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Generalized Solution to the Second-Order Quasi-Linear Elliptic Partial Differential Equation under Form-Boundary Conditions in the Euclidean Space $R^l, \quad l \geq 3$

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Abstract: We study the existence of a generalized solution of a second-order quasi-linear elliptic partial differential equation under new form-boundary conditions on its coefficients; prove the analog of the Minty-Browder theorem for operator generated by this elliptic differential equation.

Keywords: elliptic partial differential equation, general solution, Minty-Browder's theorem, form-boundary condition, semi-group, monotone operator

1 Introduction and main results

In this paper, we study a new class of differential operators and apply these operators to establish the existence of a weak general solution of a classic second-order quasi-linear elliptic partial differential equation under the new conditions on its coefficients [1 - 5, 21-24] ([1 - 42]).

Partial differential equations have been being studied for a long time and there is extensive literature on the conditions on their coefficients under which there are the solutions of these equations in a specific functional space [6, 18, 44, 23]. Let us accentuate several fundamental works the most relevant to the present article. A general framework was established in the Hilbert program in 1900 in so-called Hilbert's problems 19 and 20, these problems address the questions about existence and regularity of solutions of boundary value problems. These problems were studied and partially solved by S. Bernstein, J. Serrin [37], G. Stampacchia, Poincare and officially resolved by Ennio de Giorgi and, John Forbes Nash [29]. Ennio de Giorgi's method was being developed by O. Ladyzhenskaya, N. Uraltseva, O.A. Solonnikov [21-23]. In 1960, J. Moser applied the maximum principle and created a new method of studying the regularity of the solutions of elliptic differential equations and Harnack's inequality [27, 23] under the assumption that the coefficients are bounded measurable and satisfy a uniform ellipticity condition, these results also were developed by O. Ladyzhenskaya, N. Uraltseva, O.A. Solonnikov.

John Forbes Nash's method has been less popular for a long time and only relatively recently obtain due attention in works U.A. Semenov, L. Hormander, M. Clement, C. Villani, H. Lindblad, and others. Operator approaches were developed by G.Minty in the 1960s [24, 25], he studied maximal monotone operators, in 1963 M. I. Visik introduced the class of elliptic differential operators in the generalized divergence form, F.Browder and H. Brezis [25-27] studied pseudo-monotone operators, T. Kato, Y. Komura [16-20], M. Crandall and A. Pazy [9] generalized the Hille-Yosida-Komura theory in Hilbert spaces [16, 19, 46], which proves correspondence between continuous semi-groups of contractions and maximal monotone operators in Hilbert space, next, was proven that in Banach space m-dissipative operators generate contraction semi-groups [26, 46]. I. Miyadera explored the Komura theorem and the Crandall-Liggett theorem, studied the Kobayashi generation theorem of nonlinear semigroups [26]. The important results of nonlinear problems have been

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obtained in the works of I.V. Skripnik, M.M. Kukharchuk, who introduced the operators that are studying in this paper [39, 40].

Let X ***be given Banach space*** and its dual or adjoint space X^* that consists of all bounded linear functionals from X to R and endowed with the operator norm defined by

$$\|f\|^* = \sup_{0\neq x\in X} \frac{|\langle f,x\rangle|}{\|x\|} = \sup_{\|x\|=1} |\langle f,x\rangle| .$$

The Banach space is called reflexive if the natural map

$$\begin{cases} F_X : X \to X^{**} \\ F_X(x)(f) = f(x) \quad \forall x\in X,\ \forall f\in X^* \end{cases}$$

is surjective or mapping "onto" [46].

The relevant exemplars of Banach reflexive spaces are Lebesgue and Sobolev spaces, their importance can be justified by their extensive and fundamental applications in the theory of differential equations and other branches of mathematics.

A Lebesgue space $L^p\left(R^l,\, d^l x\right)$ for $1<p<\infty$ can be defined as a set of all real-valued measurable functions defined almost everywhere such that the Lebesgue integral of its absolute value raised to the p- th power is a finite number with its natural norm

$$\|u\|_{L^p} = \left(\int |u(x_1,...,x_n)|^p\, d^l x\right)^{\frac{1}{p}} =$$

$$= \left(\int_{R^l} |u(x)|^p\, d^l x\right)^{\frac{1}{p}} = \langle |u|^p\rangle^{\frac{1}{p}} .$$

The dual or adjoint space of $L^p\left(R^l,\, d^l x\right)$ for $1<p<\infty$ has a natural isomorphism with $L^q\left(R^l,\, d^l x\right)$, where $\frac{1}{p}+\frac{1}{q}=1$ or $q=\frac{p}{p-1}$.

We will use the inequality

$$\langle f,g\rangle \le \|f\|_p \|g\|_q \le \frac{\varepsilon^p}{p}\|f\|_p^p + \frac{1}{\varepsilon^q q}\|g\|_q^q ,$$

where $f\in L^p(R^l),\ g\in L^q(R^l),\ \varepsilon>0$, and its consequence

$$\left\langle f, f|f|^{p-2}\right\rangle = \|f\|_{L^p(R^l)} \left\|f|f|^{p-2}\right\|_{L^q(R^l)} =$$

$$= \frac{1}{p}\|f\|_{L^p(R^l)}^p + \frac{1}{q}\left\|f|f|^{p-2}\right\|_{L^q(R^l)}^{p-1} = \|f\|_{L^p(R^l)}^p ,$$

the $f\in L^p$ yields $f|f|^{p-2}\in L^q$ that justify the last equation [46].

Let us denote $W_k^p\left(R^l,\, d^l x\right)$ given Sobolev space for $1<p<\infty$ with a natural norm

$$\|u\|_{W_k^p} = \left(\sum_{i=0}^{k}\int \left|u^{(i)}(x_1,...,x_n)\right|^p d^l x\right)^{\frac{1}{p}} =$$

$$= \left(\|u\|_p^p + \sum_{1\le|s|\le m}\|D^s u\|^p\right)^{\frac{1}{p}} = \left(\sum_{i=0}^{k}\left\|u^{(i)}\right\|_p^p\right)^{\frac{1}{p}}$$

and if $p=\infty$ we have

$$\|u\|_{W_k^\infty} = \max_{i=0,1,..,k}\left(ess\ \sup_{R^l}\left|u^{(i)}(x_1,...,x_n)\right|\right) =$$

$$= \max_{i=0,1,..,k}\left\|u^{(i)}\right\|_\infty .$$

The norm of $W_k^p\left(R^l,\, d^l x\right)$ space is equivalent to the norm

$$\|u\|_{\tilde{W}_k^p} = \|u\|_p + \left\|u^{(i)}\right\|_p$$

that takes into consideration only the firth and the last summands of the $W_k^p\left(R^l,\, d^l x\right)$ norm.

The dual space of $W_k^p\left(R^l,\, d^l x\right)$ for $1<p<\infty$ is $W_{-k}^q\left(R^l,\, d^l x\right)$, and the dual space of $W_{-k}^p\left(R^l,\, d^l x\right)$ for $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$ is $W_k^q\left(R^l,\, d^l x\right)$, Sobolev spaces are reflexive.

To explain the difference between classical theory and the method which is developing in present work we need to formulate Minty-Browder's theorem.

Theorem 1.*(Minty-Browder).*

A bounded, continuous, coercive and monotone operator Q ***from a real, separable reflexive Banach space*** X***into its adjunct space*** X^* ***is surjective. That means that for each continuous linear functional*** $f\in X^*$ ***exists a solution*** $g\in X$ ***of the equation***

$$Q(g) = f.$$

In the conditions of Minty-Browder's theorem, the operator $Q:\ X\to X^*$ maps a separable reflexive Banach space Xinto its dual space X^*. If we assume $X=W_1^p\left(R^l,\, d^l x\right)$ it is automatically following that X^* must be in $W_{-1}^q\left(R^l,\, d^l x\right)$, and let denote $Q=A$ then we have that in order Minty-Browder's theorem was true the operator must be bounded, continuous, coercive and monotone operator and mapping as

$$A:\ W_1^p\left(R^l,\, d^l x\right) \to W_{-1}^q\left(R^l,\, d^l x\right).$$

To define conditions on the coefficients under which second-order quasi-linear elliptic partial differential equation will have a general solution we need to describe several functional classes.

We are defining a functional class form-bounded functions PK_β by formula

$$PK_\beta(A) = \left\{f\in L^1_{loc}(R^l, d^l x) :\right.$$

$$\left|\left\langle f|h|^2\right\rangle\right| \le \beta\left\langle A^{\frac{1}{2}}h, A^{\frac{1}{2}}h\right\rangle + c(\beta)\,||h||_2^2\Big\},$$

where a $h \in D(A^{\frac{1}{2}})$ and $\beta > 0$ is a form-boundary and $c(\beta) \in R^1$.

To understand this condition, we consider in Euclidean space R^l, $l \ge 3$ the simple parabolic partial differential equation

$$\partial_t u = \Delta u,$$

which has explicit heat kernel

$$p_0(t,x,y) = (4\pi t)^{-\frac{l}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \quad t>0, \quad x,y \in R^l.$$

Applying this formula one can study more general heat equation presented as

$$Lu = \left[\frac{\partial}{\partial_t} - \sum_{i,k=1,..,l} a_{kj}(t,x)\nabla_k\nabla_j - \sum_{k=1,...,l} b_k(t,x)\nabla_k\right] u(t,x) = 0$$

with conditions $\exists \nu,\mu: \quad 0<\nu\le\mu<\infty$ such that

$$\nu\sum_{i=1}^{l}\xi_i^2 \le \sum_{ij=1,...,l} a_{ij}(t,x)\xi_i\xi_j \le \mu\sum_{i=1}^{l}\xi_i^2$$

these are usual boundary conditions and linear perturbation-potential $b_k(t,x): R^l \mapsto R^l$.

We will use the notations

$$\nabla\circ a\circ\nabla u = \sum_{i,j=1,...,l}\frac{\partial}{\partial x_i}a_{ij}\frac{\partial}{\partial x_j}u,$$

$$b\nabla u = b\circ\nabla u = \sum_{i=1,...,l} b_i\frac{\partial}{\partial x_i}u.$$

Let us consider fundamental solutions

$$p_0(t,x;\tau,y) = (2\pi)^{-l}\int \exp\left(ix\eta - \int_\tau^t a(\gamma,y)\,\eta^2 d\gamma\right) d\eta,$$

of parametric equation

$$[\partial_t - a_{kj}(t,y)\nabla_k\nabla_j]u(t,x) = 0.$$

It can be shown that

$$p_0(t,x;\tau,y) = (2\pi)^{-l}\int \exp\left(ix\eta - \int_\tau^t a(\gamma,y)\,\eta\cdot\eta d\gamma\right) d\eta =$$

$$= (2\sqrt{\pi})^{-l}\left(\det\left(\int_\tau^t a(\gamma,y)\,d\gamma\right)\right)^{-\frac{1}{2}} \exp\left(\left(-\int_\tau^t a(\gamma,y)\,d\gamma\right)^{-1}\frac{(x,x)}{4}\right).$$

The elliptic condition gives us estimations

$$\nu\sum_{i=1}^{l}\xi_i^2(t-\tau) \le \int_\tau^t a(\gamma,y)\,d\gamma\eta\cdot\eta \le \mu\sum_{i=1}^{l}\xi_i^2(t-\tau),$$

$$\nu\sum_{i=1}^{l}\xi_i^2(t-\tau)^{-1} \le \left(\int_\tau^t a(\gamma,y)\,d\gamma\right)^{-1}\eta\cdot\eta \le \mu\sum_{i=1}^{l}\xi_i^2(t-\tau)^{-1},$$

and we are obtaining Gaussian estimations [6, 10, 16] for our heat kernel as

$$(2\sqrt{\pi})^{-l}\,\nu^{\frac{-l}{2}}(t-\tau)^{\frac{-l}{2}}\exp\left(\frac{-\nu|x|^2}{4(t-\tau)}\right) \le p_0(t,x;\tau,y) \le$$

$$\le (2\sqrt{\pi})^{-l}\,\mu^{\frac{l}{2}}(t-\tau)^{\frac{-l}{2}}\exp\left(\frac{-|x|^2}{4\mu(t-\tau)}\right).$$

The fundamental solution of the last equation is

$$p_1(t,x;\tau,z) = p_0(t,x-z;\tau,z) + \int_\tau^t d\eta\int p_0(t,x-y;\eta,y)F(\eta,y;\tau,z)dy,$$

where the $F(\eta,y;\tau,z)$ is heat kernel density, fundamental solution. It can be rewritten as

$$p_1(t,x;\tau,z) = p_0(t,x-z;\tau,z) + \int_\tau^t d\eta\int p_0(t,x-y;\eta,y)b\circ\nabla p_0(\eta,y;\tau,z)dy.$$

Let us assume $b\circ a^{-1}\circ b \in PK_\beta(A)$ for some $\beta < 1$ then

$$|\langle\nabla h\circ bh\rangle| \le \sqrt{\beta}\,\langle Ah,h\rangle + c(\beta)\frac{1}{2\sqrt{\beta}}||h||_2^2, \quad h\in D\left(A^{\frac{1}{2}}\right)$$

and according to the KLMN-theorem [46], there is a preserving C_0- semigroups of L^∞-contraction $e^{-t\Lambda_n}$, $\frac{2}{2-\sqrt{\beta}} \le n \le \infty$ such that $\Lambda_2 = A + b\circ\nabla$.

If we assume that A is Laplace operator $A=\Delta$ then we are obtaining a form-boundary estimation

$$|\langle\nabla h\circ bh\rangle| \le \sqrt{\beta}\,\|\nabla h\|^2 + \frac{c(\beta)}{2\beta}\|h\|^2 \quad \forall h\in D(\Delta).$$

As an exemplar, we formulate a theorem.

Theorem 2. *(a consequence of [16]). Presume that, for some* $q > \frac{l}{2}$, $\quad l > 2$,

$$a(\cdot):\ \Omega\to R^l\otimes R^l, \quad a(\cdot)\in\left[L^1_{loc}(\Omega)\right]^{l\times l},$$

$$\nu\sum_{i=1}^{l}\xi_i^2 \le \sum_{ij=1,\dots,l} a_{ij}(t,x)\xi_i\xi_j \quad for\ some\ \nu>0$$

and perturbation $b\cdot\nabla$ ***satisfies a condition***

$$b\circ a^{-1}\circ b\in L^q+L^\infty.$$

Then

1. The operator $B_1=B_1(b)=\nabla\circ b$ ***of a domain***

$$D(B_1)=\left\{u\in L^1;\ |\nabla u|\in L^1_{loc};\ b\circ\nabla u\in L^1\right\}$$

is A_1***-bunded with relative bound zero namely*** $D(B_1)\supset D(A_1)$ ***and holds***

$$\|B_1h\|_1\le\alpha\|A_1h\|_1+k(\alpha)\|h\|_1,\quad h\in D(A_1)$$

for all $\alpha>0$ ***and*** $k(\alpha)<\infty$***.***

2. There are $s>0$ ***and*** $\beta(s)<1$ ***such that***

$$\int_0^s\left\|B_1e^{-tA_1}h\right\|_1dt\le\beta(s)\|h\|_1,\quad h\in D(A_1).$$

3. The operator A_1+B_1 ***of the domain*** $D(A_1)$ ***generates*** C_0***- semigroup*** T_1^t ***consistent with*** $T^t=\exp(-t(A+b\circ\nabla))$ ***and there is an estimation***

$$\left\|T_1^t\right\|_{1\to1}\le\frac{1}{1-\beta(s)}\exp\left(-t\frac{\log(1-\beta(s))}{s}\right),\quad t>0.$$

Counterexample. Let us consider a linear operator $-\Lambda_p\supset\nabla a\nabla-b\nabla$ of the domain $D(A_p)$ that the operator generates a holomorphic semigroup in $L^p\left(R^l,d^lx\right)$-space. Let holds a condition $b\circ a^{-1}\circ b\in PK_\beta(A)$ and denote $b_n=\chi_nb$, where χ_n is an indicator of set $\left\{x\in R^l:\ \left(b\circ a^{-1}\circ b\right)(x)\le n\right\}$ and

$$strong\ L^p-\lim_{n\to\infty}\exp(-t\Lambda_p(b_n))=\exp(-t\Lambda_p(b))$$

uniformly in $t\in[0,\ 1]$. Then

if $\beta<1$, $p\in\left[\frac{2}{2-\sqrt{\beta}},\ \infty\right[$ the operator $A+b\nabla$ generates !$_0$ contraction semigroup and holds the equations

$$\left\|\exp(-t\Lambda_p)\right\|_{p\to p}\le\exp\left(\frac{c(\beta)t}{p-1}\right),$$

$$\left\|\exp(-t\Lambda_p)\right\|_{p\to s}\le C\exp\left(\frac{c(\beta)t}{\sqrt{\beta}}\right)t^{\frac{-(s-p)l}{2ps}},$$

$$\frac{2}{2-\sqrt{\beta}}<p<s\le\infty;$$

if $1\le\beta<4$, $p<s\in\left[\frac{2}{2-\sqrt{\beta}},\ \infty\right[$ the operator sum $A+b\nabla$ is not well defined however there is a semigroup that can be defined as a limit

$$\exp(-t\Lambda_p(b))\equiv strong\ L^p-\lim_{n\to\infty}\exp(-t\Lambda_p(b_n)),\quad t\ge0,$$

and this limit is a definition of our semigroup.

In contrast to the condition of Minty-Browder's theorem [40], in general, the semigroup generators map a real, separable reflexive Banach space Xinto itself as does semigroup.

The goals of the presented work are to combine the linear perturbation theory that has been stated above with the theory of quasi-linear elliptic operator and apply this theory in the case of a quasi-linear elliptic partial differential equation by using a new class of operators $A^p:\ W_1^p\left(R^l,\ d^lx\right)\to W_{-1}^p\left(R^l,\ d^lx\right)$**,** $l\ge3$ in Sobolev spaces**.**

We will consider a second-order quasi-linear elliptic partial differential equation in Euclidean space R^l, $l\ge3$

$$\lambda u-\sum_{i,j=1,\dots,l}\frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial}{\partial x_j}u\right)+b(x,u,\nabla u)=f,\quad(1)$$

where the $f\in L^p\cap L^\infty$ and the a_{ij} is an elliptic matrix that for all $\xi\in R^l$, $l>2$ satisfies an inequality

$$\nu(|u|)\sum_{i=1}^{l}\xi_i^2\le\sum_{ij=1,\dots,l}a_{ij}(x,u)\xi_i\xi_j\le\mu(|u|)\sum_{i=1}^{l}\xi_i^2,\quad(2)$$

where the $\nu(s)$ is a positive nonincreasing continuous function for $s\ge0$ and $\mu(s)$ is a positive nondecreasing continuous function for $s\ge0$ [10-12]. Function $a_{ij}(x,u)$ satisfies the conditions

$$a_{ij}(x,u)\xi_j-a_{ij}(x,v)\eta_j\ge\mu_6(x)(\xi_i-\eta_i),\qquad(3)$$

where the μ_6 is a measurable function such that [12]

$$0<\delta<\langle\mu_6(\cdot)|u|\rangle<\infty\qquad(4)$$

Function $b(x,u,\nabla u)$ satisfies the conditions

$$|b(x,u,\nabla u)|\le\mu_1(x)|\nabla u|+\mu_2(x)|u|+\mu_3(x)\qquad(5)$$

$$|b(x,u,\nabla u)-b(x,v,\nabla v)|\le$$

$$\mu_4(x)|\nabla(u-v)|+\mu_5(x)|u-v|\qquad(6)$$

where the $\mu_1^2\in PK_\beta$, $\mu_2\in PK_\beta$, $\mu_4^2\in PK_\beta$, $\mu_5\in PK_\beta$, $\mu_3\in L^q\left(R^l\right)$.

Exemplar. Let us consider an equation with Gilbarg-Serrin's matrix [10, 12]

$$a\circ d^2u\equiv\sum_{i,j=1}^{l}a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u=0,$$

where $a_{ij}=\delta_{ij}+b\frac{x_ix_j}{|x|^2}$, $b=-1+\frac{l-1}{1-\chi}$, $\chi<1$, $l\ge3$.

It can be shown that

$$da=b(l-1)\frac{x}{|x|^2},$$

$$(a_{ij})^{-1} = \delta_{ij} - \frac{b}{b+1}\frac{x_i x_j}{|x|^2},$$

$$da \circ a^{-1} \circ da = (1+b)^{-1}\left(\frac{l-1}{|x|}\right)^2$$

and holds an estimation

$$\langle \nabla\varphi \circ a \circ \nabla\varphi \rangle \geq (1+b)\frac{l-2}{2}\left\|\frac{\varphi}{|x|}\right\|_2^2 \quad \forall \varphi \in W_1^2(R^l),\, l \geq 3,$$

so $\beta = 4\left(1+\frac{\chi}{l-2}\right)^2$ we have $\nabla a \circ a^{-1} \circ \nabla a \in PK_\beta(A)$ here $c(\beta)=0$, for $\beta < 4$ it is necessary that $\chi \in (-2(l-2),\, 0)$.

Let us consider the boundary condition $u(|x|=1) = 1$ for the equation with Gilbarg-Serrin's matrix. It is easy to see that two functions $u \equiv 1$ and $u = |x|^\chi$ convert our equation into tautology.

If $\chi = -\frac{l-2}{s}$ then $\beta = 4\left(1+\frac{\chi}{l-2}\right)^2$ and $\beta \leq 4$ when $p > s$ and function $u = |x|^\chi \in L^p$ in the ball $K_1(0)$, on another hand the operator estimation

$$\left\|\exp(-t\Lambda_p)\right\|_{p\to s} \leq C\exp\left(\frac{c(\beta)t}{\sqrt{\beta}}\right) t^{\frac{-(s-p)l}{2ps}},$$

$$\frac{2}{2-\sqrt{\beta}} < p < s \leq \infty$$

must be true so $|x|^\chi \in L^{\frac{pl}{l-2}}(K_1(0))$ however, it is impossible since $|x|^\chi \notin L_{loc}^{\frac{pl}{l-2}}$ the function $|x|^\chi$ does not belong to the class of possible solutions and is not a solution.

If $\beta > 4$ then the equation $a \circ d^2 u = 0$ has always two solutions.

The main result of this paper is an analog of Minty-Browder's theorem [40], which states the operator A^p associated with the equation (1) and mapping $W_1^p\left(R^l,\, d^l x\right)$ space into $W_{-1}^p\left(R^l,\, d^l x\right)$ space under the conditions (2), (3), (4), (5), (6) and $\mu_1^2 \in PK_\beta$, $\mu_2 \in PK_\beta$, $\mu_4^2 \in PK_\beta$, $\mu_5 \in PK_\beta$, $\mu_3 \in L^q\left(R^l\right)$, and for $\lambda > \lambda_0$ is subjective if $l \geq 3$. That means that a second-order quasi-linear elliptic partial differential equation has a solution belonging to $W_1^p\left(R^l,\, d^l x\right)$, $l \geq 3$ space under the conditions (2), (3), (4), (5), (6) and $\mu_1^2 \in PK_\beta$, $\mu_2 \in PK_\beta$, $\mu_4^2 \in PK_\beta$, $\mu_5 \in PK_\beta$, $\mu_3 \in L^q\left(R^l\right)$, and for $\lambda > \lambda_0$.

2 Properties of $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ operator generated by quasi-linear elliptic partial differential equation

The introduction of operators $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ was motivated by the definition of a general weak solution in $W_1^p\left(R^l,\, d^l x\right)$ functional spaces. This type of nonlinear operators was introduced by Mykola Makarovich Kukharchuk.

Definition 1.*(of a general solution.)* ***A general solution of a second-order quasi-linear elliptic partial differential equation (1) from $W_1^p\left(R^l,\, d^l x\right)$ functional space can be defined as an element of $W_1^p\left(R^l,\, d^l x\right)$, which satisfies an integral identity***

$$\lambda\,\langle u,v\rangle + \left\langle \sum_{i,j=1,\dots,l} a_{ij}\nabla_j u, \nabla_i v \right\rangle + \langle b(\cdot,u,\nabla u), v\rangle = \langle f,v\rangle$$

for all elements v from $W_{1,0}^q(R^l,d^l x)$ $l \geq 3$.

Applying this definition of a general solution of a second-order quasi-linear elliptic partial differential equation one can construct the differential form $h_\lambda^p : W_1^p \times W_1^q \to R$ as follow

$$h_\lambda^p(u,v) \equiv \lambda\,\langle u,v\rangle + \langle \nabla v \circ a \circ \nabla u\rangle + \langle b(\cdot,u,\nabla u),v\rangle,$$

this form well defined on $u \in W_1^p(R^l,d^l x)$, $v \in W_1^q(R^l,d^l x)$. Since the differential equation is quasi-linear, the form $h_\lambda^p : W_1^p \times W_1^q \to R$ is not linear on the first argument, the function b can be nonlinear, and linear and continuous on the second argument.

For any fixed element $u \in W_1^p(R^l,d^l x)$ (first argument), this form determines a continuous linear functional on $W_1^q(R^l,d^l x)$ (as a function of a second argument) and so the element of $W_{-1}^p(R^l,d^l x)$, thus we have that for any fixed element $u \in W_1^p(R^l,d^l x)$ form $h_\lambda^p : W_1^p \times W_1^q \to R$ defines the element of $W_{-1}^p(R^l,d^l x)$ space, consequently, there is a mapping or operator A^p from $W_1^p(R^l,d^l x)$ into $W_{-1}^p(R^l,d^l x)$ such that

$$h_\lambda^p(u,v) = \langle A^p(u), v\rangle.$$

So constructed operators are different from both operator types (operators of Minty-Browder's theorem and semi-groups generators), first, they do not map $W_1^p(R^l,d^l x)$ into its dual as operators in Minty-Browder's theorem [40] so the conditions of continuity, coerciveness, monotony must be revised and the analog of Minty-Browder theorem requires additional investigation; second, these operators do not map the reflexive Banach space into itself as do operators of semigroups theory [20].

Let us estimate the form $h_\lambda^p(u,v)$ on arbitrary elements $u \in W_1^p(R^l,d^l x)$, $v \in W_1^q(R^l,d^l x)$, $l \geq 3$

$$\left|h_\lambda^p(u,v)\right| \equiv$$

$$|\lambda\,\langle u,v\rangle + \langle \nabla v \circ a \circ \nabla u\rangle + \langle b(\cdot,u,\nabla u),v\rangle| \leq$$

$$\leq \lambda\,\|u\|_p\|v\|_q + \langle \nabla v \circ a \circ \nabla u\rangle +$$

$$\langle \mu_1(\cdot)\,|\nabla u| + \mu_2(\cdot)\,|u| + \mu_3(\cdot), v\rangle \leq$$

$$\leq \lambda\,\|u\|_p\|v\|_q + \langle \nabla v \circ a \circ \nabla u\rangle + \|\nabla u\|_p\|\mu_1(\cdot)v\|_q +$$

$$\|u\|_p\|\mu_2(\cdot)v\|_q + \|\mu_3(\cdot)\|_p\|v\|_q.$$

Next, we can write

$$\|\nabla u\|_p \|\mu_1(\cdot)v\|_q \le \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\mu_1(\cdot)v\|_q^q,$$

$$\|\mu_1(\cdot)v\|_q^q \le \left\langle |\mu_1|^q \left(|v|^{\frac{q}{2}}\right)^2 \right\rangle \le$$

$$\beta\left\langle \nabla\left(|v|^{\frac{q}{2}}\right)\circ a\circ\nabla\left(|v|^{\frac{q}{2}}\right)\right\rangle + c(\beta)\|v\|_q^q$$

$$\left\langle \nabla\left(|v|^{\frac{q}{2}}\right)\circ\nabla\left(|v|^{\frac{q}{2}}\right)\right\rangle \le \|v^{q-2}\|_{\frac{q}{q-2}} \left\||\nabla v|^2\right\|_{\frac{q}{2}} =$$

$$\|v\|_q^{q-2}\|\nabla v\|_q^2$$

$$\|v\|_q^{q-2}\|\nabla v\|_q^2 \le \frac{q-2}{q\sigma}\|v\|_q^q + \frac{2\sigma}{q}\|\nabla v\|_q^q$$

for $\sigma > 0$. Next, we estimate

$$\|u\|_p \|\mu_2(\cdot)v\|_q \le \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|\mu_2(\cdot)v\|_q^q$$

$$\|\mu_2(\cdot)v\|_q^q \le \left\langle |\mu_2|^q \left(|v|^{\frac{q}{2}}\right)^2 \right\rangle \le$$

$$\beta\left\langle \nabla\left(|v|^{\frac{q}{2}}\right)\circ a\circ\nabla\left(|v|^{\frac{q}{2}}\right)\right\rangle + c(\beta)\|v\|_q^q.$$

We have obtained that

$$\left|h_\lambda^p(u,v)\right| \le \lambda\|u\|_p\|v\|_q + \langle\nabla v\circ a\circ\nabla u\rangle +$$

$$\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{2}{q}\beta\left(\left(\frac{q-2}{q\sigma} + c(\beta)\right)\|v\|_q^q + \right.$$

$$\left.\frac{2\sigma}{q}K\langle\nabla v\circ a\circ\nabla v\rangle\right) + \|\mu_3(\cdot)\|_p\|v\|_q,$$

where the K depends on the matrix $[a_{ij}]$.

Assuming that $v = u|u|^{p-2}$ we have

$$\left|h_\lambda^p\left(u, u|u|^{p-2}\right)\right| \equiv \left|\lambda\left\langle u, u|u|^{p-2}\right\rangle + \right.$$

$$\left.\left\langle \nabla\left(u|u|^{p-2}\right)\circ a\circ\nabla u\right\rangle + \left\langle b(\cdot,u,\nabla u), u|u|^{p-2}\right\rangle\right| \le$$

$$\lambda\|w\|^2 + (p-1)\left\langle\left(|u|^{\frac{p-2}{2}}\nabla u\right)\circ a\circ\left(|u|^{\frac{p-2}{2}}\nabla u\right)\right\rangle +$$

$$\left\langle \mu_1(\cdot)|\nabla u| + \mu_2(\cdot)|u| + \mu_3(\cdot), |u|^{p-1}\right\rangle \le$$

$$\lambda\|w\|^2 + \frac{4(p-1)}{p^2}\langle\nabla w\circ a\circ\nabla w\rangle + \frac{2}{p}\langle\mu_1|\nabla w|, |w|\rangle +$$

$$\beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2 + \|\mu_3\|\|u\|^{p-1},$$

where we have denoted $w = u|u|^{\frac{p-2}{2}}$ and $\nabla w = \frac{p}{2}|u|^{\frac{p-2}{2}}\nabla u$.

Next, we can write

$$\left\langle \mu_1|\nabla u|, |u|^{p-1}\right\rangle = \left\langle \mu_1|u|^{\frac{p-2}{2}}|\nabla u|, |u|^{\frac{p}{2}}\right\rangle \le$$

$$\frac{2}{p}\langle\mu_1|\nabla w|, |w|\rangle,$$

$$\left\langle \mu_2(\cdot), w^2\right\rangle \le \beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2,$$

$$\left\langle \mu_3(\cdot), |u|^{p-1}\right\rangle \le \|\mu_3\|\left\||u|^{p-1}\right\| = \|\mu_3\|\|u\|^{p-1},$$

by Holder's inequality

$$\frac{2}{p}\langle\mu_1|\nabla w|, |w|\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\|,$$

$$\|\mu_1 w\| = \left\langle(\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le \left(\beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}},$$

So

$$\frac{2}{p}\langle\mu_1|\nabla w|, |w|\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\| =$$

$$\frac{2}{p}\|\nabla w\|\left\langle(\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le$$

$$\frac{2}{p}\|\nabla w\|\left(\beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}} \le$$

$$\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2\right)\right).$$

We can conclude that

$$\left|h_\lambda^p\left(u, u|u|^{p-2}\right)\right| \le \lambda\|w\|^2 + \frac{4(p-1)}{p^2}\langle\nabla w\circ a\circ\nabla w\rangle +$$

$$\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2\right)\right) +$$

$$\beta\langle\nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2 + \frac{\sigma^p}{p}\|\mu_3\|^p + \frac{1}{\sigma^q q}\|u\|^p,$$

or in a more compact form, we have had

$$\left|h_\lambda^p\left(u, u|u|^{p-2}\right)\right| \le$$

$$\left(\lambda + \left(\frac{\varepsilon^2}{p} + 1\right)c(\beta) + \frac{1}{\sigma^q q}\right)\|w\|^2 +$$

$$+\left(\frac{4(p-1)}{p^2} + \frac{\beta\varepsilon^2}{p} + \beta\right)\langle\nabla w\circ a\circ\nabla w\rangle +$$

$$\frac{1}{p}\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \frac{\sigma^p}{p}\|\mu_3\|^p,$$

for $\varepsilon,\ \sigma > 0$.

The form h_λ^p generates mapping A^p from W_1^p into W_{-1}^p that can be written as

$$h_\lambda^p(u,v) = \langle A^p(u), v\rangle.$$

Definition 2. *An operator $A^p : W_1^p \to W_{-1}^p$ is said to be coercive operator if and only if the form*

$$h_\lambda^p(u,v) = \langle A^p(u), v\rangle$$

satisfies a condition

$$\lim_{\|u|u|^{p-2}\|_{W_1^q}\to\infty} \frac{h_\lambda^p\left(u,u|u|^{p-2}\right)}{\|u|u|^{p-2}\|_{W_1^q}} = \infty.$$

Definition 3. *An operator $A^p : W_1^p \to W_{-1}^p$ is called accretive operator in $L^p\left(R^l, d^l x\right)$ if and only if this operator for any $u, v \in W_1^p\left(R^l, d^l x\right)$ satisfies an inequality*

$$\left\langle A^p(u) - A^p(v), (u-v)|u-v|^{p-2}\right\rangle \geq$$

$$-c\left(l, \|u\|_{W_1^p}, \|u-v\|_{W_1^p}\right),$$

where the $c\left(l, \|u\|_{W_1^p}, \|u-v\|_{W_1^p}\right)$ is a continuous positive function and such that

$$\lim_{t\to o} c\left(l, \|u\|_{W_1^p}, \rho t\right) t^{1-p} = 0,$$

where the ρ and t are positive real numbers.

Definition 4. *An operator $A^p : W_1^p \to W_{-1}^p$ is called quasi-continuous if and only if for any functions u, v that belong $W_{1,0}^p\left(R^l, d^l x\right)$ space, there is a limit*

$$\omega - \lim_{t\to 0} \mathrm{A}^p(u+tv) = A^p(u)$$

with respect to the weak topology of the $W_{-1}^p\left(R^l, d^l x\right)$.

Proposition 1. *An operator $A^p : W_1^p \to W_{-1}^p$ is a coercive operator.*

Proof. To prove that $A^p : W_1^p \to W_{-1}^p$ the operator is a really coercive operator, we have to estimate form $h_\lambda^p\left(u, u|u|^{p-2}\right)$ from below

$$h_\lambda^p\left(u, u|u|^{p-2}\right) = \lambda\left\langle u, u|u|^{p-2}\right\rangle +$$

$$\left\langle \nabla\left(u|u|^{p-2}\right)\circ a\circ\nabla u\right\rangle + \left\langle b(\cdot,u,\nabla u), u|u|^{p-2}\right\rangle \geq$$

$$\geq \lambda\|w\|^2 + \frac{4(p-1)}{p^2}\langle \nabla w\circ a\circ\nabla w\rangle$$

$$-\left\langle \mu_1(\cdot)|\nabla u| + \mu_2(\cdot)|u| + \mu_3(\cdot), |u|^{p-1}\right\rangle \geq$$

$$\geq \left(\lambda - \left(\left(\frac{\varepsilon^2}{p}+1\right)c(\beta) + \frac{1}{\sigma^q q}\right)\right)\|w\|^2 +$$

$$+\left(\frac{4(p-1)}{p^2} - \left(\frac{\beta\varepsilon^2}{p} + \beta + \frac{1}{\nu p}\frac{1}{\varepsilon^2}\right)\right)\langle \nabla w\circ a\circ\nabla w\rangle -$$

$$\frac{\sigma^p}{p}\|\mu_3\|^p,$$

where the β is a form-bound, the measure of singularity; $\frac{1}{p}+\frac{1}{q}=1$, $\sigma>0$, exists $\varepsilon > 0$; and we have denoted the $w=(u-v)|u-v|^{\frac{p-2}{2}}$ and $\nabla w = \frac{p}{2}|u-v|^{\frac{p-2}{2}}\nabla(u-v)$.

Proposition 2. *An operator $A^p : W_1^p \to W_{-1}^p$ is an accretive operator.*

To prove this statement, we are estimating

$$\left\langle A^p(u) - A^p(v), (u-v)|u-v|^{p-2}\right\rangle =$$

$$= \lambda\left\langle u, (u-v)|u-v|^{p-2}\right\rangle +$$

$$\left\langle \nabla\left((u-v)|u-v|^{p-2}\right)\circ a(\cdot,u)\circ\nabla u\right\rangle +$$

$$+\left\langle b(\cdot,u,\nabla u), (u-v)|u-v|^{p-2}\right\rangle -$$

$$-\lambda\left\langle v, (u-v)|u-v|^{p-2}\right\rangle -$$

$$\left\langle \nabla\left((u-v)|u-v|^{p-2}\right)\circ a(\cdot,v)\circ\nabla v\right\rangle +$$

$$-\left\langle b(\cdot,v,\nabla v), (u-v)|u-v|^{p-2}\right\rangle =$$

$$= \lambda\left\langle u-v, (u-v)|u-v|^{p-2}\right\rangle +$$

$$\left\langle \nabla_i\left((u-v)|u-v|^{p-2}\right)\left(a_{ij}(\cdot,u)\nabla_j u - a_{ij}(\cdot,v)\nabla_j v\right)\right\rangle +$$

$$+\left\langle b(\cdot,u,\nabla u) - b(\cdot,v,\nabla v), (u-v)|u-v|^{p-2}\right\rangle \geq$$

$$\geq \lambda\left\langle u-v, (u-v)|u-v|^{p-2}\right\rangle +$$

$$\left\langle \mu_6(\cdot)\nabla(u-v)\circ\nabla\left((u-v)|u-v|^{p-1}\right)\right\rangle -$$

$$-\left\langle \mu_4(\cdot)|\nabla(u-v)| + \mu_5(\cdot)|u-v|, (u-v)|u-v|^{p-2}\right\rangle \geq$$

$$\geq \lambda\|w\|^2 + \frac{4(p-1)}{p^2}M_1\langle \nabla w\circ a\circ\nabla w\rangle -$$

$$-\left\langle \mu_4(\cdot)|\nabla(u-v)| + \mu_5(\cdot)|u-v|, (u-v)|u-v|^{p-2}\right\rangle,$$

where $w = (u-v)|u-v|^{\frac{p-2}{2}}$ and $\nabla w = \frac{p}{2}|u-v|^{\frac{p-2}{2}}\nabla(u-v)$.

Next, we have

$$\frac{2}{p}\langle \mu_4(x)|\nabla w|, |w|\rangle \leq \frac{2}{p}\|\mu_4 w\|\,\|\nabla w\| =$$

$$\frac{2}{p}\|\nabla w\|\left\langle(\mu_4 w)^2\right\rangle^{\frac{1}{2}} \leq$$

$$\leq \frac{2}{p}\|\nabla w\|\left(\beta\langle \nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\langle \nabla w\circ a\circ\nabla w\rangle + c(\beta)\|w\|^2\right)\right),$$

and the last term can be estimated as

$$\left\langle \mu_5(\cdot)\,|u-v|\,,(u-v)\,|u-v|^{p-2}\right\rangle = \left\langle \mu_5, w^2\right\rangle \le$$

$$\beta\left\langle \nabla w \circ a \circ \nabla w\right\rangle + c(\beta)\,\|w\|^2.$$

Thus, we are obtaining

$$\left\langle A^p(u) - A^p(v)\,,(u-v)\,|u-v|^{p-2}\right\rangle \ge$$

$$\lambda\,\|w\|^2 + \frac{4(p-1)}{p^2}K\left\langle \nabla w \circ a \circ \nabla w\right\rangle -$$

$$-\left(\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\left\langle \nabla w \circ a \circ \nabla w\right\rangle + c(\beta)\,\|w\|^2\right)\right)+\right.$$

$$\left.+\beta\left\langle \nabla w \circ a \circ \nabla w\right\rangle + c(\beta)\,\|w\|^2\right) \ge$$

$$\ge\left(\lambda - \frac{\varepsilon^2 c(\beta)}{p} - c(\beta)\right)\|w\|^2 +$$

$$\left(\frac{4(p-1)}{p^2}K - \beta\frac{\varepsilon^2}{p} - \frac{1}{p\varepsilon^2\nu} - \beta\right)\left\langle \nabla w \circ a \circ \nabla w\right\rangle \ge 0,$$

where the constant K depends on the ellipticity of the matrix a.

The last estimation is proving statement 2.

Proposition 3. ***An operator*** $A^p : W_1^p \to W_{-1}^p$ ***is a quasi-continuous operator.***

Proof. Assuming that $u,\ v \in W_{1,0}^p\left(R^l, d^l x\right)$ and $w \in W_1^q\left(R^l, d^l x\right)$, we are obtaining

$$\left\langle A^p(u+tv) - A^p(u)\,, w\right\rangle =$$

$$\lambda\left\langle u+tv, w\right\rangle + \left\langle \nabla w \circ a(\cdot, u+tv) \circ \nabla(u+tv)\right\rangle +$$

$$+\left\langle b(\cdot, u+tv, \nabla(u+tv))\,, w\right\rangle - \lambda\left\langle u, w\right\rangle -$$

$$\left\langle \nabla w \circ a(\cdot, u) \circ \nabla u\right\rangle - \left\langle b(\cdot, u, \nabla u)\,, w\right\rangle =$$

$$= \lambda t\left\langle v, w\right\rangle +$$

$$\left\langle \nabla_i w, \left(a_{ij}(\cdot, u+tv)\nabla_j(u+tv)\right) - \left(a_{ij}(\cdot, u)\nabla_j(u)\right)\right\rangle +$$

$$+\left\langle b(\cdot, u+tv, \nabla(u+tv)) - b(\cdot, u, \nabla u)\,, w\right\rangle =$$

$$= \lambda t\left\langle v, w\right\rangle + t\left\langle \mu_6(\cdot)\nabla w \nabla v\right\rangle +$$

$$t\left\langle \mu_4(\cdot)\,|\nabla v| + \mu_5(\cdot)\,|v|\,, w\right\rangle \underset{t\to 0}{\to} 0.$$

Statement 3 is proven.

Proposition 4. ***Let us assume*** $S_R = \left(\vec{C} : |\vec{C}| = R\right)$ ***is a sphere, where*** $R > 0$ ***is a given number a radius of the sphere, and assume that*** $\vec{B} : R^n \to R^n$ ***is given continuous mapping such that satisfies a condition*** $\left\langle \vec{B}\left(\vec{C}\right), \vec{C}^*\right\rangle \ge 0$. ***Then there is at least one point*** $\vec{C} : \quad |\vec{C}| \le R$ ***such that*** $\vec{B}\left(\vec{C}\right) = 0$.

Proof. Let us prove this statement by assuming the opposite, which asserts that for all $\vec{C}$ such that $|\vec{C}| \le R$ the inequality

$$\vec{B}\left(\vec{C}\right) \ne 0$$

is hold. In ball $B_R = \left(\vec{C} : |\vec{C}| \le R\right)$, let us consider the mapping

$$\vec{A}\left(\vec{C}\right) = -R\frac{\vec{B}\left(\vec{C}\right)}{\left|\vec{B}\left(\vec{C}\right)\right|},$$

it is obvious that the map $\vec{A} :\ V_R \to S_R$ is continuous mapping thus according to the Brouwer's fixed-point theorem there is a fixed point $\vec{C}$ such that

$$\vec{A}\left(\vec{C}\right) = \vec{C}.$$

However, it is impossible since

$$\left\langle \vec{A}\left(\vec{C}\right), \vec{C}^*\right\rangle = -R\frac{\left\langle \vec{B}\left(\vec{C}\right), \vec{C}^*\right\rangle}{\left|\vec{B}\left(\vec{C}\right)\right|} \le 0$$

and simultaneously we have

$$\left\langle \vec{A}\left(\vec{C}\right), \vec{C}^*\right\rangle = \left\langle \vec{C}, \vec{C}^*\right\rangle = R^n > 0.$$

The obtaining contradiction proves statement 4.

3 Existence of a weak generalized solution of a second-order quasi-linear elliptic partial differential equation

We are going to establish the existence of a general solution of a second-order quasi-linear elliptic partial differential equation by applying the Galerkin method [12], which consists of choosing approximations by restricting the operator that is generated by this equation to some finite-dimensional subspaces and then pass to the limit.

Theorem 3. ***(analog of Minty-Browder theorem). Let operator*** $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ ***be generated by differential form*** $h_\lambda^p : W_1^p \times W_1^q \to R$ ***associated with a second-order quasi-linear elliptic partial differential equation*** (1) ***with conditions*** (2), (3), (4), (5), (6) ***and*** $\mu_1^2 \in PK_\beta,\ \ \mu_2 \in PK_\beta,\ \ \mu_4^2 \in PK_\beta,\ \ \mu_5 \in PK_\beta,\ \ \mu_3 \in L^q\left(R^l\right)$, $l > 2$ ***and*** $\lambda > \lambda_0$; ***assume operator*** $A^p : W_1^p \to W_{-1}^p$ ***satisfies the coercive, accretive and quasi-continuous conditions then the operator*** $A^p : W_1^p \to W_{-1}^p$ ***is surjective.***

Theorem 4. *Theorem (the existence of a general solution of a second-order quasi-linear elliptic partial differential equation). A second-order quasi-linear elliptic partial differential equation (1) under the conditions (2), (3), (4), (5), (6) and $\mu_1^2 \in PK_\beta$, $\mu_2 \in PK_\beta$, $\mu_4^2 \in PK_\beta$, $\mu_5 \in PK_\beta$, $\mu_3 \in L^q\left(R^l\right)$, $f \in L^p \cap L^\infty$, $l > 2$ and $\lambda > \lambda_0$ has a general solution $u \in W_1^p(R^l, d^l x)$.*

Proof. Let us assume that $\{v_i\}$ and $\{v_i^*\}$ are two smooth bases of $W_1^p(R^l, d^l x)$ and $W_1^q(R^l, d^l x)$ spaces respectively, and let $[v_1, \ldots v_k]$ be a linear span of the elements of the bases with the property

$$\langle u_k, u_k^* \rangle = \|u_k\|_p^p,$$

where we denoted $u_k = \sum_{i=1}^k c_i v_i$ and $u_k^* = \sum_{i=1}^k c_i^* v_i^*$ by the definition.

Next step is to construct a Galerkin approximate solution, in order to achieve this we are writing a system of k equations as follow

$$\langle A^p(u_k) - f, v_i^* \rangle = 0, i = 1, \ldots, k,$$

and show that this system has a solution in the linear span of the first n elements of our basis $\{v_i\}$. Indeed, this system defines a continuous mapping of a sphere in Euclidean space into itself

$$\vec{B}\left(\vec{C}\right) : B_i\left(\vec{C}\right) = \langle A^p(u_k) - f, v_i^* \rangle$$

so operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ according to statement 4 and statement 1 satisfies the following condition

$$\left\langle \vec{B}\left(\vec{C}\right), \vec{C}^* \right\rangle = \left\langle A^p\left(\sum_{i=1}^k c_i v_i\right) - f, \sum_{i=1}^k c_i^* v_i^* \right\rangle =$$

$$\left\langle A^p(u_k) - f, u_k|u_k|^{p-2} \right\rangle \geq$$

$$\geq \left(\frac{h_\lambda^p\left(u_k, u_k|u_k|^{p-2}\right)}{\|u_k|u_k|^{p-2}\|_{W_1^q}} - \|f\|_p\right) \|u_k|u_k|^{p-2}\|_{W_1^q} \geq 0.$$

Since mapping $A^p : W_1^p \to W_{-1}^p$ is continuous on finite-dimensional subspaces of the $W_1^p\left(R^l, d^l x\right)$ space hence there is an element $\vec{C}$, $|\vec{C}| = R$ such that $\vec{B}\left(\vec{C}\right) = 0$, if $R > 0$ large enough. Thus, there is a sequence $\{u_k(x)\}$ such that

$$\langle A^p(u_k) - f, v_i^* \rangle = 0.$$

Coerciveness of the operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ applies an inequality

$$\|A^p(u_k)\|_{W_{-1}^p} \leq \|f\|_{L^p}.$$

Now we have to show that this sequence converges to the solution of the quasi-linear elliptic partial differential equation. To achieve this goal we consider the integral tautology

$$\lambda \langle u_k, \xi \rangle + \langle d\xi \circ a \circ du_k \rangle + \langle b(x, u_k, \nabla u_k), \xi \rangle \equiv \langle f, \xi \rangle,$$

where we presume $\xi = u_k|u_k|^{p-2}$ thus it yields the equation

$$\lambda \langle u_k, u_k|u_k|^{p-2} \rangle +$$

$$\frac{4(p-1)}{p^2} \left\langle \nabla\left(u_k|u_k|^{\frac{p-2}{2}}\right) \circ a \circ \nabla\left(u_k|u_k|^{\frac{p-2}{2}}\right) \right\rangle +$$

$$\left\langle b, u_k|u_k|^{p-2} \right\rangle \equiv \left\langle f, u_k|u_k|^{p-2} \right\rangle.$$

Easy to see that

$$|\langle b, u_k|u_k|^{p-2} \rangle| \leq$$

$$\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\langle\nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2\right)\right) +$$

$$+\beta\langle\nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2 + \frac{\sigma^p}{p}\|\mu_3\|^p + \frac{1}{\sigma^q q}\|w\|^2,$$

where the $w = u_k|u_k|^{\frac{p-2}{2}}$ and $\nabla w = \frac{p}{2}|u_k|^{\frac{p-2}{2}}\nabla u_k$.

Let estimate the term on the right as

$$|\langle f, u_k|u_k|^{p-2} \rangle| \leq \|f\|_p \left\|u_k|u_k|^{p-2}\right\|_q \leq \|f\|_p \|u_k\|_p^{p-1}$$

and for$\rho > 0$ it gives next estimation

$$\|f\|_p \|u_k\|_p^{p-1} \leq \frac{\rho^p}{p}\|f\|_p^p + \frac{1}{q\rho^q}\|u_k\|_p^p.$$

Correspondingly we are obtaining an estimation

$$\|u_k\| + \|\nabla u_k\| \leq c(\lambda, p, l, \lambda_0, N)\|f\|$$

and as a result, we have

$$||u_k||_{W_1^p} < C,$$

where constant depends only on equation coefficients.

Since the operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ is a quasi-continuous operator there is a converge subsequence $\{u_{k'}(x)\}$ such that there are the weak limits $u_{k'} \underset{W_1^p}{\longrightarrow} u_0$ and $A^p(u_{k'}) \underset{W_{-1}^p}{\longrightarrow} y$.

Now we have to show that $y = A^p(u_0) = f$, to achieve this goal let us consider an integral tautology

$$\langle A^p(u_{k'}), v_i^* \rangle = \langle f, v_i^* \rangle, i = 1, \ldots, k'$$

and find the limit in $W_{-1}^p\left(R^l, d^l x\right)$ topology when $k' \to +\infty$ we are obtaining

$$\lim_{k' \to \infty} A^p(u_{k'}) = y = f.$$

Since the operator $A^p(\cdot) : W_1^p(R^l, d^l x) \to W_{-1}^p(R^l, d^l x)$ is accretive in $L^p(R^l, d^l x)$ space we are passing to the limit under integral

$$\lim_{k' \to \infty} \left\langle A^p(u_{k'}) - A^p(v), (u_{k'} - v)|u_{k'} - v|^{p-2} \right\rangle =$$

$$\left\langle y - A^p(v), (u_0 - v)\,|u_0 - v|^{p-2} \right\rangle \geq 0.$$

Presuming that $v = u_0 - tz$, $t > 0$, $z \in W_1^p\left(R^l, d^l x\right)$ and dividing by t^{p-1} one obtains

$$\left\langle y - A^p(u_0 - tz), z\,|z|^{p-2} \right\rangle \geq 0.$$

Since vector $z \in W_1^p\left(R^l, d^l x\right)$ is an arbitrary element of $W_1^p\left(R^l, d^l x\right)$, $l > 2$ space and operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ is a quasi-continuous, we have obtained

$$y = A^p\left(u_0\right) = f,$$

and have proven that for given initial conditions we constructed the functional sequence $\{u_{k'}\}$ that converges to the element $u_0 \in W_1^p\left(R^l, d^l x\right)$ so vector $u_0 \in W_1^p\left(R^l, d^l x\right)$ is a general solution of given second-order quasi-linear elliptic partial differential equation with form-boundary conditions.

4 Conclusions

In this paper, we have studied the connection between parabolic and elliptic differential equations, clarified the definition of $A^p(\cdot) : W_1^p(R^l, d^l x) \to W_{-1}^p(R^l, d^l x)$ operators and investigated their properties, applied these results we proved the analog of Minty-Browder's theorem and established the existence of general solution of a second-order quasi-linear elliptic partial differential equation under fair weak conditions. In future works, we are going to construct a semigroup of contraction that is generated by $A^p(\cdot) : W_1^p(R^l, d^l x) \to W_{-1}^p(R^l, d^l x)$ operators and study its properties.

5 Perspective

The theoretical approach proposed in this article could be also adapted for the analysis of partial differential equations of parabolic and hyperbolic types, and can be developed to account of high order derivatives.

This is an interesting research perspective since, it will allow to study the equations, which describes the evolutionary processes and the propagation of waves.

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