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The Epsilon-Skew Exponentiated Beta Distribution

Chary Akmyradov[1,*] *and Hassan Elsalloukh*[2]

[1] Arkansas Children's Research Institute, 13 Children's Way Slot 842, Little Rock, AR, 72202, U.S.
[2] Mathematics and Statistics Department, University of Arkansas at Little Rock, 2801 S University Ave., Little Rock, AR, 72204, U.S.

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Abstract: In this paper, we develop the Epsilon-Skew Exponentiated Beta Distribution (ESEB). This distribution is a bimodal skew distribution with shape, location, and skewness parameters. This distribution is useful in applications with bimodal and skew data, such as in the biological data. The shape properties and the effects of parameters are discussed. Finally, we derived the mode, the expected values, and mean and variance.

Keywords: Exponentiated Beta, Epsilon Skew, Bimodal, Distribution theory, Gamma functions

1 Introduction

The Standard Beta distribution, which was first introduced by [23] as Pearson's Type I distribution, has applications in a variety of disciplines such as genetics, time allocation in control systems, geology, etc. It was generalized by [4] to a more flexible domain. [13] studied the Beta-Normal distribution as a special case of a more general class of distributions which is generated from the logit of the beta random variable. The Beta-Normal distribution has four parameters and can model both symmetric heavy-tailed and bimodal distributions. The Exponentiated Exponential distribution was introduced by [15] as a generalization of the Standard Exponential distribution and as an alternative to the Weibull and Gamma distributions. [20] used the transformation of $Y = -logX$ for the random variable $X \sim Beta(\alpha, \beta)$ and derived the Exponentiated Beta Distribution which has a skew unimodal shape. The Modified Beta distribution family is another family of distributions obtained by [21]. If we set $\beta = 1$ and use the Standard Normal distribution in the Modified Beta family we derive the Beta-Normal distribution. Similarly, Skew Normal distribution [2] and the Exponentiated Beta distribution are the special cases of the Modified Beta distribution family. The Epsilon Skew Normal distribution was introduced by [19], in which its explicit skewness parameter, ε, provides flexibility to controlling the skewness. This distribution was later generalized to the Epsilon Skew Exponential Power distribution family by [12]. Later, [14] developed the Epsilon Skew Exponential Power Tobit regression model. The Epsilon Skew Laplace Distribution was developed by [6] and was generalized later to the Multivariate Epsilon Skew Laplace distribution by [9]. A test for skewness within the Univariate and Multivariate Epsilon Skew Laplace distribution was developed by [11]. Also, a new regression model was developed by [10] using the Epsilon Skew Laplace distribution for the errors which made it easier to fit data when residuals are not normal or heavy tailed. [5] studied the conditions of bimodality for the mixture of two normal distributions. The Bimodal Skew-Symmetric Normal distribution was studied by [16]. Another bimodal distribution called Exponentiated Log-Sinh Cauchy distribution is proposed by [24]. [1] proposed several bimodal distributions such as the Bimodal Asymmetric Power Normal, Bimodal Beta-generated Power Normal, Asymmetric Rathie-Swamee, and Beta-generated Rathie-Swamee distributions which were generated either by using the asymmetrization strategy of [2] or [18]. Bimodal Birnbaum-Sanders distribution was proposed by [22]. In addition, the Odd Log-Logistic Skew-Normal distribution was proposed by [3] to model asymmetric and bimodal data. We have seen that embedding a skewness parameter ε may result in a bimodal distribution. Asymmetric and bimodal distributions such as the Epsilon Skew Gamma and Epsilon Skew Inverted Gamma distributions were developed by [7] and [8].

* Corresponding author e-mail: akmyradovc@archildrens.org

2 The Epsilon Skew Exponentiated Beta Distribution

2.1 Probability Density Function of ESEB

Definition 1.*The random variable X has ESEB distribution denoted by $X \sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, if there exist parameters $0<\alpha$, $0<\beta$, $\theta \in \mathbb{R}$, $0<\sigma$, and $-1<\varepsilon<1$ such that the pdf of X is*

$$f(x) = \frac{1}{2B(\alpha,\beta)\sigma} \begin{cases} \exp(-\frac{(x-\theta)\alpha}{(1-\varepsilon)\sigma})\left(1-\exp(-\frac{x-\theta}{(1-\varepsilon)\sigma})\right)^{\beta-1} & ,x \geq \theta \\ \exp(-\frac{(\theta-x)\alpha}{(1+\varepsilon)\sigma})\left(1-\exp(-\frac{\theta-x}{(1+\varepsilon)\sigma})\right)^{\beta-1} & ,x < \theta, \end{cases} \quad (1)$$

where θ, σ, and ε are location, scale, and skewness parameters respectively. Note that the pdf in (1) satisfies the following two properties.

i. $f(x) \geq 0$
ii. $\int_{-\infty}^{\infty} f(x)dx = 1$

Figure (1) - (3) are the graphs of (1) with different parameters and they verify that the value of (1) is non-negative. In addition, when we integrate (1) over $(-\infty,\infty)$ we get $P(X<\theta) = \frac{1+\varepsilon}{2}$ and $P(X \geq \theta) = \frac{1-\varepsilon}{2}$. Therefore, the second property is also valid.

*Proof.*Substituting $u_1 = \exp\left(-\frac{\theta-x}{(1+\varepsilon)\sigma}\right)$ and $u_2 = \exp\left(-\frac{x-\theta}{(1-\varepsilon)\sigma}\right)$ when $x<\theta$ and $x>\theta$ in (1) respectively results in $P(X<\theta) = \frac{1+\varepsilon}{2}$ and $P(X \geq \theta) = \frac{1-\varepsilon}{2}$.

*Property 1.*Here we can also conclude that

$$P(X<\theta) = \frac{1+\varepsilon}{2} \quad (2)$$

and

$$P(X \geq \theta) = \frac{1-\varepsilon}{2}. \quad (3)$$

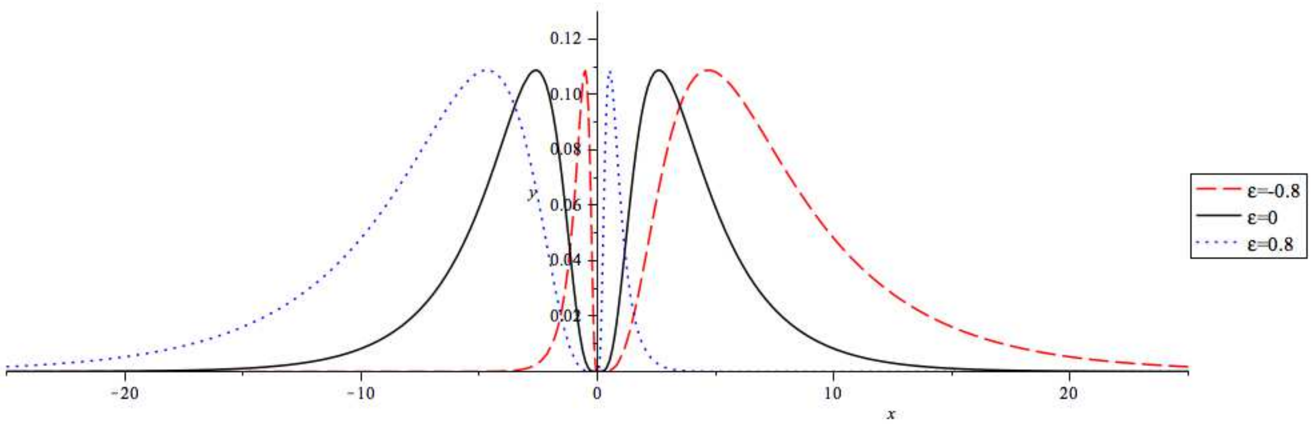

Fig. 1: ESEB with $\theta = 0, \sigma = 1, \alpha = 0.4, \beta = 6$

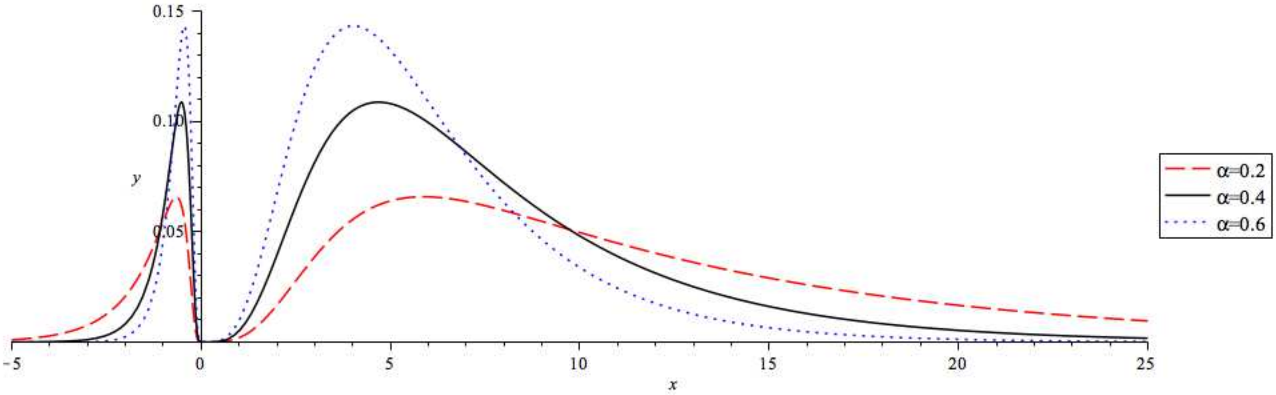

Fig. 2: ESEB with $\theta=0,\sigma=1,\varepsilon=-0.8,\beta=6$

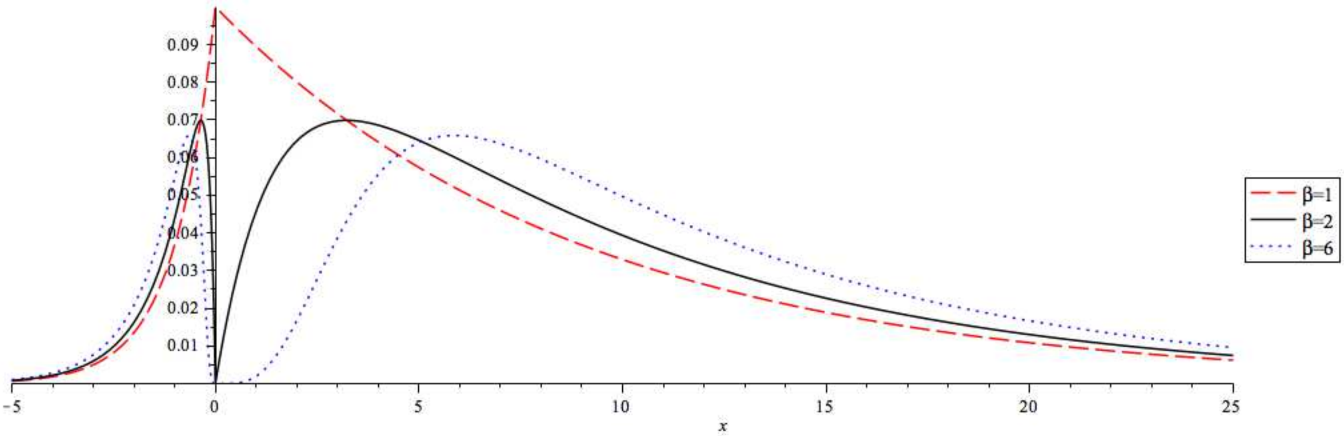

Fig. 3: ESEB with $\theta=0,\sigma=1,\varepsilon=-0.8,\alpha=0.2$

2.2 Cumulative Distribution Function of ESEB

Definition 2.*Let $X\sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, then the cumulative distribution function of ESEB is defined as*

$$F(x)=\begin{cases}1-\frac{1-\varepsilon}{2}I(\exp(-\frac{x-\theta}{(1-\varepsilon)\sigma});\alpha,\beta) & ,x\geq\theta\\ \frac{1+\varepsilon}{2}I(\exp(-\frac{\theta-x}{(1+\varepsilon)\sigma});\alpha,\beta) & ,x<\theta\end{cases}, \quad (4)$$

where $I(z;\alpha,\beta)$ is called Regularized Beta function [25] which is defined as

$$I(z;\alpha,\beta)=\frac{B(z;\alpha,\beta)}{B(\alpha,\beta)},$$

with Incomplete Beta function defined as

$$B(z;\alpha,\beta)=\int_0^z u^{\alpha-1}(1-u)^{\beta-1}du.$$

*Proof.*Considering the two cases of ESEB at the left and right hand side of θ and use of (2) simplify the CDF. Further, with the $u=\exp(-\frac{\theta-t}{(1+\varepsilon)\sigma})$ and $u=\exp(-\frac{t-\theta}{(1-\varepsilon)\sigma})$ substitutions in both cases simplify the integral into the form of Regularized Beta function.

3 Digamma and Polygamma Functions

3.1 Gamma Function

One of the earliest definitions of the Gamma function as it was mentioned in the book by [17] is

$$\Gamma(x) = \lim_{r\to\infty} \Gamma_r(x),$$

where

$$\Gamma_r(x) = \frac{r!r^x}{x(1+x)(2+x)\cdots(r+x)}.$$

3.2 Digamma and Polygamma Functions

The digamma function is defined as the logarithmic derivative of Gamma function. In the book by [17], the digamma function $\psi(x)$ is clearly defined and simplified as

$$\psi(x) = -\gamma + \sum_{r=1}^{\infty}\left(\frac{1}{r} - \frac{1}{r+x-1}\right) \tag{5}$$

with

$$\gamma = \lim_{r\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{r} - \ln r\right).$$

Then

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ln B(\alpha,\beta) &= \frac{\partial}{\partial\alpha}\ln\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\partial}{\partial\alpha}\ln\Gamma(\alpha) - \frac{\partial}{\partial\alpha}\ln\Gamma(\alpha+\beta).\\ &= \psi(\alpha)-\psi(\alpha+\beta).\end{aligned} \tag{6}$$

Let's notate (6) as $\Psi(\alpha,\beta) = \psi(\alpha)-\psi(\alpha+\beta)$ and $\Psi^{(n)}(\alpha,\beta) = \frac{\partial^n}{\partial\alpha^n}\Psi(\alpha,\beta)$ for convenience where the latter is called the Polygamma function of order n. We make use of the function (5) in the following lemmas to simplify the expression in (6) and the Polygamma function of order n.

Lemma 1.*Difference of two digamma functions at α and $\alpha+\beta$ is*

$$\Psi(\alpha,\beta) = \psi(\alpha)-\psi(\alpha+\beta) = \sum_{r=1}^{\infty}\left(\frac{1}{\alpha+\beta+r-1} - \frac{1}{\alpha+r-1}\right). \tag{7}$$

*Proof.*Plugging α and $\alpha+\beta$ in (5) then subtracting them from each other results in (7).

Lemma 2.*nth degree partial derivative of (7) with respect to α is*

$$\Psi^{(n)}(\alpha,\beta) = (-1)^n n! \sum_{r=1}^{\infty}\left(\frac{1}{(\alpha+\beta+r-1)^{n+1}} - \frac{1}{(\alpha+r-1)^{n+1}}\right). \tag{8}$$

*Proof.*Taking the n^{th} derivative of (7) with respect to α concludes the proof.

Lemma 3.*Let $X \sim Beta(\alpha,\beta)$, then logarithmic n^{th} moment of Beta distribution can be written as*

$$E(\ln X)^n = \sum_{i=0}^{n-1}\Psi^{n-1-i}(\alpha,\beta)\Psi^{(i)}(\alpha,\beta). \tag{9}$$

Proof.

$$\begin{aligned}E(\ln X)^n = \frac{1}{B(\alpha,\beta)}\int_0^1(\ln u)^n u^{\alpha-1}(1-u)^{\beta-1}du &= \frac{1}{B(\alpha,\beta)}\frac{\partial^n}{\partial\alpha^n}\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du\\ &= \frac{1}{B(\alpha,\beta)}\frac{\partial^n}{\partial\alpha^n}B(\alpha,\beta).\end{aligned}$$

Let $B(n) = E(\ln X)^n$ for non-negative integers n, then for $n = 0$

$$B(0) = \frac{1}{B(\alpha,\beta)}\frac{\partial^0}{\partial\alpha^0}B(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}B(\alpha,\beta) = 1, \tag{10}$$

and for $n = 1$

$$B(1) = \frac{1}{B(\alpha,\beta)}\frac{\partial}{\partial\alpha}B(\alpha,\beta) = \Psi(\alpha,\beta), \tag{11}$$

then from (11):

$$\frac{\partial}{\partial\alpha}B(\alpha,\beta) = B(\alpha,\beta)\Psi(\alpha,\beta). \tag{12}$$

Next, we have

$$B(n) = \frac{1}{B(\alpha,\beta)}\frac{\partial^n}{\partial\alpha^n}B(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\left(\frac{\partial}{\partial\alpha}B(\alpha,\beta)\right) = \frac{1}{B(\alpha,\beta)}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\left(B(\alpha,\beta)\Psi(\alpha,\beta)\right),$$

which results in the following recursive relationship:

$$B(n) = \Psi^2(\alpha,\beta)B(n-2) + \Psi(\alpha,\beta)\Psi^{(n-2)}(\alpha,\beta) + \Psi^{(n-1)}(\alpha,\beta).$$

Then, for a non-negative integer $i \le n-1$ and by (10) and (11)

$$\begin{aligned} B(n) &= \Psi^i(\alpha,\beta)B(n-i) + \Psi^{i-1}(\alpha,\beta)\Psi^{(n-i)}(\alpha,\beta) + \cdots + \Psi^{(n-1)}(\alpha,\beta) \\ &= \sum_{i=0}^{n-1}\Psi^{n-1-i}(\alpha,\beta)\Psi^{(i)}(\alpha,\beta). \end{aligned}$$

4 Properties of ESEB

4.1 The Modes of ESEB

Proposition 1. *If $X \sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, then the modes when $x < \theta$ and when $x \ge \theta$ are respectively at*

$$x = \theta + (1+\varepsilon)\sigma\ln\left(\frac{\alpha}{\alpha+\beta-1}\right), \tag{13}$$

and

$$x = \theta + (\varepsilon-1)\sigma\ln\left(\frac{\alpha}{\alpha+\beta-1}\right). \tag{14}$$

Proof. When $x < \theta$, setting and solving $\left.\frac{\partial f}{\partial x}\right|_{x<\theta} = 0$ in (1) and when $x > \theta$, setting and solving $\left.\frac{\partial f}{\partial x}\right|_{x>\theta} = 0$ in (1) we have (13) and (14).

4.2 Nth Moment of the ESEB

Theorem 1. *If $X \sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, then*

$$E(X^n) = \sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\sum_{i=0}^{n-k-1}\binom{n}{k}\binom{n+1-k}{2j-1} \theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\Psi^{n-k-1-i}(\alpha,\beta)\Psi^{(i)}(\alpha,\beta). \tag{15}$$

*Proof.*Substituting $u_1 = \exp\left(-\frac{\theta-x}{(1+\varepsilon)\sigma}\right)$ and $u_2 = \exp\left(-\frac{x-\theta}{(1-\varepsilon)\sigma}\right)$ in the first and second integrals of $E(X^n)$ and reorganizing the terms, simplifications with the binomial expansion, and converting both u_i-s into u for simplicity we obtain

$$\begin{aligned}
E(X^n) &= \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\binom{n}{k}\binom{n+1-k}{2j-1}\theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\int_0^1(\ln u)^{n-k}u^{\alpha-1}(1-u)^{\beta-1}du \\
&= \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\binom{n}{k}\binom{n+1-k}{2j-1}\theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\int_0^1\frac{\partial^{n-k}}{\partial\alpha^{n-k}}u^{\alpha-1}(1-u)^{\beta-1}du \\
&= \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\binom{n}{k}\binom{n+1-k}{2j-1}\theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\frac{\partial^{n-k}}{\partial\alpha^{n-k}}\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du \\
&= \frac{1}{B(\alpha,\beta)}\sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\binom{n}{k}\binom{n+1-k}{2j-1}\theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\frac{\partial^{n-k}}{\partial\alpha^{n-k}}B(\alpha,\beta) \\
&= \sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\binom{n}{k}\binom{n+1-k}{2j-1}\theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\frac{1}{B(\alpha,\beta)}\frac{\partial^{n-k}}{\partial\alpha^{n-k}}B(\alpha,\beta).
\end{aligned}$$

By Lemma (3)

$$E(X^n) = \sum_{k=0}^{n}\sum_{j=1}^{\lceil\frac{n+1-k}{2}\rceil}\binom{n}{k}\binom{n+1-k}{2j-1}\theta^k\sigma^{n-k}\varepsilon^{n+2-k-2j}\sum_{i=0}^{n-k-1}\Psi^{n-k-1-i}(\alpha,\beta)\Psi^{(i)}(\alpha,\beta),$$

further organization of terms leads to the expression at (15).

4.3 Mean of the ESEB

Corollary 1.*Setting $n = 1$ in Theorem* (1) *leads to*

$$E(X) = \theta + 2\sigma\varepsilon\Psi(\alpha,\beta). \tag{16}$$

Proposition 2.*If $X \sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, then*

$$E(X) = \theta + 2\sigma\varepsilon\sum_{r=1}^{\infty}\left(\frac{1}{\alpha+\beta+r-1} - \frac{1}{\alpha+r-1}\right). \tag{17}$$

*Proof.*Plugging expression (7) in (16) concludes our proof.

Corollary 2.*Setting $n = 2$ in Theorem* (1) *leads to*

$$E(X^2) = \theta^2 + 4\theta\varepsilon\sigma\Psi(\alpha,\beta) + \sigma^2(1+3\varepsilon^2)(\Psi^{(1)}(\alpha,\beta) + \Psi^2(\alpha,\beta)). \tag{18}$$

4.4 Variance of the ESEB

Proposition 3.*If $X \sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, then*

$$Var(X) = \sigma^2(1+3\varepsilon^2)\Psi^{(1)}(\alpha,\beta) + \sigma^2(1-\varepsilon^2)\Psi^2(\alpha,\beta). \tag{19}$$

*Proof.*By plugging the results of corollaries (1) and (2) in $Var(X) = E(X^2) - E^2(X)$ and by reorganizing the terms, we derive the variance of X.

4.5 Moment Generating Function of the ESEB

Theorem 2. *If $X \sim ESEB(\alpha,\beta,\theta,\sigma,\varepsilon)$, then the moment generating function is*

$$M_X(t) = \frac{e^{t\theta}}{2}\left(\frac{(1+\varepsilon)B(\alpha+\beta,(1+\varepsilon)\sigma t)}{B(\alpha,(1+\varepsilon)\sigma t)} - \frac{(\varepsilon-1)B(\alpha+\beta,(\varepsilon-1)\sigma t)}{B(\alpha,(\varepsilon-1)\sigma t)}\right). \tag{20}$$

Proof. If we replace X^n with e^{tX}, perform the same u substitution similarly in the proof of Theorem 1, and after further simplifications we obtain:

$$\begin{aligned} M_X(t) &= \frac{e^{t\theta}}{2B(\alpha,\beta)}\left((1+\varepsilon)B((1+\varepsilon)\sigma t+\alpha,\beta)-(\varepsilon-1)B((\varepsilon-1)\sigma t+\alpha,\beta)\right) \\ &= \frac{e^{t\theta}\Gamma(\alpha+\beta)}{2\Gamma(\alpha)\Gamma(\beta)}\left(\frac{(1+\varepsilon)\Gamma((1+\varepsilon)\sigma t+\alpha)\Gamma(\beta)}{\Gamma((1+\varepsilon)\sigma t+\alpha+\beta)} - \frac{(\varepsilon-1)\Gamma((\varepsilon-1)\sigma t+\alpha)\Gamma(\beta)}{\Gamma((\varepsilon-1)\sigma t+\alpha+\beta)}\right). \end{aligned}$$

When we reorganize and insert $\Gamma((1+\varepsilon)\sigma t)$ and $\Gamma((\varepsilon-1)\sigma t)$ to the respective fractions' numerator and denominator we obtain (20).

Theorem 3. *If $X \sim ESEB(\alpha,\beta,\varepsilon,\theta,\sigma)$, then the central n^{th} moment is*

$$E(X-\theta)^n = \sigma^n \sum_{j=1}^{\lceil \frac{n+1}{2} \rceil} \sum_{i=0}^{n-1} \binom{n+1}{2j-1} \varepsilon^{n+2-2j} \Psi^{n-1-i}(\alpha,\beta)\Psi^{(i)}(\alpha,\beta). \tag{21}$$

Proof. Performing the same u substitution as in Theorem 1's proof, and after reorganizing the terms we obtain:

$$\begin{aligned} E(X-\theta)^n &= \frac{\sigma^n}{2B(\alpha,\beta)}\left((1+\varepsilon)^{n+1}-(\varepsilon-1)^{n+1}\right)\int_0^1 (\ln u)^n u^{\alpha-1}(1-u)^{\beta-1}du \\ &= \frac{\sigma^n}{2B(\alpha,\beta)} 2\sum_{j=1}^{\lceil \frac{n+1}{2} \rceil}\binom{n+1}{2j-1}\varepsilon^{n+2-2j}\int_0^1 (\ln u)^n u^{\alpha-1}(1-u)^{\beta-1}du, \end{aligned}$$

then by Lemma 3 we conclude the proof.

5 Discussion

Overall, in this paper we develop the ESEB distribution. We derive the properties of ESEB. The *ESEB* density function has both modes at the equal height. This is due to the fact that we weighted each side of the function equally. For further studies, we may develop generalizations of the *ESEB* by introducing a weight parameter to control each mode's height. Additionally, application of the ESEB is being developed.

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Chary Akmyradov is a Senior Biostatistician, Arkansas Children's Research Institute, 13 Children's Way, Slot 842, Little Rock, Arkansas, 72202

Hassan Elsalloukh is an Associate Professor, Department of Mathematics and Statistics, University of Arkansas at Little Rock, 2801 South University Avenue, 616 DKSN, Little Rock, Arkansas, 72204