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Symmetric and Asymmetric Bayesian Estimation For Lindley Distribution Based on Progressive First Failure Censored Data

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Abstract: In this paper, we made point and interval estimation for Lindley distribution based on progressive first failure censoring by two methods: Maximum likelihood estimation (Mle) and Bayesian estimation . A comparison between Bayesian estimation under Symmetric and Asymmetric Loss Functions are obtained. Highest Posterior Density (HPD) interval and Approximate Confidence Interval (CI) are obtained.

Keywords: Dalgaard-Strulik model, energy, economic growth, time delay, limit cycle

1 Introduction

Censoring is very common in life tests. There are different types of censored tests. One of the most common censored test is Type-II censoring. It is noted that one can use Type-II censoring for saving time and money. However, when the lifetimes of products are very high, the experimental time of a Type-II censoring life test can be still too long. A generalization of Type-II censoring is the progressive Type-II censoring. [1] described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units and then run all the test units simultaneously until occurrence the first failure in each group. Such a censoring scheme is called first-failure censoring, [2] and [3] obtained Mle, exact confidence intervals and exact confidence regions for the parameters of the Gompertz and Burr Type-XII distributions based on first failure-censored sampling, respectively. For more reading one can refer to [4] and [5]. The first-failure censoring does not allow for sets to be removed from the test at the points other than the final termination point. however, this allowance will be desirable in practice. This leads us to the area of progressive censoring. [6] combine the concepts of first-failure censoring and progressive censoring to develop a new life test plan called a progressive first-failure censoring scheme. [7] studied the coefficient of variation of Gompertz distribution under progressive first-failure censoring. [8] and [9] introduced Mle, Bayesian estimates, exact confidence intervals and exact confidence regions for the parameters of Gompertz and Burr Type-XII distributions under progressive first failure-censored sampling.

Suppose that n independent groups with k items within each group are put on a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $Y^R_{1;m,n,k}$ has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure occurred $Y^R_{2;m,n,k}$, and finally when the m-th failure $Y^R_{m;m,n,k}$ is observed, the remaining groups $R_m, (m \leq n)$ are removed from the test. Then $Y^R_{1;m,n,k} < \dots < Y^R_{m;m,n,k}$ are called progressively first-failure censored order statistics with the progressive censored scheme,

$R = (R_1, R_2, ..., R_m)$, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with distribution function $F(y)$ and probability density function $f(y)$, the joint

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probability density function for $Y^R_{1;m,n,k}, Y^R_{2;m,n,k}, \ldots, Y^R_{m;m,n,k}$ is defined as follows:

$$f_{1,2,\ldots,m}(Y^R_{1;m,n,k}, Y^R_{2;m,n,k}, \ldots, Y^R_{m;m,n,k}) = A(n,m-1)k^m \prod_{i=1}^{m} f(Y^R_{i;m,n,k}) \left[1-F(Y^R_{i;m,n,k})\right]^{k(R_i+1)-1} \quad (1)$$

where $A(n,m-1) = n(n-R_1-1)\ldots(n-R_1-R_2-\ldots-R_{m-1}-(m-1))$, $0 < x_1 < x_2 < \ldots < x_m < \infty$.

Special cases:

1. Putting $k=1$ gives the progressively Type-II censored order statistics.
2. The complete sample case when $R = \{0,0,\ldots,0\}$ and $k=1$.

2 Lindley Distribution

The Lindley distribution was originally proposed by [10] in the context of Bayesian statistics, as a counter example of fudicial statistics. Assume that the random variable X representing the lifetime of a product has Lindley distribution with parameters θ. Lindley distribution has the following probability density function and cumulative distribution function respectively:

$$f(x) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}, x>0, \theta>0, \quad (2)$$

and

$$F(x) = 1-\left(1+\frac{\theta x}{1+\theta}\right)e^{-\theta x}, x>0, \theta>0. \quad (3)$$

Lindely distribution has many real life applications for example [11] have introduced real data represent the waiting times and fitting them. They proved that the Lindely distribution is better model than the exponential distribution. They also found that the maximum likelihood has a standard error reduced than the exponential distribution.

3 Maximum Likelihood Estimation

This section discussed the process of obtaining point and interval estimations of the parameter based on progressive first-failure censored data. Let $y_i = Y^R_{i;m,n,k}$ be the observed values of the lifetime y obtained from a progressive first-failure censoring scheme $R = (R_1, \ldots, R_m)$. Then the maximum likelihood function of the observations is:

$$L(\theta) = Ak^m \prod_{i=1}^{m} f(y_i)\left[1-F(y_i)\right]^{k(R_i+1)-1} \quad (4)$$

$$L(\theta) = Ak^m \prod_{i=1}^{m} \left[\frac{\theta^2(1+y_i)e^{-\theta y_i}}{1+\theta}\right]\left[\left(1+\frac{\theta y_i}{1+\theta}\right)e^{-\theta y_i}\right]^{k(R_i+1)-1} \quad (5)$$

The log likelihood function may have the form:

$$\begin{aligned}\ell(\theta) &= \log A + m\log k + 2\sum_{i=1}^{m}\log\theta - \sum_{i=1}^{m}\log(1+\theta)\\ &+ \sum_{i=1}^{m}\log(1+y_i) + \sum_{i=1}^{m}(k(R_i+1)-1)\\ &\times \log\left(1+\frac{\theta y_i}{1+\theta}\right) - \sum_{i=1}^{m}\theta y_i(k(R_i+1))\end{aligned} \quad (6)$$

Differentiating equation (6) with respect to θ and equating the equation to zero.

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial\theta} &= \frac{2m}{\theta} - \sum_{i=1}^{m} y_i k(R_i+1)\\ &+ \sum_{i=1}^{m}(k(R_i+1))\left(\frac{y_i}{((1+\theta)(1+\theta+\theta y_i))}\right)\\ &- \sum_{i=1}^{m}\left(\frac{y_i}{(1+\theta)(1+\theta+\theta y_i)} - \frac{m}{1+\theta}\right) = 0.\end{aligned} \quad (7)$$

Equation (7) can't be solved analytically, but can be solved by using Newton-Raphson method.

3.1 Approximate confidence interval

In this section we obtained the Approximate confidence interval for Lindely distribution parameter. The observed Fisher's information is given by $I(\hat{\theta}) = -\frac{\partial^2\ell(\theta)}{\partial\theta^2}$ at $\theta = \hat{\theta}$. The sampling distribution of $\frac{\hat{\theta}-\theta}{\sqrt{var(\hat{\theta})}}$ can be approximated by a standard normal distribution. When the sample size is large, the $(1-\gamma)$ confidence interval bounds for θ can be computed by : $(\hat{\theta}_L, \hat{\theta}_U) = \hat{\theta} \pm Z_{\frac{\gamma}{2}}\sqrt{var(\hat{\theta})}$.

4 Bayes Estimators under Symmetric and Asymmetric Loss Function

This section deals with obtaining the Bayesian estimation for the Lindely distribution parameter under different loss functions. [12] had studied the Bayesian estimation using Squared (SE) Error loss function for lindely distribution by using important sampling technique and Metropolis-Hasting algorithm .

In practical works the parameters cannot be treated as a constant during the life testing time. Therefore, it would be a fact to assume the parameters used in the life time model as random variables. We have also conducted a Bayesian study by assuming the following independent gamma prior for θ:

$$g(\theta)\alpha\theta^{a-1}e^{-b\theta}, \theta > 0 \tag{8}$$

Where a and b are hyperparameters and $a, b > 0$

4.1 Symmetric loss function (Squared Error Loss (SEL) function)

In this subsection, we made Bayesian estimation using Squared (SE) Error loss function. The likelihood function has the form :

$$L(\theta) = Ak^m \prod_{i=1}^{m} \left[\frac{\theta^2 (1+y_i) e^{-\theta y_i}}{1+\theta}\right] \left[\left(1+\frac{\theta y_i}{1+\theta}\right) e^{-\theta y_i}\right]^{k(R_i+1)-1} \tag{9}$$

Thus, the posterior density function of θ, given the data, is given by

$$\pi(\theta \mid x) = \frac{L(\theta)g(\theta \mid a,b)}{\int_0^\infty L(\theta)g(\theta \mid a,b)d\theta} \tag{10}$$

Therefore, the Bayes estimate of any function of θ say $h(\theta)$ under Squared Error loss function is

$$\hat{\theta}_{SE} = E_{(\theta|data)}[h(\theta)] = \frac{\int_0^\infty h(\theta)L(\theta)g(\theta \mid a,b)d\theta}{\int_0^\infty L(\theta)g(\theta \mid a,b)d\theta.} \tag{11}$$

The posterior density function is:

$$\pi(\theta \mid x) \propto L(\theta)g(\theta \mid a,b) \tag{12}$$

$$\pi(\theta \mid y) \propto \theta^{2m+a-1}\frac{Ak^m}{(1+\theta)^m}\prod_{i=1}^{m}(1+y_i)(1+\frac{\theta y_i}{(1+\theta)})^{k(R_i+1)-1} \times [\exp(-\theta(b+k\sum_{i=1}^{m} y_i(R_i+1)))]. \tag{13}$$

It is not possible to compute equation (11) analytically. The posterior density function cannot be reduced analytically to well known distributions. But its plot shows that it is similar to normal distribution. So, to calculate the integral that we cannot calculate it exact, we use the Metropolis-Hasting Algorithm with normal proposal distribution.

4.2 Asymmetric loss function

Asymmetric loss function may be more appropriate in some fields. Recently, many authors considered asymmetric loss functions in reliability and life testing. one of the most popular asymmetric loss functions is linear-exponential (LINEX) loss function (LI) which was introduced by [13]. It used in several papers, for example, [14], [15], [16] and [17]. This function is approximately linearly on one side and rises approximately to zero on the other side. Under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$, the LINEX loss function can be expressed as:

$$L_1(\delta) \propto \exp(c\delta) - c\delta - 1, \tag{14}$$

where $\delta = \hat{\theta} - \theta, \hat{\theta}$ is the estimate of $\theta, c \neq 0$.
The magnitude of c represent the direction, and degree of symmetry. Where $c > 0$ means overestimation is more serious than underestimation, and $c < 0$ means the opposite. For c close to zero the LINEX loss function is approximately the Squared Error Loss (SEL) function. The posterior expectation of the LINEX loss function of is :

$$E_\theta(L1(\hat{\theta} - \theta)) \propto \tag{15}$$

$$((\exp(c\theta))E_\theta[\exp(-c\theta)] - c(\hat{\theta} - E_\theta[\theta]) - 1. \tag{16}$$

The Bayes estimator under the LINEX loss function is the value of

$$\hat{\theta}_{LI} = \frac{-1}{c}\log(E_\theta[\exp(-c\theta)]), \tag{17}$$

such that $E_\theta[\exp(-c\theta)]$ exists.

Another asymmetric loss function called a General Entropy (GE) Loss function was proposed by [18] which can be expressed as:

$$L_2(\hat{\theta}, \theta) \propto [\frac{\hat{\theta}}{\theta}]^q - q\log\frac{\hat{\theta}}{\theta} - 1. \tag{18}$$

The weighted SEL function results from $q = -1$. The Bayes estimate $\hat{\theta}_{GE}$ under GEL function is

$$\hat{\theta}_{GE} = (E_\theta[\theta^{-q}])^{\frac{-1}{q}} \tag{19}$$

such that $E_\theta[\theta^{-q}]$ exists.

Since it is not possible to compute equation (16) and (18) analytically. we used the Markov chain Mont-Carlo (MCMC) method such as Metropolis-Hastings algorithm. to draw samples from the posterior density function and then to compute the Bayes estimate.

4.3 Metropolis-Hasting algorithm

The Metropolis algorithm was originally introduced by [19]. Suppose that our goal was to draw samples from some distributions $f(x|\theta) = vg(\theta)$, where v is the normalizing constant which may not be known or very difficult to compute. The MH algorithm provided a way of sampling from $f(x|\theta)$ without requiring us to know v. Let $q\left(\theta^{(b)}|\theta^{(a)}\right)$ be an arbitrary transition kernel: that is the probability of moving, or jumping, from current state $\theta^{(a)}$ to $\theta^{(b)}$. This is sometimes called the proposal distribution. The following algorithm generated a sequence of values $\theta^{(1)},\theta^{(2)},\dots,\theta^{(n)}$ which form a Markov chain with stationary distribution given by $f(x|\theta)$.

4.4 Algorithm

1. Start with $\theta^{(0)} = \theta_{MLE}$.
2. Set i=1.
3. Generate $\theta^{(*)}$ from the proposal distribution $N(\theta^{(i-1)}, var\theta^{(i-1)})$.
4. Calculate the acceptance probability $r(\theta^{(i-1)},\theta^{(*)}) = \min\left[1, \dfrac{\pi(\theta^{(*)})}{\pi(\theta^{(i-1)})}\right]$.
5. Generate U from uniform on (0, 1).
6. If $U < r(\theta^{(i-1)},\theta^{(*)})$ accept the proposal distribution and set $\theta^{(i)} = \theta^{(*)}$. Otherwise, reject the proposal distribution and set $\theta^{(i)} = \theta^{(i-1)}$.
7. Set $i = i+1$.
8. Repeat Steps $3-9$ N times.
9. Obtain the BEs of θ using MCMC under SEL function as $\hat{\theta}_{SE} = \sum_{i=M+1}^{N} \frac{1}{N-M}\theta^{(i)}$.
10. Obtain the BEs of θ using MCMC under LINEX function as $\hat{\theta}_{LI} = \frac{-1}{c}\log\frac{\sum_{i=M+1}^{N}\exp(-c\theta^{(i)})}{N-M}$.
11. Obtain the BEs of θ using MCMC under GE function $\hat{\theta}_{GE} = [\frac{\sum_{i=M+1}^{N}(\theta^{(i)})^{-q}}{N-M}]^{\frac{-1}{q}}$, where M is nburn units and N is the number of mcmc iterations.

4.5 Highest Posterior Density (HPD) interval algorithm

In Bayesian statistics, a credible interval is an interval in the domain of a posterior probability distribution used for interval estimation .The credible intervals are analogous to confidence intervals in frequentist statistics although they differ on a philosophical basis Bayesian intervals that treat their bounds as fixed and the estimated parameter as a random variable, whereas frequentist confidence intervals treat their bounds as random variables and the parameter as a fixed value. In this section, we described the algorithm for finding $(1-\gamma)$ HPD interval for θ (credible intervals). This algorithm proposed by [20]

1. Arrange the values of $\theta^{(*)}$ in ascending order.
2. Find the position of the lower bound which is $N[(N-M)*\gamma/2]$, M is the nburn.
3. The lower bound of θ is the observed value has the number in arrangement $\theta_{low}^{(*)} = \theta_{(\ N\ -\ M)*(\gamma/2)}^{(*)}$.
4. Find the position of upper bound which is $N[(N-M)*(1-(\gamma/2))]$.
5. The upper bound of θ is the observed value has the number in arrangement $\theta_{upp}^{(*)} = \theta_{(N-M)*(1-(\gamma/2))}^{(*)}$.
6. Repeat thea bove steps N times and every time find the average value of $\theta_{low}^{(*)}$ and $\theta_{upp}^{(*)}$.

Table 1: Censoring schemes

C.S	[1]	$m=20, n=50, R_1=R_2=R_3=10, R_4 = R_5=R_6=R_7=\dots=R_m=0$
C.S	[2]	$m=20, n=50, R_1=R_2=R_3=R_4 = R_5=R_6=5, R_7=R_8=\dots=R_m=0$
C.S	[3]	$m=30, n=70, R_1=R_2=R_3=R_4=10, R_5=\dots=R_m=0$
C.S	[4]	$m=30, n=70, R_1=R_2=20, R_3=R_4 =\dots=R_m=0$
C.S	[5]	$m=40, n=100, R_1=R_2=R_3=R_4=R_5 = R_6=10, R_7=\dots=R_m=0$
C.S	[6]	$m=60, n=140, R_1=R_2=R_3=R_4=R_5 = R_6=, R_7=R_8=10, R_9=\dots=R_m=0$

5 Simulation Study

This section deals with obtaining some numerical results. Maximum likelihood estimate (Mle) and Bayes estimates using LINEX (LI), Mean Square Error (SE) and General entropy (GE) loss functions with their Mean square error and 95 % confidence intervals (CI) and Highest Posterior Density Interval (HPD) with their widths for the parameter θ when $N = 10000$, $M = 1000, \theta = 3, a = 5, b = 1, A = 2, k = 2,3,5, c = 1, q = 1$.

Using the fact that the progressive first-failure censored sample with distribution function $F(x)$, can be viewed as a progressive Type-II censored sample from a population with distribution function $1-(1-F(x))^k$, we generate a progressively first-failure censored samples from the continuous random variable using the algorithm described in [21].

Table (1) contains censoring schemes (C.S) used in the simulation , Table(2) and Table(3), Table(4) contains the results concluded from the simulation study.

6 Conclusion

Point and interval estimation using Symmetric and Asymmetric Bayesian Estimation by two methods For Lindley Distribution parameter based on progressive first failure samples are derived and computed. Asymmetric Bayesian Estimation is always better than symmetric Bayesian Estimation, The Mean square error (Mse) of $\hat{\theta}_{LI}$ is always smaller than Mse of $\hat{\theta}_{SE}$, the HPD interval length and the Confidence Interval length of the parameter decreases as n, m increase, olso as the diffrence

Table 2: Estimators and MSE, CI, HPD

C.S	[1]	[2]	[3]	[4]	[5]	[6]
n	50	50	70	70	100	140
m	20	20	30	30	40	60
k	2	2	2	2	2	2
θ_{MlE}	3.091	3.131	3.077	3.047	3.041	3.031
θ_{SE}	2.896	2.921	2.952	2.926	2.954	2.97
θ_{LI}	2.771	2.799	2.867	2.926	2.891	2.928
θ_{GE}	2.807	2.836	2.893	2.933	2.911	2.941
Mse_{Mle}	0.3729	0.3472	0.2202	0.2574	0.1765	0.1106
Mse_{SE}	0.2689	0.2585	0.1963	0.2054	0.1537	0.1059
Mse_{LI}	0.263	0.2441	0.187	0.2013	0.1512	0.1037
Mse_{GE}	0.2758	0.257	0.1964	0.2083	0.1552	0.1059
$95\%CI$	$\{2.054,4.392\}$	$\{1.967,4.1\}$	$\{2.23,4.11\}$	$\{2.14,3.82\}$	$\{2.25,3.73\}$	$\{2.407,3.646\}$
Length	2.3376	2.21447	1.8758	1.792	1.542	1.275
$95\%CI\,forHPD$	$\{2.039,4.16\}$	$\{1.97,3.98\}$	$\{2.14,3.82\}$	$\{2.255,3.730\}$	$\{2.407,3.646\}$	$\{2.232,3.92\}$
Length	2.12906	2.01199	1.69547	1.678	1.474	1.239

Table 3: Estimators and MSE, CI, HPD

C.S	[1]	[2]	[3]	[4]	[5]	[6]
n	50	50	70	70	100	140
m	20	20	30	30	40	60
k	3	3	3	3	3	3
θ_{Mle}	3.06	3.192	3.083	3.063	3.023	3.044
θ_{SE}	2.905	2.973	2.984	2.964	2.931	2.98
θ_{LI}	2.777	2.838	2.936	2.895	2.863	2.932
θ_{GE}	2.814	2.88	2.913	2.917	2.884	2.947
Mse$_{Mle}$	0.4203	0.4581	.01941	0.1518	0.164	0.103
Mse$_{SE}$	0.2787	0.2869	0.155	0.1256	0.1443	0.09
Mse$_{LI}$	0.2691	0.2554	0.1465	0.1238	0.1439	0.0879
Mse$_{GE}$	0.2848	0.2768	0.1529	0.1271	0.1475	0.0899
95% CI	$\{1.991,4.217\}$	$\{2.051,4.357\}$	$\{2.113,3.850\}$	$\{2.23,4.100\}$	$\{2.13,3.88\}$	$\{2.41,3.66\}$
Length	2.22659	2.30549	1.73705	1.861	1.753	1.255
95% CI for HPD	$\{1.97,3.997\}$	$\{2.041,4.111\}$	$\{2.219,3.93\}$	$\{2.13,3.78\}$	$\{2.119,3.734\}$	$\{2.40,3.623\}$
Length	2.018	2.070	$\{2.219,3.93\}$	$\{2.13,3.78\}$	$\{2.119,3.734\}$	$\{2.40,3.623\}$

Table 4: Estimators and MSE, CI, HPD

C.S	[1]	[2]	[3]	[4]	[5]	[6]
n	50	50	70	70	100	140
m	20	20	30	30	40	60
k	5	5	5	5	5	5
θ_{MlE}	3.104	3.083	3.055	3.062	3.041	3.055
θ_{SE}	2.898	2.8854	2.924	2.962	2.954	2.992
θ_{LI}	2.926	2.759	2.835	2.894	2.891	2.945
θ_{GE}	2.77	2.796	2.862	2.915	2.911	2.96
Mse_{Mle}	0.502	0.381	0.2347	0.145	0.1179	0.0956
Mse_{SE}	0.309	0.2711	0.1875	0.1189	0.1015	0.0829
Mse_{LI}	0.2761	0.2682	0.1849	0.1171	0.1	0.0799
Mse_{GE}	0.3007	0.2792	0.1912	0.1202	0.1023	0.0821
$95\%CI$	1.98,4.19	1.99,4.2	2.16,3.94	2.291,3.83	2.359,3.727	2.427,3.682
Length	2.20442	2.21509	1.7755	1.54008	1.36855	1.254
$95\%CI\,forHPD$	$\{1.98,3.98\}$	$\{2.02,3.84\}$	$\{2.154,3.822\}$	$\{2.279,3.739\}$	$\{2.3,3.6\}$	$\{2.416,3.628\}$
Length	1.9965	1.8175	1.66847	1.46057	1.31821	1.2121

between n,m deacreses the Mse error of the paramter deacreses, all the results concludede are :

1. For all censoring schemes and $k = 2,3,5$ as n,m increase the Mse of $\hat{\theta}_{Mle}$, $\hat{\theta}_{SE}$, $\hat{\theta}_{LI}$, $\hat{\theta}_{GE}$ decrease.
2. For all censoring schemes and $k = 2,3,5$ as n,m increase the HPD interval length and the Confidence Interval length of the parameter decreases.
3. The Mse of $\hat{\theta}_{SE}$, $\hat{\theta}_{LI}$, $\hat{\theta}_{GE}$ (bayesian estimators) is always smaller than Mse of $\hat{\theta}_{Mle}$.
4. The Mse of $\hat{\theta}_{SE}$, $\hat{\theta}_{LI}$ is always smaller than Mse of $\hat{\theta}_{GE}$.
5. The Mse of $\hat{\theta}_{LI}$ is always smaller than Mse of $\hat{\theta}_{SE}$.

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