

Optimal Protocol for Polarization Ququart State Tomography

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We develop a practical quantum tomography protocol and implement measurements of pure states of ququarts realized with polarization states of photon pairs (biphotons). The method is based on an optimal choice of the measuring scheme's parameters that provides better quality of reconstruction for the fixed set of statistical data. A high accuracy of the state reconstruction (above 0.99) indicates that developed methodology is adequate.

Keywords: Quantum tomography, biphotons, polarization ququart.

1 Introduction.

For the last several years many elegant experiments were performed in which different kinds of multi-dimensional quantum states (qudits) were introduced [17–21, 23–26] (for more details see the review [11]). Most of them are based on the states of light emitted via spontaneous parametric down-conversion (SPDC). In this process photons of the laser pump decay on the pairs of photons (biphotons) inside the crystal possessing non-zero quadratic susceptibility. In stationary case the sum of daughter photons frequencies coincides with the frequency of the pump $w_s + w_i = w_p$ and intensity of the biphoton light emitted from the crystal is maximal when phase matching condition $k_s + k_i = k_p$ holds. Choosing particular regime of SPDC one can prepare a whole family of the biphoton states with different properties. In this paper we focus upon polarization states of biphotons although some other qudits can be realized with photon pairs. Selecting so called frequency degenerate ($w_s = w_i$) and collinear ($k_s || k_i$) regime of SPDC, polarization qutrits ($d = 3$) can be realized. Preparation of arbitrary polarization qutrit ($d = 3$) was reported in [6]. The experimental method for engineering with pure states of ququarts ($d = 4$) was presented in [9, 10].

Quality assurance of preparation and transformation requires a complete characterization of these states, which can be accomplished through a procedure known as quantum process tomography. The first protocol for reconstruction of polarization qutrits was introduced in [7, 16]. The protocol for quantum tomography of polarization states of photon pair propagating along two spatial modes was suggested in [12]. Further this protocol was implemented and modified for the case of single spatial mode [9]. However in order to distinguish the photons forming biphoton, non-degenerate regime of SPDC was accomplished to realize a ququart. This sort of states seems to be promising for quantum communication problems because it allows one to pass the states of two photons (both entangled and product) along single spatial mode for example in optical fiber.

Although tomographic procedure has been used in many experiments and right now serves as an “application tool”, still there are several problems related to simple and in some sense optimal choice of protocol. By “optimal” protocol we mean firstly how to implement the procedure with minimal number of measurements for achieving highest accuracy. For example the remarkable paper [22] analyzes a minimal measurement scheme for single-qubit tomography. Their analysis showed that the scheme is efficient in the sense that it enables one to estimate the qubit state without enormous number of qubits - a few thousand are sufficient for most practical applications. Also the work [22] indicates algorithms for manipulations with qudits. But from practical point of view there is the second reason to introduce an optimality of the protocol. Usually the experimentalist possess by limited resources to perform the measurements. For example he has a set of retardant plates with fixed optical thickness but he can not access any other particular plates which are necessary to perform the protocol according to an optimal way (see previous point). Moreover sometimes an experimentalist has a limited time for doing the measurement and he is not able to accumulate as many statistical data as it would be necessary. One of the trivial reason for that might be instability of the experimental set-up. So practically any measurement set has a limited size and it is not convenient (or even is not possible) to increase it for achieving complete volume. That is why it would be useful to take into account the set of available tools and develop a protocol which gives the highest accuracy for fixed experimental resources.

The present work is to address the experimental problem of realization of the optimal state reconstruction for biphoton-based polarization ququarts. We restrict ourselves with the protocol of quantum tomography suggested and tested earlier [9]. Basically the method is based on an optimal choice of the measuring scheme’s parameters that provides better quality of reconstruction with the fixed set of statistical data.

2 Quantum Tomography, Principle of Realization

An arbitrary quantum state is completely determined by a wave vector for pure state, or by a density matrix for mixed state. To measure the quantum state one needs to perform

a set of projective measurements and then to apply some computation procedure to the data obtained at the previous stage. It is well known that the number of real parameters characterizing a quantum state is determined by the dimension of the Hilbert space d . For a pure state,

$$N_{pure} = 2d - 2, \quad (2.1)$$

and for a mixed state,

$$N_{mixed} = d^2 - 1. \quad (2.2)$$

However in practice the normalization is necessary to be established with the data so the total number of measurements increases by one. According to Bohr's complementarity principle, it is impossible to measure all projections simultaneously, operating with single quantum state only. So, first of all, one needs to generate a lot of the same representatives of a quantum ensemble [10]. In our experiment for preparation such states we used the process of spontaneous parametric down-conversion (SPDC). So fixing the conditions under which SPDC takes place we achieve the initial states which serve as a base for further manipulations.

As it was already mentioned above we deal with the collinear and frequency non-degenerate regime of biphoton field for which $k_s \parallel k_i$, and $w_s \neq w_i$. From the point view of polarization there are four natural states of photons pairs: $|H_s H_i\rangle$, $|H_s V_i\rangle$, $|V_s H_i\rangle$, $|V_s V_i\rangle$. Then any pure polarization state of biphoton can be expressed as superposition of four basis states:

$$|C\rangle = c_1 |H_s H_i\rangle + c_2 |H_s V_i\rangle + c_3 |V_s H_i\rangle + c_4 |V_s V_i\rangle. \quad (2.3)$$

Here $c_i = |c_i| e^{i\phi_i}$, $\sum_{i=1}^4 |c_i|^2 = 1$ are complex probability amplitudes. Thus ququart represents a quantum polarization state of two qubits (photons), whose states can be either entangled or non-entangled. For complete characterization of polarization ququarts and their properties including methods for preparation, transformation and measurement we refer to our previous work [9].

It is worthy to note that the universally accepted method for describing the multi-mode quantum polarization states of photons is based on P-quasispin approach [13]. Application of P-quasispin concept to the polarization ququart has been done in [14]. In paper [15] it has been shown that polarization properties of two-mode biphoton field are completely defined by the coherency matrix. It is a matrix consisting of fourth-order moments in the electromagnetic field

$$K_4 = \begin{pmatrix} A & E & F & G \\ E^* & B & I & K \\ F^* & I^* & C & L \\ G^* & K^* & L^* & D \end{pmatrix}, \quad (2.4)$$

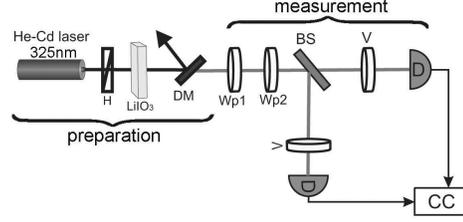


Figure 2.1: Setup for preparation and measurement of ququarts

$$\begin{aligned}
A &\equiv \langle a_s^\dagger a_i^\dagger a_s a_i \rangle = |c_1|^2, & B &\equiv \langle a_s^\dagger b_i^\dagger a_s b_i \rangle = |c_2|^2, \\
C &\equiv \langle b_s^\dagger a_i^\dagger b_s a_i \rangle = |c_3|^2, & D &\equiv \langle b_s^\dagger b_i^\dagger b_s b_i \rangle = |c_4|^2. \\
E &\equiv \langle a_s^\dagger a_i^\dagger a_s b_i \rangle = c_1^* c_2, & F &\equiv \langle a_s^\dagger a_i^\dagger b_s a_i \rangle = c_1^* c_3, \\
G &\equiv \langle a_s^\dagger a_i^\dagger b_s b_i \rangle = c_1^* c_4, & I &\equiv \langle a_s^\dagger b_i^\dagger b_s a_i \rangle = c_2^* c_3, \\
K &\equiv \langle a_s^\dagger b_i^\dagger b_s b_i \rangle = c_2^* c_4, & L &\equiv \langle b_s^\dagger a_i^\dagger b_s b_i \rangle = c_3^* c_4.
\end{aligned} \tag{2.5}$$

The averaging in (2.5) is taken over the state (2.3). The polarization density matrix of ququart state coincides with coherency matrix K_4 and completely determines an arbitrary ququart state. Thus in order to reconstruct the unknown ququart state all moments in (2.5) have to be measured.

As it was mentioned earlier, to measure an unknown state it is necessary to perform a set of projective measurements. At present, the only realistic way to register fourth-order moments is using the Hanbury Brown-Twiss scheme. In order to be able to measure polarization moments (2.5) we supplied this scheme with retardant plates and polarization prisms. Basically two protocols for quantum state reconstruction of ququarts can be applied [9]. In the first protocol the ququart is divided into two spacial/frequency modes by means of dichroic mirror and then each photon of the pair is subjected by polarization transformations separately. In the second protocol the ququart undergoes polarization (linear) transformations as a whole before the beamsplitter. Here we consider only the second protocol since it seems to be more practical. The idea of the protocol is straightforward. The ququart state is transformed by two retardant plates Wp1 and Wp2, putting in series and after that it is split by beamsplitter into two spatial modes ended with single-photon detectors D (Fig.2.1). Polarization prisms project the state onto vertical polarization. Pulses coming from detectors are coupled on the coincidence scheme, which selects only those of them coinciding in time with the accuracy of coincidence window (about 3 nsec in our case). So each pulse coming from the coincidence scheme associates with projection of the initial ququart subjected to given polarization transformation. Finally the output pulses of the coincidence scheme accumulated during fixed time interval (coincidence rate) serve as statistical data to be analyzed for the state reconstruction. The transformations performed by the plates Wp1 and Wp2 are expressed in the form

$$|\Psi^{out}\rangle_{kl} = \hat{G}(\delta_{1(s,i)}, \theta_k) \hat{G}(\delta_{2(s,i)}, \theta_l) |\Psi^{in}\rangle. \tag{2.6}$$

Matrix G is given by 4×4 matrix which is obtained by a direct product of two 2×2 matrices describing the $SU(2)$ transformation performed on each photon [15]:

$$\begin{aligned} \hat{G} &\equiv \begin{pmatrix} t_s t_i & t_s r_i & r_s t_i & r_s r_i \\ -t_s r_i^* & t_s t_i^* & -r_s r_i^* & r_s t_i^* \\ -r_s^* t_i & -r_s^* r_i & t_s^* t_i & t_s^* r_i \\ r_s^* r_i^* & -r_s^* t_i^* & -t_s^* r_i^* & t_s^* t_i^* \end{pmatrix} \\ &= \begin{pmatrix} t_s & r_s \\ -r_s^* & t_s^* \end{pmatrix} \otimes \begin{pmatrix} t_i & r_i \\ -r_i^* & t_i^* \end{pmatrix}, \end{aligned} \quad (2.7)$$

with complex coefficients of effective transmission

$$t_{1,2(s,i)} = \cos \delta_{1,2(s,i)} + i \sin \delta_{1,2(s,i)} \cos 2\theta_{k,l}, \quad (2.8)$$

and effective reflection

$$r_{1,2(s,i)} = i \sin \delta_{1,2(s,i)} \sin 2\theta_{k,l}. \quad (2.9)$$

Here $\theta_{k,l}$ are the orientation angles of the first or second retardant plates. The parameters of the plates, i.e. optical thicknesses for different wavelengths $\delta_{1,2(s,i)}$ and their orientations, are supposed to be known with high accuracy which relates to the final accuracy of the state reconstruction. Disregarding the normalization, the number of events detected in the experiment, i.e. coincidence rate R_{kl} is the projection of the transformed state $|\Psi_{out}\rangle$ onto the state $|V_s V_i\rangle$ determined by the orientation of the polarization prisms. This projection is given by the expression

$$R_{kl} \propto |\langle V_s V_i | \Psi_{out} \rangle_{kl}|^2. \quad (2.10)$$

Thus the joint action of two retardant plates and polarization prisms provides the basis for projective measurements.

The intensity of the event generation in each process can be expressed in terms of squared modulus of the amplitude of a quantum process [1–3]

$$R_\nu = M_\nu^* M_\nu, \quad (2.11)$$

Although the amplitudes of the processes cannot be measured directly, they are of the greatest interest as these quantities describe fundamental relationships in quantum physics. It follows from the superposition principle that the amplitudes are linearly related to the state-vector components [1–3]. So the main purpose of quantum tomography is the reproduction of the amplitudes and state vectors, which are hidden from direct observation. The linear transformation of the state vector $c = (c_1, c_2, c_3, c_4)$ into the amplitude of the process M is described by a certain matrix X

$$Xc = M. \quad (2.12)$$

The matrix X is a so called instrumental matrix for a set of mutually complementary measurements. Suppose that protocol contains m steps, which means that m consistent transformations should be done with the initial state. Consequently the matrix X contains m rows. So each row corresponds to various projection measurements under a quantum state. To each row X_j ($j = 1, 2, \dots, m$) of a length d we will construct a new row of a length d^2 which is determined as a direct product of row X_j and row X_j^* . Then let us compose another matrix B with rows introduced above. The size of the matrix B is $m \times d^2$ and we assume that $m \geq d^2$. The important property of the protocol of measurement is its completeness. The protocol is supposed to be complete if all d^2 singular eigenvalues of the matrix B are strictly positive. Such protocol provides with a reconstruction of an arbitrary quantum state at sufficiently large sample size. However if several (or even single) singular eigenvalues of the matrix are close to zero then the matrix becomes degenerate. Hence in order to reconstruct unknown ququart state with this quasi-degenerate matrix one needs to increase number of statistical data. We have found rank and singular eigenvalue of the matrix B and introduced parameter R which is defined as the ratio between the minimal nonzero singular eigenvalue and maximal one. The R value lies in a range from 0 to 1. It is important to notice that the ratio R depends on parameters of experimental set-up only (in our case these parameters are optical thicknesses of the quartz plates) and does not depend at all on the state to be measured. Therefore the ratio R can serve as a testing parameter of the protocol in the sense whether the protocol is optimal or not. Namely, the smaller R the worse the protocol and the quality of the reconstructed state is supposed to be lower. In the present paper we are not going to prove this statement mathematically. We just suggest the empirical parameter and test its validity with particular experiments. In nearest future we will develop this concept and give strict arguments related to the completeness of statistical state reconstruction protocol [8]. In our experiment each run is specified by the orientation angle of the Wp1 plate $\theta_1 = 0^\circ, 15^\circ, 30^\circ, 45^\circ$ for the complete rotation of the Wp2 plate by 360° with step 10° ; i.e., 144 measurements have totally been made. Consequently in our case matrix X consists of 144 rows (the total number of different orientations for both plates in experiment) and 4 columns (the dimension of Hilbert space for ququarts). Each row is formed in the following way. The initial state $|\Psi_{in}\rangle$ is transformed by the two quarts retardant plates Wp1 and Wp2 and projected onto the vertical state $|V_1 V_2\rangle$. Thus using the formulas (2.7-2.9) the four-element row can be re-written in the form

$$X_j = 1/2(\alpha_s \alpha_i \quad \alpha_s \beta_i \quad \beta_s \alpha_i \quad \beta_s \beta_i), \quad (2.13)$$

where the values of the complex parameters $a_{s,i}, b_{s,i}$ are different for particular rows of the measurement protocol:

$$\begin{aligned} \alpha_{s,i} &= -t_{1(s,i)}^*(\theta_k) r_{2(s,i)}(\theta_l) - r_{1(s,i)}(\theta_k) t_{2(s,i)}(\theta_l), \\ \beta_{s,i} &= -r_{1(s,i)}^*(\theta_k) r_{2(s,i)}(\theta_l) + t_{1(s,i)}(\theta_k) t_{2(s,i)}(\theta_l). \end{aligned} \quad (2.14)$$

Here indexes 1, 2 relate to the first(Wp1) or second(Wp2) retardant plates correspondingly. Within the bounds of the proposed method of measurement, thicknesses of plates do not seem to play a significant role and can be chosen arbitrary. Of course the experimentalist should know the exact optical thickness of each plate to be used in further calculations. However, below (see Section 3) we will show that only particular sets of plates with certain thicknesses provide the optimal reconstruction of an unknown state.

3 Quantum Tomography, Simulation and Experiment

Let us analyze how the choice of plates's thicknesses affect on the quality of state reconstruction. We have calculated the ratio R for set of thicknesses of the first and second quartz retardant plates. Specifically first plate varied within the limits from 0.8mm to 1mm with the step 0.002mm and second plate varied within the limits from 0.5mm to 1mm with the same step. Corresponding picture is shown on Fig.3.1. Thicknesses of the first and second quartz retardant plates lie along the axis x, y correspondingly.

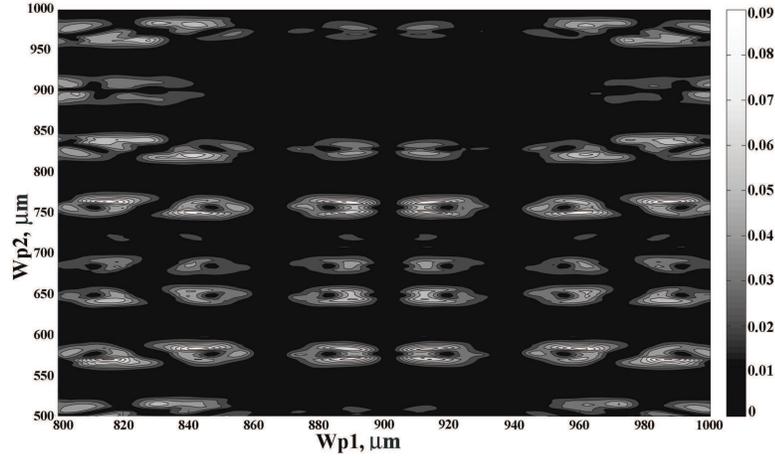


Figure 3.1: Dependence of ratio R on plates thicknesses

In Fig.3.1 light color indicates areas with high value of ratio R , and dark color indicates areas with low value. It is obvious, that the rash choice of plates thicknesses most likely will be wrong from the point of view of completeness of matrix B . For offered protocol the maximum achievable parameter R takes on a value 0.09 and as it will be shown below, it is quite enough for good quantum state reconstruction. For example, to receive such value one can choose plates with thicknesses $Wp1=0.988\text{mm}$ and $Wp2=0.570\text{mm}$. Particularly for our experiment we have chosen the following two sets of plates: a) $Wp1=0.988\text{mm}$ and $Wp2=0.836\text{mm}$ (optimal); b) $Wp1=0.836\text{mm}$ and $Wp2=0.536\text{mm}$ (non-optimal). For the first set of plates the ratio is $R = 0.0554$ and for the second set $R = 0.0013$. Unfortunately,

our choice has been limited by available plates, so we were not able to check their extremal values.

In the simplest case, for example for state $|V_s\rangle|V_i\rangle$ we can simulate the procedure of the basis state reconstruction. The difference between reconstructed state (which was numerically simulated) and theoretical one is caused by information loss due to the finite number of statistical data. As estimated parameters, we have considered the average information losses

$$L = \log_{10} \frac{1}{1 - F}, \quad (3.1)$$

where $F(\textit{fidelity})$ represents a correspondence between the theoretical and experimental (or numerically simulated) state vectors at different thicknesses of the first and second quartz retardant plates. Corresponding plot is shown on Fig.3.2. Here the thicknesses of the first and second quartz retardant plates lie along the axis x, y . Thicknesses of both plates were varied within the limits indicated on previous plot.

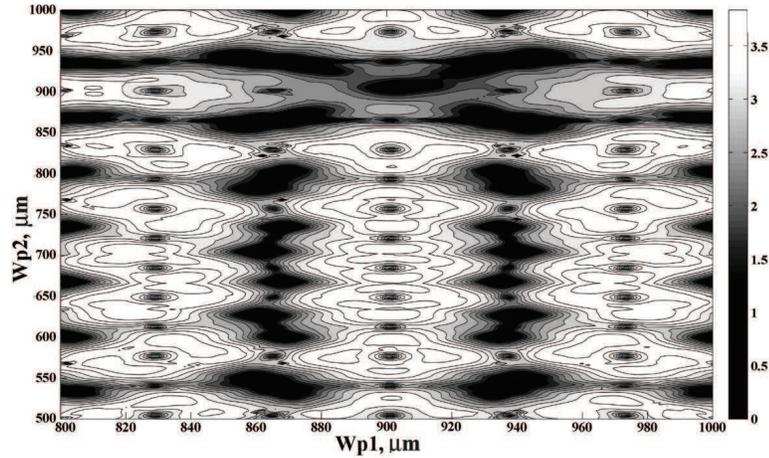


Figure 3.2: Dependence of average information losses L on plates thicknesses

In Fig.3.2 light color indicates areas with low value of information losses, and dark color indicates areas with high value of information losses. Numbers at the right bar show the accuracy level of the tomography protocol. For example, value 3 means that correspondence between the theoretical and experimental state vectors (*fidelity*) is about 0.999. Black “holes” in Fig.3.2 set areas, where matrix B becomes ill-conditioned. In other words the figure presented at the plot serves as some sort of a “navigation map” for measurement protocol. To achieve good quality of the measurement one should select the plates thicknesses in light areas and avoid dark areas. It is clearly seen that minima (dark areas) on both plots (Fig.3.1 and Fig.3.2) coincide. Moreover Fig.3.1 shows that choosing the plates one meets more limitations in terms of optimal protocol since the ratio R does not depend

on the state to be measured. At the same time the parameter “average information losses L ” does. This fact has to be taken into account by performing statistical state reconstruction. The dependence of density of distribution on the accuracy information losses $1 - F$ is shown in Fig.3.3. We see that accuracy information losses for the optimal tomography are characterized by density of distribution which is determined by the narrow high peak concentrated at low values of information losses (obviously less than 0.005). On the contrary, non-optimal tomography is characterized by the wide density of distribution stretched up to values 0.05. These differences disappear at sufficiently large sample size (which is related directly to the number of statistical data), but for equal quality of state reconstruction the right choice of plates allows one to make smaller set of statistical data, i.e. reduce time exposition.

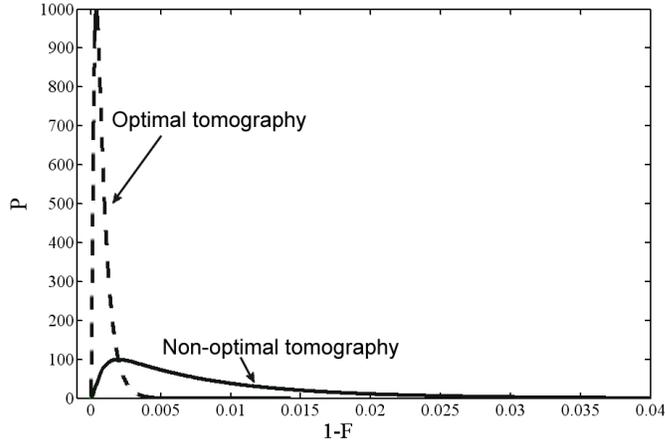


Figure 3.3: Dependence of density of distribution on the accuracy information losses $1 - F$. Dot line corresponds to optimal set of plates (optimal tomography), solid line corresponds to non-optimal set (non-optimal tomography).

We applied suggested protocol to measure some particular set of ququart states Ψ . For the generation of biphoton-based ququarts we used lithium-iodate 15 mm crystal (with type-I phase matching) pumped with 5 mW cw horizontally polarized helium-cadmium laser operating at 325 nm. The angle between the pumping wavevector and optical axis of the crystal is equal to 58° . Under these conditions, the product state $|V_{702nm}V_{605nm}\rangle$ is generated in the crystal. The initial state was subjected to transformations done by dichroic retardant plate in order to prepare some subset of ququart. This subset is known as a product of two polarization qubits [27]. That subset of ququart states was to be reconstructed. In particular, we used the 0.441mm length quartz plate whose orientation varied as $\alpha = 0^\circ, -60^\circ$. Since the thickness of the plate, quartz dispersion and orientation were supposed to be known, we were able to calculate the result of the state transformation with

high accuracy. In measurement part of setup we used two sets of retardant plates with different thicknesses, optimal and non-optimal, and made the reconstruction procedure of ququarts states at the fixed set of statistical data for each set. In both series of experiments we gathered identical statistic equal 30-35 thousands events. According to protocol, four sets of measurements were performed for each input state, so totally we performed 144 measurements of the coincidence rate as a function of orientation angles of Wp1 and Wp2. While accumulating experimental data, the accidental coincidence rate N_{acc} , which is expressed in terms of the rate of averaged single counts from each photodetector and the coincidence-window width (T) of the scheme as $N_{acc} = \langle N_1 \rangle \langle N_2 \rangle T$, is extracted from the total coincidence rate. For state reconstruction we used the maximum likelihood method, that was developed in [4] and was successively applied to reconstruct states of optical qutrits [5]. The result of ququarts reconstruction is given in Table 3.1. The value

Table 3.1: The result of ququarts reconstruction

	fidelity	
α (deg.)	optimal set	non-optimal set
0	0.999	0.974
-60	0.993	0.975

of parameter F is defined as $F = |\langle c_{theory} | c_{exp} \rangle|^2$. It is clearly seen that obtained fidelity values for optimal set of thicknesses is much higher than for non-optimal set. Fidelity for all states in the first case (optimal) was above 0.99, in the second case (non-optimal) fidelity was about 0.97.

4 Conclusion

In conclusion, we have suggested and tested an optimal protocol for polarization ququarts state tomography. The protocol allows one to achieve highest accuracy of the state reconstruction with available resources which experimentalist holds in his hands while doing particular measurements. Then we investigated theoretically and experimentally the accuracy of polarization ququart reconstruction depending on parameters available in experiment (quartz plates thicknesses) at fixed set of statistical data. The developed methodology can be extended easily to other quantum states reconstruction as well as used for optimization of various technological parameters of quantum tomography protocols.

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