Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/110101

A Modified Shifted Gegenbauer Polynomials for the Numerical Treatment of Second-Order BVPs

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Received: 5 Dec. 2021, Revised: 25 Dec. 2021, Accepted: 30 Dec. 2021
Published online: 1 Jan. 2022

Abstract: The current article is devoted to constructing modified shifted Gegenbauer polynomials and employing them to treat the second-order BVPs. The operational matrices of these polynomials are established. These operational matrices generalize some introduced operational matrices in the literature. Furthermore, they are employed along with suitable spectral methods to handle the linear and non-linear second-order BVPs. More precisely, the two spectral methods, namely, Petrov-Galerkin and collocation methods are applied to treat respectively the linear and non-linear BVPs. The feasibility of the presented algorithms is checked through some examples with individual kinds namely: singular, singularly perturbed, and Bratu-type equations.

Keywords: shifted normalized Gegenbauer polynomials, second-order BVPs, singular and singularly perturbed problems, Bratu equation, Petrov-Galerkin method, collocation method

1 Introduction

Spectral methods are among the most methods used in discretizing the numerical solution of differential equations (see, [1–3]). Here to detect a smooth solution, we use smooth Gegenbauer polynomials. The main assumption of these methods aims to obtain approximate solutions for various types of differential equations. The three methods namely, collocation, Galerkin, and tau methods are the different used spectral methods to obtain the desired numerical solutions, see for example [4–6]. By ensuring that the approximate expansion gives a true statement for the problem at some internal points (i.e., the residual at specific nodes), the expansion coefficients will be obtained, and the trustee solution will be obtained too. The general Galerkin method can be efficiently employed to handle the linear single and multidimensional domain problems (see, [7, 8]). By applying the weighted inner product for the residual which approaches zero, one can get the expansion solution. General Galerkin method was modified to a Petrov-Galerkin method which is characterized by its flexibility in choosing the trial and test functions (see, for example, [9–11]).

Differential equations models simulate many real-life phenomena and cover many applied sciences such as biology, medicine, engineering, and physics [12–16]. Solving linear and non-linear differential equations is a spotlight for scientists as they cover applications in various fields. Various numerical methods were proposed to solve such equations. For example, in [17], an operational matrix method is followed to solve the initial value problems based on employing a certain harmonic numbers operational matrix of derivatives. Some other operational matrix methods are followed in [6, 18, 19]. Operational matrices of derivatives and integrals are useful in handling various types of differential equations along with the application of suitable spectral methods.

The employment of spectral methods for the numerical treatments of ordinary differential equations has occupied considerable interest by many authors. For example, the authors in [20] developed some spectral solutions to the one and two-dimensional second-order BVPs based on utilizing certain modified shifted Chebyshev polynomials of the third- and fourth-kinds. The authors in [21] treated the higher-order differential equations using the monic Chebyshev polynomials. Recently, another approach to approximate some BVPs is

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followed in [22] based on the application of the pseudo-Galerkin method. The authors in [23] have developed two spectral Legendre's derivative algorithms for the Lane Emden and Bratu type equations. In [24], the authors applied the Kansa-radial basis function (RBF) collocation method to the two-dimensional fourth-order BVPs.

The objective problems are the one-dimensional second-order linear and non-linear BVPs. The linear equations are treated by the Petrov-Galerkin method, while the non-linear equations are treated with the collocation method.

Among the important second-order BVPs is the Bratu type equation. This type of equation was handled by many techniques. Among these used techniques are the differential transform method [25], variational method [26], compact exponentially fitted method [27], the homotopy analysis method [28], Adomian decomposition method [29], Petrov–Galerkin matrix method (PGMM), collocation matrix method (CMM) [30], the Lie-group shooting method (LGSM) [31], B-spline method [32], the Laplace transform decomposition numerical algorithm [33], and the decomposition technique [34]. It is also worthy to mention that the singularly perturbed problems are specific important problems that have many applications. These problems were investigated by Mohsen and El-Gamel using shooting method [35], Sinc-collocation, and Sinc–Galerkin methods [36].

In this article, the focus tunnels are:

– Establishing a new type of orthogonal polynomials, namely, modified shifted Gegenbauer polynomials.
– Constructing the operational matrices of such polynomials.
– Employing the operational matrices of derivatives along with the application of the Petrov-Galerkin method to solve the linear second-order BVPs.
– Employing the operational matrices of derivatives along with the application of the collocation method to solve the non-linear second-order BVPs. Specifically, the singular problems, the singularly perturbed problems, Bratu's problems will be handled.

The construction of the paper is as follows. Section 2 presents an overview of Gegenbauer polynomials and their shifted ones. In Section 3, we construct the shifted modified Gegenbauer polynomials. In addition, the operational matrices of derivatives of such polynomials are established in this section. In Section 4, the second-order BVPs are treated. Both linear and non-linear second-order BVPs are solved by employing respectively the Petrov-Galerkin and collocation spectral methods. The introduced operational matrix serves in reducing the equations governed by their underlying conditions to linear or non-linear algebraic systems of equations that can be treated with suitable numerical solvers. The convergence of the methods is highlighted in Section 5. Section 6 offers some numerical examples to demonstrate that the proposed algorithms are effective and applicable. Section 7 includes some remarks.

2 An overview on Gegenbauer polynomials

In this section, we give an overview on Gegenbauer polynomials and their shifted polynomials. The set of Gegenbauer polynomials $\left\{C_k^{(\rho)}(u)\right\}_{k\geq 0}$, $\rho > \frac{-1}{2}$ constitutes a linearly independent orthogonal set on $[-1,1]$ with respect to the weight function $(1-u^2)^{\rho-\frac{1}{2}}$ in the sense that:

$$\int_{-1}^{1}(1-u^2)^{\rho-\frac{1}{2}}\,C_k^{\rho}(u)\,C_m^{\rho}(u)\,du = \begin{cases} \dfrac{2^{1-2\rho}\,\pi\,\Gamma(m+2\rho)}{m!\,(m+\rho)\,(\Gamma(\rho))^2}, & k=m,\\ 0, & k\neq m \end{cases} \tag{1}$$

It is preferable to introduce the normalized Gegenbauer polynomials $U_k^{\rho}(u)$ defined by:

$$U_k^{\rho}(u) = \frac{C_k^{(\rho)}(u)}{C_k^{(\rho)}(1)},$$

where $C_k^{(\rho)}(1) = \frac{(2\rho)_k}{k!}$.
It is to be noted here that the three well-known families of polynomials, namely, Legendre polynomials $P_k(u)$ and Chebyshev polynomials of the first- and second-kinds $T_k(u)$ and $U_k(u)$ are special ones of the polynomials $U_k^{\rho}(u)$. In fact, we have

$$P_k = U_k^{(\frac{1}{2})}(u), \quad T_k(u) = U_k^{(0)}(u),$$
$$U_k(u) = (k+1)\,U_k^{(1)}(u).$$

In addition, the normalized Gegenbauer polynomials may be generated by means of the following recursive formula

$$\begin{aligned}(k+2\,\rho)\,U_{k+1}^{\rho}(u) &= 2\,u\,(k+\rho)\,U_k^{\rho}(u)\\ &-k\,U_{k-1}^{\rho}(u), \quad k\geq 2,\\ U_0^{\rho}(u)&=1,\ U_1^{\rho}(u)=u.\end{aligned} \tag{2}$$

Now, we define the shifted normalized Gegenbauer polynomials $\tilde{U}_k^{\rho}(u)$ on $[c,d]$ defined as:

$$\tilde{U}_k^{\rho}(u) = \frac{C_k^{(\rho)}\left(\frac{2u-c-d}{d-c}\right)}{C_k^{(\rho)}(1)}, \quad k=0,1,\ldots.$$

The shifted polynomials may be generated by the following recurrence relation:

$$\begin{aligned}\tilde{U}_k^{\rho}(u) =&2\left(\frac{2u-d-c}{d-c}\right)\frac{k+\rho-1}{k+2\rho-1}\tilde{U}_{k-1}^{\rho}(u)\\&-\frac{k-1}{k+2\rho-1}\tilde{U}_{k-2}^{\rho}(u),\quad k\geq 2,\\&\tilde{U}_0^{\rho}(u)=1,\,\tilde{U}_1^{\rho}(u)=\frac{2u-d-c}{d-c}.\end{aligned} \tag{3}$$

The polynomials $\tilde{U}_k^{\rho}(u)$ are orthogonal on $[c,d]$ with respect to the weight function: $(u-c)^{\rho-\frac{1}{2}}(d-u)^{\rho-\frac{1}{2}}$. Explicitly, we have the following orthogonality relation:

$$\begin{aligned}&\int_c^d (u-c)^{\rho-\frac{1}{2}}(d-u)^{\rho-\frac{1}{2}}\,\tilde{U}_k^{\rho}(u)\,\tilde{U}_m^{\rho}(u)\,du\\&=\begin{cases}h_m^{\rho}, & k=m,\\ 0, & k\neq m,\end{cases}\end{aligned} \tag{4}$$

with

$$h_m^{\rho}=\left(\frac{d-c}{2}\right)^{2\rho}\frac{2^{1-2\rho}\,\pi\,\Gamma(m+2\rho)}{m!\,(m+\rho)\,\left(\Gamma(\rho)\right)^2}. \tag{5}$$

For other properties of Gegenbauer polynomials, one can consult [37, 38].

3 Construction of the shifted modified Gegenbauer polynomials and their operational matrices of derivatives

In this section, we construct a new type of orthogonal polynomials, namely, shifted modified Gegenbauer polynomials.

Now, consider the following space (see, [39])

$$\begin{aligned}&L_0^2[c,d]=\\&\left\{p_k(u)\in\, L^2[c,d]: p_k(c)=p_k(d)=0, k=0,1,2,...\right\},\end{aligned} \tag{6}$$

and consider the following polynomials:

$$\begin{aligned}H_i^{\rho}(u) =&(d-u)(u-c)\,\tilde{U}_i^{\rho}(u),\\&i=0,1,2,\ldots,u\in[c,d].\end{aligned} \tag{7}$$

It is clear that $H_i^{\rho}(u)\in L_0^2[c,d]$. Furthermore, they constitute a linearly independent set of orthogonal polynomials. They are orthogonal with respect to the weight function: $ß(u)=(u-c)^{\rho-\frac{5}{2}}(d-u)^{\rho-\frac{5}{2}}$. Explicitly, we have the following orthogonality relation:

$$\begin{aligned}&\int_c^d (u-c)^{\rho-\frac{5}{2}}\,(d-u)^{\rho-\frac{5}{2}}\,H_i^{\rho}(u)\,H_m^{\rho}(u)\,du\\&=\begin{cases}0, & i\neq m,\\ \left(\dfrac{d-c}{2}\right)^{2\rho}\dfrac{2^{1-2\rho}\,\pi\,\Gamma(m+2\rho)}{m!\,(m+\rho)\,\left(\Gamma(\rho)\right)^2}, & i=m.\end{cases}\end{aligned}$$

For a given function $\zeta(u)\in L_0^2[c,d]$, it is possible to express it as

$$\zeta(u)=\sum_{i=0}^{\infty} b_i\,H_i^{\rho}(u), \tag{8}$$

where

$$\begin{aligned}b_i =&\frac{1}{h_i^{\rho}}\int_r^q \zeta(u)\,H_i^{\rho}(u)\,(u-c)^{\rho-\frac{5}{2}}\,(d-u)^{\rho-\frac{5}{2}}\,du\\=&\left(\zeta(u),H_i^{\rho}(u)\right)_{ß(u)},\end{aligned}$$

and h_i^{ρ} is as given in (5).
Approximating the expansion in (8) by the first $(K+1)$ terms in the series, we can write

$$\zeta(u)\simeq\zeta_K(u)=\sum_{i=0}^{K} b_i\,H_i^{\rho}(u)=\boldsymbol{B}^T\,\boldsymbol{H}^{\rho}(u), \tag{9}$$

where

$$\boldsymbol{B}^T =[b_0,b_1,\ldots,b_K], \tag{10}$$

$$\boldsymbol{H}^{\rho}(u) =[H_0^{\rho}(u),H_1^{\rho}(u),\ldots,H_K^{\rho}(u)]^T. \tag{11}$$

Now, we implement in detail the formula that expresses the first derivative of the polynomials $H_i^{\rho}(u)$ in terms of their original ones. The following theorem exhibits this result.

Theorem 1. *Given $H_i^{\rho}(u)$ as in (7). The following formula holds:*

$$\begin{aligned}D\,H_i^{\rho}(u)=&\frac{2}{d-c}\sum_{j=0}^{i-1} w_{ij}\left(\frac{4\,(j+\rho)}{2\rho-1}\right.\\&\left.+\frac{2\,i!\,(j+\rho)(2\rho-3)\,\Gamma(j+2\rho)}{j!\,(2\rho-1)\,\Gamma(2\rho+i)}\right)H_j^{\rho}(u)+\delta_i(u),\end{aligned} \tag{12}$$

where $\delta_i(u)$ and w_{ij} are given by

$$\delta_i(u)=\begin{cases}c+d-2\,u, & i \text{ even},\\ c-d, & i \text{ odd},\end{cases} \tag{13}$$

$$w_{ij}=\begin{cases}0 & (i+j) \text{ even},\\ 1, & (i+j) \text{ odd}.\end{cases} \tag{14}$$

Proof. Without loss of generality, the proof can be facilitated by taking the special case of (12) corresponding to the values $c=0$ and $d=1$, So, it is enough to prove for the polynomials:

$$A_k^{\rho}(u)=u(1-u)\,U_k^{\rho}(u), \tag{15}$$

the following identity is valid:

$$D\,A_i^{\rho}(u)=2\sum_{j=0}^{i-1} w_{ij}\,\bar{Y}_{ij}^{\rho}\,A_j^{\rho}(u)+\bar{\delta}_i(u), \tag{16}$$

where $\bar{Y}^{\rho}_{ij}$ and $\bar{\delta}_i(u)$ are given by

$$\bar{Y}^{\rho}_{ij} = \frac{4(j+\rho)}{2\rho-1} + \frac{2i!(j+\rho)(2\rho-3)\Gamma(j+2\rho)}{j!(2\rho-1)\Gamma(2\rho+i)},$$

and

$$\bar{\delta}_i(u) = \begin{cases} 1-2\,u, & i \text{ even}, \\ -1, & i \text{ odd}. \end{cases} \tag{17}$$

By induction on i, we continue the proof. It is evident that the left-hand side of (16) is equal to its right-hand side, for $i=1$, which is equal to: $-6u^2+6u-1$. Assuming that relation (16) is valid for $(i-2)$ and $(i-1)$, to prove the validity of (16) itself. To complete the proof, begin with multiplying both sides of (3) (for $c=0,\ d=1$) by $u(u-1)$ and make use of relation (15), to get

$$\begin{aligned} A^{\rho}_i(u) =& 2\,(2u-1)\,\frac{i+\rho-1}{i+2\rho-1}\,A^{\rho}_{i-1}(u) \\ &- \frac{i-1}{i+2\rho-1}\,A^{\rho}_{i-2}(u),\ i=2,3,\ldots, \end{aligned} \tag{18}$$

which immediately gives

$$\begin{aligned} DA^{\rho}_i(u) =& \frac{2\,(i+\rho-1)}{i+2\rho-1}\left[(2u-1)DA^{\rho}_{i-1}(u)\right. \\ & \left. +2\,A^{\rho}_{i-1}(u)\right] - \frac{i-1}{i+2\rho-1}DA^{\rho}_{i-2}(u). \end{aligned} \tag{19}$$

Now, the induction step is applied to $DA^{\rho}_{i-1}(u)$ and $DA^{\rho}_{i-2}(u)$ in (19) along with making use of relation (3) (for $d=1$, $c=0$) yields

$$\begin{aligned} &DA^{\rho}_i(u) = \\ &\frac{2(\rho+i-1)}{2\rho+i-1}\sum_{j=0}^{i-2}\left(\frac{2\rho+j}{\rho+j}\right)\bar{Y}^{\rho}_{i-1,j}\,w_{i+j-1}\,A^{\rho}_{j+1}(u) \\ &-\frac{2(i-1)}{2\rho+i-1}\sum_{j=0}^{i-3}\bar{Y}^{\rho}_{i-2,j}\,w_{i+j-2}\,A^{\rho}_j(u) \\ &+\frac{2(\rho+i-1)}{2\rho+i-1}\sum_{j=1}^{i-2}(\frac{j}{\rho+j})\,\bar{Y}^{\rho}_{i-1,j}\,w_{i+j-1}\,A^{\rho}_{j-1}(u) \\ &+\frac{4(\rho+i-1)}{2\rho+i-1}\,A^{\rho}_{i-1}(u)+\eta_i(u), \end{aligned} \tag{20}$$

where

$$\eta_i(u) = \begin{cases} 1-2u, & i \text{ even}, \\ \dfrac{i-1-2(\rho+i-1)(2u-1)^2}{2\rho+i-1}, & i \text{ odd}. \end{cases}$$

If we make use of the following recurrence relation:

$$2\,(2\,u-1)\,A^{\rho}_i(u) = \frac{2\rho+i}{\rho+i}\,A^{\rho}_{i+1}(u) + \frac{i}{\rho+i}\,A^{\rho}_{i-1}(u), \tag{21}$$

then, we can write

$$\begin{aligned} &DA^{\rho}_i(u) = \eta_i(u) + \frac{4\,(\rho+i-1)}{2\rho+i-1}H^{\rho}_{i-1}(u) \\ &+\frac{2\,(\rho+i-1)}{2\rho+i-1}\sum_{j=0}^{i-1}w_{i,j-1}\bar{Y}^{\rho}_{i-1,j}\frac{2\rho+j}{\rho+j}H^{\rho}_{j+1}(u)+ \\ &\frac{2\,(\rho+i-1)}{2\rho+i-1}\sum_{j=1}^{i-2}w_{i,j-1}\bar{Y}^{\rho}_{i-1,j}\frac{j}{\rho+j}H^{\rho}_{j-1}(u) \\ &-\frac{2\,(i-1)}{2\rho+i-1}\sum_{j=0}^{i-3}w_{i,j-2}\bar{Y}^{\rho}_{i-2,j}H^{\rho}_j(u). \end{aligned} \tag{22}$$

For a more simplified form, expand some terms and use

$$w_i = \begin{cases} 0, & i \text{ even}, \\ 1, & i \text{ odd}. \end{cases}$$

to obtain the following relation

$$\begin{aligned} &DA^{\rho}_i(u) = \eta_i(u) \\ &+2\sum_{j=0}^{i-1}\left[\frac{\rho+i-1}{2\rho+i-1}\left(\frac{2\rho+j-1}{\rho+j-1}\right)\bar{Y}^{\rho}_{i-1,j-1}\,w_{i,j-2}\right. \\ &-\frac{i-1}{2\rho+i-1}\bar{Y}^{\rho}_{i-2,j}\,w_{i,j-2} \\ &\left.+\frac{\rho+i-1}{2\rho+i-1}\left(\frac{j+1}{\rho+j+1}\right)\bar{Y}^{\rho}_{i-1,j+1}\,w_{i,j}\right]H^{\rho}_j(u). \end{aligned} \tag{23}$$

It is not difficult to note that

$$\begin{aligned} \bar{Y}^{\rho}_{ij} =& \frac{1}{2\rho+i-1}\left((\rho+i-1)\left[\frac{2\rho+i-1}{\rho+j-1}\bar{Y}^{\rho}_{i-1,j-1}\right.\right. \\ &\left.\left.+\frac{j+1}{\rho+j+1}\bar{Y}^{\rho}_{i-1,j+1}\right]-(i+1)\bar{Y}^{\rho}_{i-2,j}\right). \end{aligned}$$

Consequently, after doing some computations, the following relation is obtained:

$$D\,A^{\rho}_i(u) = 2\sum_{j=0}^{i-1}w_{ij}\,\bar{Y}^{\rho}_{ij}\,A^{\rho}_j(u) + \bar{\delta}_i(u),$$

and therefore, Theorem 1 is now proved.

For the first derivative of the vector $\boldsymbol{H}^{\rho}(u)$, defined in (11), a matrix form is written with the use of Theorem 1 as

$$\frac{d\boldsymbol{H}^{\rho}(u)}{du} = Y\,\boldsymbol{H}^{\rho}(u) + \boldsymbol{\delta}, \tag{24}$$

where the vector $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta} = (\delta_0(u), \delta_1(u), \ldots, \delta_K(u))^T,$$

with the following components:

$$\delta_i = \begin{cases} c+d-2u, & i \text{ even}, \\ c-d, & i \text{ odd}, \end{cases}$$

In addition, the matrix Y is given by

$$Y = \frac{2}{d-c}\left(w_{ij}\bar{Y}^{\rho}_{ij}\right)_{0\leqslant i,j\leqslant K}.$$

Y is the operational matrix of derivatives of dimension $(K+1)\times(K+1)$. Its nonzero elements can be given explicitly from relation (12) as:

$$Y^{\rho}_{ij} = \frac{2}{d-c}\times \begin{cases} \dfrac{4(j+\rho)}{2\rho-1} + \dfrac{2(j+\rho)(2\rho-3)\,i!\,\Gamma(j+2\rho)}{(2\rho-1)\,j!\,\Gamma(2\rho+i)}, \\ i>j,\ (i+j)\ odd, \\ 0, \text{otherwise}. \end{cases}$$

For example, forw $K=3$, we get

$$Y = \frac{2}{d-c}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & \frac{8(1+\rho)}{1+2\rho} & 0 & 0 \\ \frac{(9+4\rho(2+\rho))}{(1+\rho)(1+2\rho)} & 0 & \frac{5(2+\rho)}{(1+\rho)} & 0 \end{pmatrix}.$$

*Remark.*The second derivative of the vector $\boldsymbol{H}^{\rho}(u)$ is given by

$$\frac{d^2\boldsymbol{H}^{\rho}(u)}{du^2} = Y^2\,\boldsymbol{H}^{\rho}(u) + Y\,\boldsymbol{\delta} + \boldsymbol{\delta'}, \tag{25}$$

where the vector $\boldsymbol{\delta'}$ is given by

$$\boldsymbol{\delta'} = \left(\delta'_0, \delta'_1, \dots, \delta'_K\right)^T,$$

with the following components

$$\delta'_i = \begin{cases} -2, & i \text{ even}, \\ 0, & i \text{ odd}. \end{cases}$$

4 Treatment of the second-order BVPs

In this section, the linear and non-linear second-order BVPs will be solved using the Petrov-Galerkin and collocation methods, respectively.

4.1 Modified Gegenbauer Petrov-Galerkin treatment of general second-order linear BVPs

Consider the following linear one-dimensional second-order BVPs

$$\epsilon\,S''(u) + J_1(u)\,S'(u) + J_2(u)\,S(u) = g_1(u), \tag{26}$$

subject to the non-homogeneous boundary conditions:

$$u\in(c,d), \quad S(c)=\alpha, \quad S(d)=\beta. \tag{27}$$

The following transformation (see, [30])

$$S(u) = \zeta(u) + \frac{\alpha\,(d-u)+\beta\,(u-c)}{d-c},$$

can be used to transform (26) into the following modified equation

$$\epsilon\,\zeta''(u) + J_1(u)\,\zeta'(u) + J_2(u)\,\zeta(u) = g(u),\ u\in(c,d), \tag{28}$$

governed by the homogeneous boundary conditions:

$$\zeta(c) = \zeta(d) = 0, \tag{29}$$

where

$$\begin{aligned} g(u) =& g_1(u) - \frac{\beta-\alpha}{d-c}\,J_1(u) \\ &- \frac{\alpha\,(d-u)+\beta\,(u-c)}{d-c}\,J_2(u). \end{aligned}$$

Approximating $\zeta(u)$, $\zeta'(u)$ and $\zeta''(u)$, using (9), (24) and (25), respectively, we can write

$$\zeta(u) \simeq \boldsymbol{B}^T\,\boldsymbol{H}^{\rho}(u), \tag{30}$$

$$\zeta'(u) \simeq \boldsymbol{B}^T\,Y\,\boldsymbol{H}^{\rho}(u) + \boldsymbol{B}^T\,\boldsymbol{\delta}, \tag{31}$$

$$\zeta''(u) \simeq \boldsymbol{B}^T\,Y^2\,\boldsymbol{H}^{\rho}(u) + \boldsymbol{B}^T\,Y\,\boldsymbol{\delta} + \boldsymbol{B}^T\,\boldsymbol{\delta'}. \tag{32}$$

By using (30),(31) and (32), the residual $R(u)$ of (28) can be written as:

$$\begin{aligned} R(u) =& \epsilon\,\boldsymbol{B}^T\ \ Y^2\boldsymbol{H}^{\rho}(u) + \epsilon\,\boldsymbol{B}^T\,Y\,\boldsymbol{\delta} \\ &+ \epsilon\,\boldsymbol{B}^T\,\boldsymbol{\delta'} + J_1(u)\,\left(\boldsymbol{B}^T\,Y\,\boldsymbol{H}^{\rho}(u) + \boldsymbol{B}^T\,\boldsymbol{\delta}\right) \\ &+ J_2(u)\,\left(\boldsymbol{B}^T\,\boldsymbol{H}^{\rho}(u)\right) - g(u). \end{aligned} \tag{33}$$

By the application of the Petrov-Galerkin method (see, [11]) , the following $(K+1)$ linear equations in the unknown expansion coefficients, b_i, can be obtained

$$\int_c^d R(u)\,U^{\rho}_i(u)\,du = 0, \quad i = 0,1,\dots,K. \tag{34}$$

Consequently, the system resulting from Eq. (34) can be solved by using any suitable solver for the unknown vector $\boldsymbol{B}$ components, and thus the approximate spectral solution $\zeta_K(u)$ that is given by (9) can be obtained.

4.2 Modified Gegenbauer collocation treatment of general second-order non-linear BVPs

Consider the following non-linear one-dimensional second-order BVP:

$$\zeta''(u) = f\left(u, \zeta(u), \zeta'(u)\right), \quad u\in(c,d), \tag{35}$$

subject to the following non-homogeneous boundary conditions:

$$\zeta(c)=\gamma_1,\zeta(d)=\gamma_2. \tag{36}$$

If we follow the same procedures of Section 4.1, taking into consideration the approximation of $\zeta(u)$ in (30), and the two relationships (24) and (25), we can write the following non-linear equation in the unknown vector **B**.

$$\begin{aligned}&\boldsymbol{B}^T\mathbf{Y}^2\boldsymbol{H}^{\rho}(\boldsymbol{u})+\boldsymbol{Y}^T\,Y\,\boldsymbol{\delta}+\boldsymbol{B}^T\,\boldsymbol{\delta}'\\&=F\left(u,\boldsymbol{B}^T\,\boldsymbol{H}^{\rho}(\boldsymbol{u}),\boldsymbol{B}^T\,Y\,\boldsymbol{H}^{\rho}(\boldsymbol{u})+\boldsymbol{B}^T\,\boldsymbol{\delta}\right).\end{aligned} \tag{37}$$

To find the numerical solution $\zeta_K(u)$, we enforce (37) to be satisfied exactly at the first $(K+1)$ roots of the polynomial $\tilde{U}^{\rho}_{K+1}(u)$. Thus, a set of $(K+1)$ non-linear equations can be obtained. Newton's iterative scheme can be employed to determine the values of the expansion coefficients b_i. Therefore, the approximate solution $\zeta_K(u)$ can be found.

4.3 Numerical treatment of the two-dimensional Bratu equation

In this section, we consider the following Bratu's type equation in two dimensions given by

$$\partial_{uu}\zeta(u,g)+\partial_{gg}\zeta(u,g)+\lambda e^{\zeta(u,g)}=0, \tag{38}$$

subject to the following homogeneous boundary conditions:

$$\zeta(u,0)=\zeta(u,1)=0,\quad \zeta(0,g)=\zeta(1,g)=0. \tag{39}$$

We assume the following approximate solution for (38)

$$\begin{aligned}\zeta_{K,K}(u,g)&=\sum_{j=0}^{K}\sum_{i=0}^{K}f_{ij}H^{\rho}_i(u)H^{\rho}_j(g)\\&=(H^{\rho})^T(u)\,\boldsymbol{F}\,H^{\rho}(g),\end{aligned}$$

where

$$\boldsymbol{F}=(f_{ij})_{0\le i,j\le K}.$$

The residual of (38) can be written in the following form:

$$\begin{aligned}R(u,g)=&\sum_{j=0}^{K}\sum_{i=0}^{K}f_{ij}\partial_{uu}H^{\rho}_i(u)\,H^{\rho}_j(g)\\&+\sum_{j=0}^{K}\sum_{i=0}^{K}f_{ij}\,H^{\rho}_i(u)\partial_{gg}\,H^{\rho}_j(g)\\&+\lambda\,e^{\sum_{j=0}^{K}\sum_{i=0}^{K}f_{ij}\,H^{\rho}_i(u)\,H^{\rho}_j(g)}.\end{aligned} \tag{40}$$

Taking into consideration the approximation of $\zeta(u)$ in (30), and the two relationships (24) and (25), we can write the following non-linear equation in the unknown vector $\boldsymbol{F}$.

$$\begin{aligned}&H^T(g)\boldsymbol{F}(\boldsymbol{F}^TY^2H(u)+\boldsymbol{F}^TY\delta+\boldsymbol{F}^T\delta')\\&+H^T(u)\boldsymbol{F}(\boldsymbol{F}^TY^2H(g)+\boldsymbol{F}^TY\delta+\boldsymbol{F}^T\delta')\\&=\lambda e^{H^T(u)\boldsymbol{F}H(g)}.\end{aligned} \tag{41}$$

To find the desired numerical solution $\zeta_{K,K}(u)$, we enforce (41) to be satisfied exactly at the first $(K+1)^2$ assumed roots, substituting explicitly with,

$$(u,g)=\left(\frac{1+cos(\frac{i+1}{2\,K}\pi)}{2},\frac{1+cos(\frac{j+1}{2\,K}\pi)}{2}\right),\quad i,j=0,1,2,\ldots,K.$$

Thus a system of $(K+1)^2$ equations can be obtained. This system can be solved with the aid of Newton's iterative method can be solved to determine the values of the expansion coefficients $f_{i,j}$. Therefore, the approximate solution $\zeta_{K,K}(u,g)$ can be found.

5 Analysis of convergence

In this section, we state and prove a theorem for the convergence of the proposed methods in Sections 3.1 and 3.2.

Theorem 2.*Let $\zeta(u)$ be the analytic solution of (26), or (35), and $\zeta_K(u)=\sum_{i=0}^{K}b_i\,H^{\rho}_i(u)$ be the suggested approximate solution, then the expansion coefficients satisfy the following estimate:*

$$|b_i|<\frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}\rho}i^{2-2\rho-q}.$$

*Proof.*Begin with the analytic solution

$$\zeta(u)=\sum_{i=0}^{\infty}b_i\,H^{\rho}_i(u), \tag{42}$$

multiply both sides by $H^{\rho}_j(u)\,\text{ß}(u),\ j\ge 0$, and integrate with respect to u from a to b to get

$$b_i=\frac{1}{h^{\rho}_i}\int_c^d \text{ß}(u)\,\zeta(u)H^{\rho}_i(u)\,du. \tag{43}$$

Noting that $\zeta(u)$ can be represented as:

$$\zeta(u)=(u-c)(d-u)\,g(u),$$

the coefficients are given as

$$b_i=\int_c^d\bar{w}(u)\,g(u)\tilde{U}^{\rho}_i(u)\,du, \tag{44}$$

and $\bar{w}(u) = ((u-c)(d-u))^{\rho-\frac{1}{2}}$. If the right-hand side of (43) is integrated twice by parts, then we get

$$b_i = -\int_c^d \bar{w'}(u)\, I_i'(u)\big(g'(u)\bar{w}(u) + g(u)\big); \quad |g^{(q)}(u)| < M_q < M. \tag{45}$$

From the last equation, we have

$$|b_i| < \left|\int_c^d I_i^{(1)}(u)\big(g'(u)\bar{w}(u) + \bar{w'}(u)g(u)\big)\right| < \int_c^d |I_i^{(1)}(u)|\, M\, \big(\bar{w}(u) + \bar{w'}(u)\big).$$

Noting the inequality:

$$|I_i^{(q)}(u)| < \left(\frac{d-c}{2i}\right)^q \lessapprox O(i^{-q}),\ \ q \geq 1,$$

the case $q = 1$ gives

$$|I_i^{(1)}(u)| < \left(\frac{d-c}{2i}\right) \lessapprox O(i^{-1}),$$

and therefore, we have

$$|b_i| < \frac{M(d-c)}{2i}\int_c^d |\bar{w}(u) + \bar{w'}(u)|du.$$

If we note the following two identities:

$$\int_c^d w'(u)du = 0, \qquad \int_c^d ß(u)du = (d-c)^{2\rho}\,\frac{\Gamma^2\left(\rho+\frac{1}{2}\right)}{\Gamma(2\,\rho+1)},$$

then we get

$$|b_i| < \frac{M(d-c)^{2\rho+1}}{2\,i\Gamma(2\rho+1)}\Gamma^2\left(\rho+\frac{1}{2}\right) \lessapprox O(i^{-1}). \tag{46}$$

Similarly, integration by parts q times will yield

$$|b_i| < \frac{M(d-c)^q}{(2i)^q}\sum_{m=0}^{q} U_m^\rho \int_c^d |\bar{w}^m(u)|du, \tag{47}$$

hence we get

$$|b_i| < \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}\rho}\, i^{2-2\rho-q}. \tag{48}$$

Lemma 1. *The following estimates are valid for $H_i^\rho(u)$ and its first and second derivatives*

1. $|H_i^\rho(u)| \lessapprox i^{-\rho}$,
2. $|DH_i^\rho(u)| \lessapprox i^{1-\rho}$,
3. $|D^2H_i^\rho(u)| \lessapprox i^{2-\rho}$.

Proof. The above estimations can be proved through the mathematical induction.

Theorem 3. *If $\zeta(u)$ satisfies the hypothesis of Theorem 2, then for $\rho > 0$, we have the following truncation error estimates:*

1. $|\zeta_i(u) - \zeta(u)| \lessapprox i^{3-3\rho-q}; q > 4$,
2. $\|\zeta_i'(u) - \zeta'(u)\| \lessapprox i^{4-3\rho-q}; q > 5$,
3. $\|\zeta_i''(u) - \zeta''(u)\| \lessapprox i^{5-3\rho-q}; q > 6$.

Proof. To prove the first inequality, using (48), and Lemma 1, we have

$$\begin{aligned}\|\zeta_i(u) - \zeta(u)\| &= \left\|\sum_{s=i+1}^{\infty} b_i H_i^\rho(u)\right\| \\ &\leq \sum_{s=i+1}^{\infty} |b_i|\big|H_i^\rho(u)\big| \\ &< \sum_{s=i+1}^{\infty} \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}\rho} i^{2-3\rho-q} \\ &< \int_i^\infty \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}\rho} p^{2-3\rho-q}dp \\ &< \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}(3\rho+q-3)\rho} i^{3-3\rho-q} \\ &\lessapprox i^{3-3\rho-q}.\end{aligned} \tag{49}$$

Based on the inequalities in Lemma 1, and using Theorem 2 we have,

$$\begin{aligned}\|\zeta_i' - \zeta'\| &= \left\|\sum_{s=i+1}^{\infty} b_i DH_i^\rho(u)\right\| \\ &\leq \sum_{s=i+1}^{\infty} |b_i|\big|DH_i^\rho(u)\big| \\ &< \sum_{s=i+1}^{\infty} \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}\rho} i^{3-3\rho-q} \\ &< \int_i^\infty \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}\rho} p^{3-3\rho-q}dp \\ &< \frac{M(d-c)^{2\rho+q}}{2^{2\rho+q}(3\rho+q-4)\rho} i^{4-3\rho-q} \lessapprox i^{4-3\rho-q}.\end{aligned} \tag{50}$$

The proof of the third inequality is similar to the proofs of the first and second inequalities but using Theorem 2, and Lemma 1.

Theorem 4. *Under the assumption of Theorems 2 and 3, then we have the following global error estimate:*

$$\epsilon\|\zeta_i''(u)-\zeta''(u)\|+\|\zeta_i'(u)-\zeta'(u)\|+\|\zeta_i(u)-\zeta(u)\| \lessapprox i^{5-2\rho-q}. \tag{51}$$

Proof. using Theorem 3 and Lagrange's notations [23], we can write

$$\begin{aligned}&\epsilon\|\zeta_i''(u)-\zeta''(u)\|+\|\zeta_i'(u)-\zeta'(u)\|+\|\zeta_i(u)-\zeta(u)\|\\ &\lessapprox \epsilon i^{3-3\rho-q}+i^{4-3\rho-q}+i^{5-3\rho-q}\\ &\lessapprox i^{5-3\rho-q}.\end{aligned}$$

6 Numerical results and discussions

This section is devoted to the application of our two proposed algorithms, namely, the modified Gegenbauer Petrov-Galerkin method ($MGPGM$) and the modified Gegenbauer collocation method ($MGCM$) to treat some specific second-order BVPs. More definitely, non-singular, singular, and singularly perturbed BVPs will be solved via the presented algorithms. The emphasis is on small errors to show the efficiency of the technique.

Example 1. (see, [36])
Consider the following second-order BVP:

$$\begin{aligned}&\zeta''(u)=\frac{1}{2}(\zeta+u+1)^3, \quad u\in(0,1),\\ &\zeta(0)=\zeta(1)=0.\end{aligned} \tag{52}$$

The exact solution of (52) is

$$\zeta(u)=\frac{2}{2-u}-u-1.$$

Table 1 shows the maximum absolute error E for various values of K and ρ, while Table 2 shows a comparison between the approximate solution of problem (52) gotten by the application of $MGCM$ with the two approximate solutions gotten by the Sinc-collocation method (SCM) and the Sinc-Galerkin method (SGM) in Ref. [36]. Figure 1 shows the resulting absolute errors from the application of $MGCM$ for the case corresponding to $\rho=\frac{1}{2}$.

Table 1: Maximum absolute error (MAE) for Ex. 1

ρ/K	10	14	18	22
0	$1.7\,.10^{-9}$	$6.1\,.\,10^{-13}$	$1.0\,.10^{-15}$	$1.4\,.10^{-16}$
0.5	$7.4\,.10^{-10}$	$4.3\,.\,10^{-13}$	$4.6\,.10^{-16}$	$1.1\,.10^{-16}$
1	$1.1\,.10^{-9}$	$5.7\,.\,10^{-13}$	$7.3\,.10^{-16}$	$3.8\,.10^{-16}$

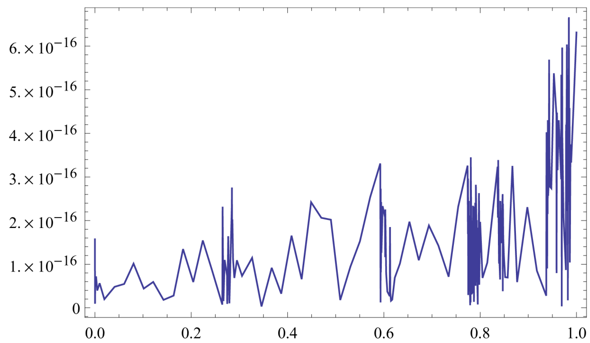

Fig. 1: The best absolute error for Ex. 1 for $\rho=0.5$

Table 2: Comparison between different solutions for Ex. 1

Method	$MGCM$	SCM in [36]	SGM in [36]
K	22	130	130
MAE	$1.04\,.10^{-16}$	$9.159\,.10^{-16}$	$9.992\,.10^{-16}$

Example 2. (see, [36])
Consider the following second-order BVP:

$$\begin{aligned}&\zeta''(u)=\\ &\frac{1}{(u^\sigma)'}\left(\frac{s\,u^{\sigma+s-2}}{4+u^s}\left(s\,u^s\,e^{\zeta}(u)-\sigma-s+1\right)-(u^\sigma)'\,\zeta(u)'\right),\\ &u\in(0,1),\ \zeta(0)=\ln\left(\frac{1}{4}\right),\ \zeta(1)=\ln\left(\frac{1}{5}\right),\\ &s=3-\sigma,\ \sigma\in(0,1),\end{aligned} \tag{53}$$

with the exact solution

$$\zeta(u)=-\ln\left(4+u^s\right).$$

Maximum absolute error (MAE) is listed in Table 3 for various values of σ and K, while Table 4 displays a comparison between the $MGCM$ with those obtained by the Sinc-collocation method (SCM) and the Sinc-Galerkin method (SGM) in [36] for the case corresponding to the case $\sigma=\frac{1}{4}$. Furthermore, Figure 2 shows the absolute error arising using $MGCM$ corresponding to $K=10,\ \sigma=\frac{1}{4}$ and $K=15,\ \sigma=\frac{1}{4}$. The results of Table 4 shows that our obtained numerical results are more accurate than those obtained by the two methods followed in [36].

Example 3. Consider the following second-order BVP [21]:

$$\begin{aligned}&\epsilon S''(u)+S'(u)-S(u)=0,\ u\in(0,1),\\ &S(0)=\frac{2\bar{\epsilon}\,e^{\frac{1+\bar{\epsilon}}{2\epsilon}}}{(\bar{\epsilon}+2\epsilon+1)e^{\frac{\bar{\epsilon}}{\epsilon}}+\bar{\epsilon}-2\epsilon-1},\quad S(1)=1,\end{aligned} \tag{54}$$

Table 3: MAE for Ex. 2

σ	K	10	15	20
0.01	$\rho=0$	$2.74\ .10^{-8}$	$1.47\ .10^{-11}$	$2.44\ .\ 10^{-14}$
	$\rho=0.5$	$1.79\ .10^{-8}$	$9.45\ .10^{-12}$	$8.88\ .\ 10^{-15}$
	$\rho=1$	$1.21\ .10^{-8}$	$7.75\ .10^{-12}$	$6.23\ .\ 10^{-15}$
0.25	$\rho=0$	$2.79\ .10^{-8}$	$1.57\ .10^{-11}$	$9.77\ .\ 10^{-15}$
	$\rho=0.5$	$1.83\ .10^{-8}$	$9.53\ .10^{-12}$	$2.22\ .\ 10^{-14}$
	$\rho=1$	$1.23\ .10^{-8}$	$7.84\ .10^{-12}$	$2.02\ .\ 10^{-14}$
0.5	$\rho=0$	$2.87\ .10^{-8}$	$1.58\ .10^{-11}$	$1.11\ .\ 10^{-14}$
	$\rho=0.5$	$1.87\ .10^{-8}$	$1.04\ .10^{-11}$	$4.26\ .\ 10^{-14}$
	$\rho=1$	$1.27\ .10^{-8}$	$7.93\ .10^{-12}$	$3.26\ .\ 10^{-14}$
0.99	$\rho=0$	$2.96\ .10^{-8}$	$1.61\ .10^{-11}$	$9.99\ .\ 10^{-15}$
	$\rho=0.5$	$1.95\ .10^{-8}$	$1.06\ .10^{-11}$	$2.55\ .\ 10^{-14}$
	$\rho=1$	$1.35\ .10^{-8}$	$8.09\ .10^{-12}$	$7.99\ .\ 10^{-15}$

Table 4: Comparison between different solutions for Ex. 2, $\sigma=\frac{1}{4}$

Method	$MGCM$	SCM in [36]	SGM in [36]
K	20	100	100
MAE	$6.23\ .10^{-15}$	$1.55\ .10^{-6}$	$1.55\ .10^{-6}$

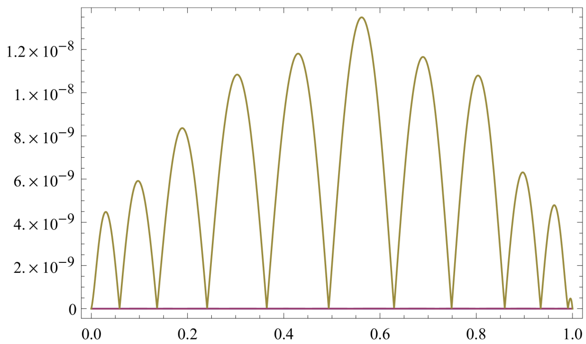

Fig. 2: $\zeta_{20}(u), \zeta_{15}(u), and\, \zeta_{10}(u)$ for Ex. 2 for $\rho=1$

where $\bar{\epsilon}=\sqrt{4\epsilon+1}$, with the exact solution

$$S(u)=e^{\frac{(\bar{\epsilon}+1)(1-u)}{2\epsilon}}\frac{(\bar{\epsilon}+2\epsilon+1)\,e^{\frac{u\bar{\epsilon}}{\epsilon}}+\bar{\epsilon}-2\epsilon-1}{(\bar{\epsilon}+2\epsilon+1)\,e^{\frac{\bar{\epsilon}}{\epsilon}}+\bar{\epsilon}-2\epsilon-1}.$$

The maximum absolute errors E for various values of ϵ, K, and ρ are shown in Table 5. Table 6 presents a comparison the best errors resulted from the application of our method with those obtained the the method followed in between our proposed method and the proposed method in [23], while in Table 7 a comparison is presented between the approximate solutions of Ex. 3 obtained by $MGPGM$ and those obtained by the shooting method in [35].

*Remark.*It is clear from the results presented in Tables 6 and 7 that our results are more accurate if compared with those obtained in [23] and [35]. This demonstrates the advantage of our proposed method.

Table 5: MAE for Ex. 3

ϵ	ρ/K	7	9	11
10^{-3}	0	$6.8\ .10^{-6}$	$2.5\ .\ 10^{-9}$	$3.6\ .10^{-15}$
	0.5	$3.3\ .10^{-6}$	$2.6\ .10^{-9}$	$4.3\ .\ 10^{-15}$
	1	$6.2\ .10^{-7}$	$6.4\ .10^{-10}$	$9.2\ .\ 10^{-16}$
10^{-4}	0	$4.6\ .10^{-8}$	$5.3\ .10^{-10}$	$1.1\ .\ 10^{-14}$
	0.5	$3.6\ .10^{-7}$	$7.5\ .10^{-10}$	$4.6\ .\ 10^{-15}$
	1	$9.3\ .10^{-9}$	$1.2\ .\ 10^{-11}$	$2.2\ .10^{-16}$

Table 6: Comparison between the best errors for Ex. 3

ϵ	Our Method	[23]
10^{-3}	$9.2\ .10^{-16}$	$3.5\ .10^{-14}$
10^{-4}	$2.2\ .10^{-16}$	$3.2\ .10^{-14}$

Table 7: Comparison between the best errors for Ex. 3

Method	$MGPGM$	Shooting method [35]
Best error	$3.57\ .10^{-9}$	$3.677\ .10^{-5}$

*Example 4.*Consider the following second-order BVP:

$$\begin{aligned}&\zeta''=\left(\zeta'(u)\right)^2-16\,\zeta(u)+2-16\,u^6;u\in(-1,1),\\&\zeta(-1)=\zeta(1)=0,\end{aligned}\tag{55}$$

with the exact solution:
$\zeta(u)=u^2-u^4$.
Making use of (9) with $K=2$, yields

$$\begin{aligned}\zeta_N(u)=&\boldsymbol{B}^T\,\boldsymbol{H^{\rho}(u)}\\=&(1-u^2)\left(b_0\,U_0^{\rho}(u)+b_1\,U_1^{\rho}(u)+b_2\,U_2^{\rho}(u)\right).\end{aligned}$$

Moreover, in this case the two matrices Y and Y^2 are given as

$$Y=\begin{pmatrix}0&0&0\\3&0&0\\0&\frac{28}{5}&0\end{pmatrix},\quad Y^2=\begin{pmatrix}0&0&0\\0&0&0\\\frac{84}{5}&0&0\end{pmatrix}.$$

Now, with the aid of Eq. (37), we have

$$\begin{aligned}\zeta_N(u) =&16(1-u^2)\left(b_0+b_1\,u+\frac{8b_2}{15}\left(\frac{21}{8}u^2-\frac{3}{4}\right)\right)\\&-\left(2b_0+6b_1\,u+\frac{66}{5}b_2-\frac{84b_2}{5}(1-u^2)\right)\\&+\left(2b_0\,u+2b_1-3\,b_1\,(1-u^2)+2\,b_2\,u\right.\\&\left.-\frac{28}{5}b_2\,u(1-u^2)\right)^2\end{aligned}$$

Begin to generate three non-linear algebraic equations using (37) by substituting with the roots of $U_3^{\frac{3}{4}}(u)$, namely $-\sqrt{\frac{6}{11}}$, 0, $\sqrt{\frac{6}{11}}$. Solving these equations to evaluate c_0, c_1 and c_2, we get

$$b_0=\frac{2}{7},\quad b_1=0,\quad b_2=\frac{5}{7},$$

and hence

$$\begin{aligned}\zeta(u)&=\left(\frac{2}{7},\ 0,\ \frac{5}{7}\right)\begin{pmatrix}1-u^2\\ u-u^3\\ +\frac{1}{5}\left(-2+9u^2-7u^4\right)\end{pmatrix}\\&=u^2-u^4,\end{aligned}$$

which is the exact solution.

*Example 5.*Bratu equation, (see, [31–34]).
Consider the following second-order BVP of Bratu's Type:

$$\begin{aligned}&\zeta''=-\lambda\,e^{\zeta(u)},\quad u\in(0,1)\\&\zeta(0)=\zeta(1)=0,\end{aligned}\tag{56}$$

where $\lambda>0$. With the analytical solution [30]

$$\zeta(u)=-2\ln\left[\frac{\cosh\left(\frac{\theta}{4}(2u-1)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right].\tag{57}$$

Knowing that θ is the solution of the non-linear equation: $\theta=\sqrt{2\lambda}\cosh\theta$, and using $\lambda=1,2,3.51$, θ for our example will hold three corresponding values, namely 1.51716, 2.35755 and 4.66781. The maximum absolute error E is stated in Table 8. We include a comparison for different values of K, and σ. Table 9 is a comparison between the best errors obtained by various methods used to solve (56). This table illustrates that our strategy is more accurate when compared to the techniques developed in [31–34].

*Example 6.*Consider the non-linear second-order BVP [40]

$$\begin{aligned}&\zeta''=e^{\zeta(u)}\left(0.5-e^{\zeta(u)}\right)-\frac{0.5}{u}\zeta'(u);u\in(0,1),\\&\zeta(0)=\ln(2),\,\zeta(1)=0,\end{aligned}\tag{58}$$

Table 8: MAE for Ex. 5

K	σ	1	2	3.51
8	$\rho=0$	$2.92\,.10^{-11}$	$4.61\,.10^{-9}$	$4.23\,.\,10^{-5}$
	$\rho=0.5$	$2.02\,.10^{-11}$	$8.67\,.10^{-11}$	$1.28\,.\,10^{-5}$
	$\rho=1$	$1.63\,.10^{-11}$	$2.73\,.10^{-9}$	$1.03\,.\,10^{-5}$
10	$\rho=0$	$3.40\,.10^{-13}$	$1.28\,.10^{-10}$	$6.98\,.\,10^{-7}$
	$\rho=0.5$	$2.04\,.10^{-13}$	$8.67\,.10^{-11}$	$4.70\,.\,10^{-7}$
	$\rho=1$	$2.02\,.10^{-13}$	$7.61\,.10^{-11}$	$1.13\,.\,10^{-6}$
12	$\rho=0$	$4.43\,.10^{-15}$	$3.46\,.10^{-12}$	$7.67\,.\,10^{-8}$
	$\rho=0.5$	$2.97\,.10^{-15}$	$2.32\,.10^{-12}$	$3.76\,.\,10^{-8}$
	$\rho=1$	$2.04\,.10^{-15}$	$2.04\,.10^{-12}$	$8.56\,.\,10^{-8}$
14	$\rho=0$	$5.31\,.10^{-16}$	$8.74\,.10^{-14}$	$6.12\,.\,10^{-9}$
	$\rho=0.5$	$4.37\,.10^{-16}$	$6.49\,.10^{-14}$	$3.31\,.\,10^{-9}$
	$\rho=1$	$5.34\,.10^{-16}$	$5.21\,.10^{-14}$	$7.02\,.\,10^{-9}$
16	$\rho=0$	$4.64\,.10^{-16}$	$2.78\,.10^{-15}$	$5.68\,.\,10^{-10}$
	$\rho=0.5$	$4.77\,.10^{-16}$	$2.28\,.10^{-15}$	$3.07\,.\,10^{-10}$
	$\rho=1$	$5.02\,.10^{-16}$	$1.78\,.10^{-15}$	$6.44\,.\,10^{-10}$
18	$\rho=0$	$4.80\,.10^{-16}$	$7.49\,.10^{-16}$	$4.86\,.\,10^{-11}$
	$\rho=0.5$	$5.33\,.10^{-16}$	$7.08\,.10^{-16}$	$3.93\,.\,10^{-11}$
	$\rho=1$	$4.31\,.10^{-16}$	$7.08\,.10^{-16}$	$6.27\,.\,10^{-11}$

Table 9: Comparison between the best errors for Ex. 5 for $\lambda=1$

Our error	[31]	[32]	[33]	[34]
$4.3\,.10^{-16}$	$1.0\,.10^{-6}$	$8.9\,.10^{-6}$	$1.4\,.10^{-5}$	$3.0\,.10^{-3}$

with the following exact solution:

$$\zeta(u)=\ln\left(\frac{2}{u^2+1}\right).$$

Making use of Section 4.2, with small values of K, and $\rho=0,0.5,1$ yields efficient approximate solutions as shown in Table 10. In addition, Table 11 presents a comparison between the best errors resulted from our method and the method developed in [40].

Table 10: MAE for Ex. 6

ρ/K	9	10	11	22
0	$7.42\,.10^{-9}$	$3.06\,.\,10^{-9}$	$7.87\,.10^{-10}$	$2.78\,.10^{-16}$
0.5	$5.61\,.10^{-9}$	$2.01\,.\,10^{-9}$	$4.99\,.10^{-10}$	$6.66\,.10^{-16}$
1	$4.16\,.10^{-9}$	$1.46\,.\,10^{-9}$	$3.14\,.10^{-10}$	$6.66\,.10^{-16}$

*Example 7.*Consider the following second-order two dimensional Bratu's type equation [41]

$$\partial_{uu}\zeta(u,g)+\partial_{gg}\zeta(u,g)+\lambda e^{\zeta(u,g)}=0,$$

Table 11: Comparison between the best errors for Ex. 6

Method	$MGCM$	Chebyshev wavelets [40]
Best error	6.66 $.10^{-16}$	1 $.10^{-9}$

with the exact analytic solution

$$\zeta(u,g) = 2\ln\left[\frac{\cosh\left(\frac{\theta}{4}\right)\cosh\left(\left(u-\frac{1}{2}\right)\left(g-\frac{1}{2}\right)\theta\right)}{\cosh\left(\left(u-\frac{1}{2}\theta\right)\cosh\left(\left(g-\frac{1}{2}\theta\right)\right.\right.}\right], \quad \forall (u,g) \in (0,1)\times(0,1). \tag{59}$$

For $u = g = \frac{1}{2}$, ones can get θ_c values which are corresponding to critical λ_c We will solve this example for the case corresponds to $\theta_c = 4.798714561$ and $\lambda_c = 7.027661438$.This example is one of the Newton Raphson applications. The initiation of the method was chosen randomly. Table 12 shows a comparison between the best errors resulted from the application of MGCM and the variational method in [41].

Table 12: Comparisons between methods for Ex. 7

method	$MGCM$	variational method [41]
Best errors	1.11 $.10^{-15}$	9.12 $.10^{-4}$

7 Concluding remarks

We have constructed a set of polynomials namely, modified Gegenbauer polynomials. We established the operational matrices of derivatives of such polynomials. The second-order BVPs were treated using these modified polynomials as basis functions. For handling the linear and non-linear second-order BVPs, the two spectral methods, namely, Petrov-Galerkin and collocation methods, were respectively utilized. Some important specific problems such as singular perturbed equations, as well as Bratu-type equations, were treated with acceptable errors. Also, with a small degree of the assumed solution, high-precision approximate solutions are possible. Therefore, numerical schemes approach the analytical results and, in some problems, analytical results are reachable. As future work, we aim to extend the construction of the modified shifted Gegenbauer polynomials as well as their operational matrices to be capable of treating high-order BVPs. All codes were written and debugged by Mathematica 11 on HP EliteBook 840 G3 with configuration, Core i5-6200U, 8 GB memory, 2.3 GHz processor Intel, and 256 GB SSD storage.

Acknowledgments: The authors would like to thank the editor for his assistance, as well as the anonymous referees for their comments.

Competing Interests Statement

The authors declare that they have no any competing interests.

Data Availability Statement

No data is associated with this research.

Funding Statement

No funding for this research.

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