.

(ii) for $\alpha = [c]$ we have $\frac{y}{-1}[\alpha]f \Longrightarrow f(y)+(-1)+1 < 0$. As $f(xy) = 0 > -1 \Longrightarrow f(xy)+(-1)+1 = 0+0 = 0 \Longrightarrow \frac{xy}{-1}\overline{[\beta]}f$. Thus $\frac{xy}{t}\overline{[\beta]}f$ for every $\beta \in \{[e],[c],[e]\vee[c],[e]\wedge[c]\}$. Hence our supposition is wrong. So $f(xy) < 0 \Longrightarrow xy \in (S,f)^*$. Thus (S,f) is a left ideal of S.

Theorem 12. *Let (S,f) be a non-empty $([\alpha],[\beta])$-right ideal of S. Then $(S,f)^* = \{x \in S : f(x) < 0\}$ is a right ideal of S.*

Theorem 13. *Let (S,f) be a left zero LA-semigroup. Let (S,f) be a non-empty $([c],[c])$-sub LA-semigroup. Then (S,f) is a constant on $(S,f)^*$.*

Proof. Let y be an element of S such that $f(y) = \wedge\{f(x) : x \in S\}$. Since $\wedge f(x) < 0$ so $y \in (S,f)^*$. Suppose that $\exists x \in (S,f)^*$s.t. $t_x = f(x) \neq f(y) = t_y$ then $t_x > t_y$. Choose $t_1,t_2 \in [-1,0)$ s.t. $-t_y-1 > t_1 > -t_x-1 > t_2$. Then $\frac{y}{t_1}[c]f$ and $\frac{x}{t_2}[c]f \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}\overline{[c]}f$. Because S is a left zero LA-semigroup. This is a contradiction. Thus $f(x) = f(e)$ $\forall x \in (S,f)^*$. Therefore, (S,f) is a constant on $(S,f)^*$.

Theorem 14. *Let (S,f) be an N-structure in an LA-semigroup. Then (S,f) is a non-empty $([c],[c])$-sub LA-semigroup iff there exist a sub LA-semigroup H of S such that*

$$f(x) = \begin{cases} t \in [-1,0) & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

*Proof.*Let (S, f) be a non-empty $([c],[c])$-sub LA-semigroup. By theorem 10, $(S, f)^*$ is a sub LA-semigroup. So it is clear that $f(x) < 0$ for all $x \in S$. And by theorem 13,

$$f(x) := \begin{cases} f(x) & \text{if} \quad x \in (S, f)^* \\ 0 & \text{otherwise} \end{cases}$$

where $f(x) \in [-1,0)$.

Conversely, let H be a sub LA-semigroup which satisfies

$$f(x) = \{.t \in [-1,0) \quad \text{if } x \in S0 \qquad \text{otherwise}$$

Assume that $\frac{x}{t_1}[c]f$ and $\frac{y}{t_2}[c]f$ for some $t_1,t_2 \in [-1,0)$. Then $f(x)+t_1+1 < 0$ and $f(y)+t_1+1 < 0 \Longrightarrow f(x) \neq 0$ and $f(y) \neq 0$. Thus $x,y \in H$ and so $xy \in H$. So $f(xy)+\max\{t_1,t_2\}+1 < 0 \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[c]f$. Thus (S, f) is a $\{[c],[c]\}$-sub LA-semigroup.

Theorem 15.*Let H be a sub LA-semigroup of S and (S, f) be an N-structure in LA-semigroup S such that*

(i) $f(x) = 0 \ \ \forall x \in S\backslash H$
(ii) $f(x) \leq -0.5 \ \ \forall x \in H$
Then (S, f) is an $([\alpha],[e] \vee [c])$ sub LA-semigroup.

*Proof.*Let H be a sub LA-semigroup of S and (S, f) be an N-structure in LA-semigroup S, such that
(i) $f(x) = 0 \ \ \forall x \in S\backslash H$
(ii) $f(x) \leq -0.5 \ \ \forall x \in H$
Let $x,y \in S$ and $t_1,t_2 \in [-1,0)$ such that $\frac{x}{t_1}[\alpha]f$ and $\frac{y}{t_2}[\alpha]f$.

Case I: For $[\alpha] = [e]$ i.e. $\frac{x}{t_1}[e]f$ and $\frac{y}{t_2}[e]f \Longrightarrow f(x) \leq t_1$ and $f(x) \leq t_2$. For $x \notin H$ or $y \notin H$ i.e. $f(x) = 0$ or $f(y) = 0 \Longrightarrow t_1 > 0$ or $t_2 > 0$ which is not true. So $x,y \in H \Longrightarrow xy \in H \Longrightarrow f(xy) \leq -0.5$. If $\max\{t_1,t_2\} < -0.5$ then $f(xy)+\max\{t_1,t_2\}+1 < -0.5-0.5+1 = 0 \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[c]f$. But if $\max\{t_1,t_2\} \geq -0.5 \Longrightarrow \max\{t_1,t_2\} \geq -0.5 \geq f(xy) \Longrightarrow f(xy) \leq \max\{t_1,t_2\} \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[e]f$. Therefore $\frac{xy}{\max\{t_1,t_2\}}[e] \vee [c]f$.

Case II: For $[\alpha] = [c]$ i.e. $\frac{x}{t_1}[c]f$ and $\frac{y}{t_2}[c]f \Longrightarrow f(x)+t_1+1 < 0$ and $f(y)+t_2+1 < 0$. If $x \notin H$ or $y \notin H \Longrightarrow xy \notin H$ and $f(x) = 0$ or $f(y) = 0 \Longrightarrow t_1 < -1$ or $t_2 < -1$ which is not true. So $x,y \in H \Longrightarrow xy \in H$. Thus $f(xy) \leq -0.5$. Now if $\max\{t_1,t_2\} \geq -0.5$ then $\max\{t_1,t_2\} \geq -0.5 \geq f(xy) \Longrightarrow f(xy) \leq \max\{t_1,t_2\} \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[e]f$. But if $\max\{t_1,t_2\} < -0.5 \Longrightarrow f(xy)+\max\{t_1,t_2\}+1 < -0.5-0.5+1 = 0 \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[c]f \Longrightarrow \frac{xy}{\max\{t_1,t_2\}}[e] \vee [c]f$.

Case III: For $\alpha \in [e] \vee [c]$, this is obvious from Case I and Case II. Therefore, (S, f) is an $([\alpha],[e] \vee [c])$-sub LA-semigroup.

Theorem 16.*Let L be a left ideal of S and (S, f) be an N-structure such that*

(i) $f(x) = 0 \ \ \forall x \in S\backslash L$
(ii) $f(x) \leq -0.5 \ \ \forall x \in L$
Then (S, f) is a $([\alpha],[e] \vee [c])$-left ideal.

*Proof.*Let L be a left ideal of S such that
(i) $f(x) = 0 \ \forall x \in S\backslash L$ and (ii) $f(x) \leq -0.5 \ \forall x \in L$
Let $x,y \in S$ and $t \in [-1,0)$ such that $\frac{x}{t}[\alpha]f$ and $y \in S$.

Case I: For $[\alpha] = [e]$ i.e. $\frac{x}{t_1}[e]f \Longrightarrow f(x) \leq t_1$. For $x \in L$ and $y \in S \Longrightarrow xy \in L$. Thus $f(xy) \leq -0.5$. If $t < -0.5 \Longrightarrow f(xy)+t+1 < -0.5-0.5+1 = 0 \Longrightarrow \frac{xy}{t}[c]f$. But if $t \geq -0.5 \Longrightarrow t \geq -0.5 \geq f(xy) \Longrightarrow f(xy) \leq t \Longrightarrow \frac{xy}{t}[e]f$. Thus $\frac{xy}{t}[e] \vee [c]f$.

Case II: For $[\alpha] = [c]$ i.e. $\frac{x}{t_1}[c]f \Longrightarrow f(x)+t_1+1 < 0$. For $x \in L$ and $y \in S \Longrightarrow xy \in L$. Thus $f(xy) \leq -0.5$. If $t \geq -0.5$ then $t \geq -0.5 \geq f(xy) \Longrightarrow f(xy) \leq t \Longrightarrow \frac{xy}{t}[e]f$. But if $t < -0.5 \Longrightarrow f(xy)+t+1 < -0.5-0.5+1 = 0 \Longrightarrow \frac{xy}{t}[c]f \Longrightarrow \frac{xy}{t}[e] \vee [c]f$.

Case III: For $\alpha \in [e] \vee [c]$, This is obvious from Case I and Case II. Therefore, (S, f) is an $([\alpha],[e] \vee [c])$-left ideal of S.

Theorem 17.*Let L be an (resp. right) ideal of S and (S, f) be an N-structure such that*

(i) $f(x) = 0 \ \ \forall x \in S\backslash L$
(ii) $f(x) \leq -0.5 \ \ \forall x \in L$
Then (S, f) is a $([\alpha],[e] \vee [c])$-(resp. right) ideal.

*Proof.*Straightforward

Theorem 18.*For any subset A of S, let χ_A denotes the N-characteristic function of S, defined as $\chi_A : S \longrightarrow \{-1,0\}$*

$$\chi_A(x) = \begin{cases} -1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Then, χ_A is an $([e],[e] \vee [c])$-sub LA-semigroup if and only if A is a sub LA-semigroup.

*Proof.*Assume that χ_A is an $([e],[e] \vee [c])$-sub LA-semigroup. Let $x,y \in A \Longrightarrow \frac{x}{-1}[e]\chi_A$ and $\frac{y}{-1}[e]\chi_A \Longrightarrow \frac{xy}{\max\{-1,-1\}}[e] \vee [c]\chi_A$. Thus either $\frac{xy}{-1}[e]\chi_A$ or $\frac{xy}{-1}[c]\chi_A \Longrightarrow \chi_A(xy) \leq -1$ or $\chi_A(xy)+(-1)+1 < 0$. If $\chi_A(xy) \leq -1 \Longrightarrow xy \in A$. But if $\chi_A(xy)+(-1)+1 < 0 \Longrightarrow \chi_A(xy) < 0 \Longrightarrow \chi_A(xy) = -1 \Longrightarrow xy \in A$. Thus A is a sub LA-semigroup of S.

Conversely, Suppose that A is a sub LA-semigroup of S.

(i) Let $x,y \in A \Longrightarrow xy \in A$, so $\chi_A(xy) = -1, \chi_A(x) = -1, \chi_A(y) = -1$. So $\chi_A(xy) = \max\{\chi_A(x),\chi_A(y)\} \leq \max\{\chi_A(x),\chi_A(y),-0.5\}$

(ii) Let either $x \notin A$ or $y \notin A \Longrightarrow \chi_A(x) = 0$ or $\chi_A(y) = 0$. So $\chi_A(xy) \leq \max\{\chi_A(x), \chi_A(y)\} \leq \max\{\chi_A(x), \chi_A(y), -0.5\}$

(iii) Let $x, y \notin A \Longrightarrow \chi_A(x) = 0, \chi_A(y) = 0$. Thus $\chi_A(xy) \leq \max\{\chi_A(x), \chi_A(y)\} = \max\{\chi_A(x), \chi_A(y), -0.5\}$

Therefore, χ_A is an $([e], [e] \vee [c])$-sub LA-semigroup.

Theorem 19. *For any subset A of S, Let χ_A denotes the N-characteristic function of S, defined as $\chi_A : S \longrightarrow \{-1, 0\}$*

$$\chi_A(x) = \{.-1 \quad \text{if } x \in A 0 \quad \text{if } x \notin A$$

Then, χ_A is an $([e], [e] \vee [c])$-left(resp. right) ideal iff A is a left(resp. right) ideal.

Proof. Assume that χ_A is an $([e], [e] \vee [c])$-left ideal. Let $y \in A$ and $x \in H$. Then $\chi_A = -1 \Longrightarrow \frac{y}{-1}[e]\chi_A \Longrightarrow \frac{xy}{-1}[e] \vee [c]\chi_A$. Thus either $\frac{xy}{-1}[e]\chi_A$ or $\frac{xy}{-1}[c]\chi_A \Longrightarrow \chi_A(xy) \leq -1$ or $\chi_A(xy) + (-1) + 1 < 0$. If $\chi_A(xy) \leq -1 \Longrightarrow xy \in A$. But if $\chi_A(xy) + (-1) + 1 < 0 \Longrightarrow \chi_A(xy) < 0 \Longrightarrow \chi_A(xy) = -1 \Longrightarrow xy \in A$. Thus A is a left ideal of S.

Conversely, Suppose that A is a left ideal of S.

(i) Let $y \in A \Longrightarrow xy \in A$ and so $\chi_A(xy) = -1$, $\chi_A(y) = -1$. Thus $\chi_A(xy) = \chi_A(y) \leq \max\{\chi_A(y), -0.5\}$.

(ii) Let $y \notin A \Longrightarrow \chi_A(y) = 0$. Thus $\chi_A(xy) \leq \chi_A(y) = \max\{\chi_A(y), -0.5\}$.

Therefore, χ_A is an $([e], [e] \vee [c])$-left ideal. Similarly, we can show this for right ideal.

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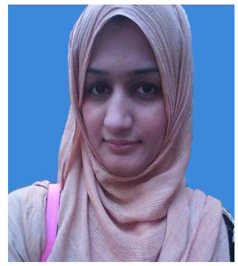

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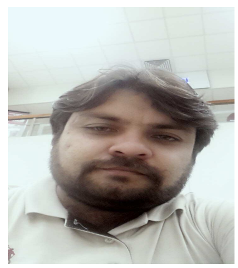

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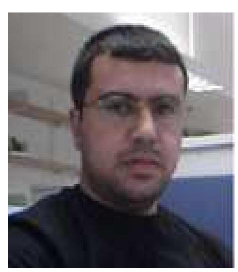

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