Generalized (1,Θ)-Derivations of Noncommutative Prime Rings

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Abstract: In the present paper, we study when a noncommutative prime ring and semiprime ring R admitting a generalized (1,Θ)-derivation F satisfying any one of the properties: (i) $F(x)F(y) + xy \in Z(R)$ for all $x,y \in I$. (ii) $F(x)F(y) + yx \in Z(R)$ for all $x,y \in I$. (iii) $F(xy) - x\alpha y \in Z(R)$ for all $x,y \in I$.

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1 Introduction

Over last few decades, several authors have investigated the relationship between the commutativity of the ring $R$ and certain specific types of derivations of $R$. The first result in this direction is due to Posner [1] who proved that if $R$ is a prime ring and $D$ a nonzero derivation on $R$ such that $[D(x),x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. In [2] M.N Daif, proved that, let $R$ be a semiprime ring and $d$ a derivation of $R$ with $d^2 \neq 0$. If $[d(x),d(y)] = 0$ for all $x,y \in R$, then $R$ contains a non-zero central ideal. H.E.Bell and W.S.Martindal III [3] proved that the center of semiprime ring contains no non-zero nilpotent elements. M.N.Daif and H.E.Bell [4] proved that, let $R$ be a semiprime ring admitting a derivation $d$ for which either $xy + d(xy) = yx + d(yx)$ for all $x,y \in R$ or $xy - d(xy) = yx - d(yx)$ for all $x,y \in R$ then $R$ is commutative. V.DeFilippis [5] proved that, when $R$ be a prime ring and let $d$ a non-zero derivation of $R$, $U \neq 0$ a two-sided ideal of $R$, such that $d([x,y]) = [x,y]$ for all $x,y \in U$. Then $R$ is commutative. A.H.Majeed and Mehsin Jabel [6], then gave some results as, let $R$ be a 2-torsion free semiprime ring and $U$ a non-zero ideal of $R$, $R$ admitting a non-zero derivation $d$ satisfying $([d(x),d(y)]) = [x,y]$ for all $x,y \in U$. If $d$ acts as a homomorphism, then $R$ contains a non-zero central ideal. Mehsin Jabel [7] proved, let $R$ be a semiprime ring and $U$ be a non-zero ideal of $R$. If $R$ admits a generalized derivation $D$ associated with a nonzero derivation $d$ such that $D(xy) = xy \in Z(R)$ for all $x,y \in U$ then $R$ contains a non-zero central ideal.L.Oukhtite and S.Salhi [8] proved, let $R$ be a 2-torsion free $\sigma$-prime ring and let $d$ be a non-zero derivation. If $[d(x),x] \in Z(R)$ for all $x \in R$, then $R$ is commutative, where if a prime ring $R$ has an involution $\sigma$, then $R$ is said to be $\sigma$-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. Obviously, every prime ring equipped with an involution $\sigma$ is $\sigma$-prime, the converse need not be true in general. M.A. Chaudhry and Allah-Bakhsh Thaheem [9] proved, let $\alpha, \beta$ be epimorphisms of a semiprime ring $R$ such that $\beta$ is centralizing. If $d$ is a commuting $(\alpha, \beta)$-derivation of $R$, then $[x,y]d(u) = 0 = d(u)[x,y]$ for all $x,y,u \in R$. In particular, $d$ maps $R$ into its center, where let $\alpha, \beta$ be mappings of $R$ into itself. An additive mapping $d$ of $R$ into itself is called an $(\alpha, \beta)$-derivation if $d(xy) = \alpha(x)(y) + d(x)\beta(y)$ for all $x,y \in R$. A number of authors have studied the commutativity theorems in prime and semiprime rings admitting derivation and generalized derivation the notion of a generalized derivation of a ring was introduced by Breˇsar [10] and Hvala [11]. They have studied some properties of such derivations. An additive mapping $g$ of $R$ into itself is called a generalized derivation of $R$, with associated derivation $\delta$, if there is a derivation $\delta$ of $R$ such that $g(xy) = g(x)y + x\delta(y)$ for all $x,y \in R$. Chang [12] introduced the notion of a generalized $(\alpha, \beta)$-derivation of a ring $R$ and investigated some properties of such derivations. Let $\alpha, \beta$ be mappings of $R$ into itself. An additive mapping $g$ of $R$ into itself is called a generalized $(\alpha, \beta)$-derivation of $R$, with associated $(\alpha, \beta)$-derivation $\delta$, if there exists an $(\alpha, \beta)$-derivation $\delta$. 

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δ of R such that g(xy) = g(x)α(y) + β(x)δ(y) for all x,y ∈ R. Obviously this notion covers the notion of a generalized derivation (in case α = β = 1), notion of a derivation (in case g = δ, α = β = 1), notion of a left centralizer (in case δ = 0, α = 1), notion of (α,β)-derivation (in case g = δ) and the notion of left α-centralizer (in case δ = 0). Thus it is interesting to investigate properties of this general notion. Recently, Mehsin Jabel [13] proved some results concerning generalized derivations on prime and semiprime rings. In this paper we shall study and investigate some results concerning a generalized (1,Θ) -derivation on noncommutative prime ring R . where I is an identity automorphism of R , we give some results about that.

2 Preliminaries

Let R be an associative ring with identity and center Z(R). A ring R is said to be prime (resp.semiprime) if aRb = 0 implies that either a = 0 or b = 0 (resp.aRa = 0 implies that a = 0). A prime ring is semiprime but the converse is not true in general. For any x,y ∈ R we shall write [x,y] = xy − yx and xoy = xy + yx. An additive mapping d : R → R is called a derivation if d(xy) = d(x)y + xd(y) for all x,y ∈ R, and is said to be n-centralizing on U (resp.n-commuting on U), if [xn,d(x)] ∈ Z(R) holds for all x ∈ U (resp.[x,n,d(x)] = 0 holds for all x ∈ U, where n is a positive integer. Also, an additive mapping d : R → R is called a left (right) centralizer if d(xy) = d(x)y(xd(y)) for all x,y ∈ R. Let Φ, Θ be endomorphisms of R. An additive mapping d : R → R is called a (Φ,Θ)-derivation if d(xy) = d(x)Φ(y) + Θ(x)d(y) for all x,y ∈ R. An additive mapping F : R → R is called a generalized (Φ,Θ)-derivation on R if there exists a (Φ,Θ)-derivation d : R → R such that F(xy) = F(x)Φ(y) + Θ(x)d(y) for all x,y ∈ R. We shall call a generalized (Φ,1)-derivation a generalized Φ-derivation, where 1 is the identity automorphism of R. Similarly a generalized (1,Θ)-derivation will be called a generalized Θ-derivation. The following lemmas are necessary for this paper.

Lemma 2.1:[14:Lemma3.1] Let R be a semiprime ring and a R some fixed element. If a[x,y] = 0 for all x,y ∈ R, then there exists an ideal U of R such that a ∈ U ⊂ Z(R) holds.

Lemma 2.2:[15, Lemma3] If a prime ring R contains a nonzero commutative right ideal I, then R is commutative.

3 The main results

Theorem 3.1: Let R be a noncommutative prime ring and I a nonzero ideal of R. Suppose that Θ is an automorphism of R. If R admits a generalized Θ-derivation F with associated Θ-derivation d such that F(xy) − xy ∈ Z(R) for all x,y ∈ I, then F is commuting on I.

Proof: By assumption, we have

\[ F(xy) − xy ∈ Z(R) \]

for all x,y ∈ I. This can be written as F(xy) + Θ(x)d(y) − xy ∈ Z(R). Replacing y by yz, we obtain

\[ F(xy) + Θ(x)d(y)z + Θ(x)y(Θ(y)d(z)) = 0 \]

for all x,y,z ∈ I. Thus, in particular

\[ [(F(xy) + Θ(x)d(y)z + Θ(x)y(Θ(y)d(z)), z) = 0 \]

for all x,y,z ∈ I. Using (3.1) and (3.2), we get

\[ Θ(x)(Θ(y)d(z), z) = 0 \]

for all x,y,z ∈ I. Replacing x by rx in the above expression, we obtain

\[ Θ(x)(Θ(y)d(z), z) = 0 \]

for all x,y,z ∈ I. Then replace y by ty, to get

\[ Θ(x)(Θ(y)d(z), z) = 0 \]

for all x,y,z ∈ I. Since Θ is an automorphism of R, then

\[ Θ(x)(Θ(y)d(z), z) = 0 \]

for all x,y,z ∈ I. Thus, the primeness of R yields that for each z ∈ I, either Θ(x), Θ(y) = 0 or Θ(y)d(z) = 0. Let

\[ I_1 = \{ z ∈ I | Θ(x), Θ(y) = 0 \} \]

and

\[ I_2 = \{ z ∈ I | Θ(y)d(z) = 0 \} \]

for all x,y,z ∈ I. Then I_1 and I_2 are two additive subgroups of Θ whose union is I. Therefore either I_1 or I_2 is I. If I_2 = I then Θ(x) = 0 for all x,y,z ∈ I. Replace y by [y,q] to get Θ(x)[Θ(y), q]d(z) = 0 for all x,y,z ∈ I. Now replacing q by sq to get

\[ Θ(x)[Θ(y), q]d(z) = 0 \]

for all x,y,z ∈ I. Since Θ is an automorphism of R, then

\[ Θ(x)[Θ(y), q]d(z) = 0 \]

for all x,y,z ∈ I, s ∈ R. But according to our hypothesis R is noncommutative. So, in the our hand d(z) = 0 for all z ∈ I, this implies that d = 0 on R. Then from the main relation, we have F(x) − x ∈ Z(R), which leads to F(x), v = [x,v] = 0 for all x,y,z ∈ I. Replacing v and y by x, gives F(x,v) = 0 for all x,y ∈ I. Thus F is commuting on I, hence we get the required result. One can note that if R admits a generalized Θ-derivation F satisfying F(xy) + xy ∈ Z(R) for all x,y ∈ I, then the generalized Θ-derivation (−F) also satisfies (−F)(xy) = xy ∈ Z(R) for all x,y ∈ I. Hence in view of Theorem 3.1 we conclude the following.

Corollary 3.2: Let R be a noncommutative prime ring and I a nonzero ideal of R. Suppose Θ is an automorphism of R. If R admits a generalized Θ-derivation F with associated Θ-derivation d such that F(xy) + xy ∈ Z(R) for all x,y ∈ I, then F is commuting on I.

Theorem 3.3: Let R be a noncommutative prime ring and I a nonzero ideal of R. Suppose Θ is an automorphism of R. If R admits a generalized Θ-derivation F with associated Θ-derivation d such that F(xy) + xy ∈ Z(R) for all x,y ∈ I, then F is commuting on I.
R. If $F$ is a generalized $\Theta$-derivation with associated $\Theta$-derivation $d$ such that $F(xy) - yx \in Z(R)$ for all $x, y \in I$, then

$$ (i) d(R) = 0, $$

(1)

(ii) $F$ is left centralizer of $I$.

**Proof:** (i) For any $x, y \in I$, we have $F(xy) - yx \in Z(R)$. This can be written as $F(x)y + \Theta(x)d(y) - yx \in Z(R)$ for all $x, y \in I$. Substituting $xy$ for $x$, we obtain

$$ F(x)yy + \Theta(x)d(y)y + \Theta(x)\Theta(y)d(y) - yxy \in Z(R), $$

(3.4)

for all $x, y \in I$. In particular

$$ [(F(x)y + \Theta(x)d(y))\Theta(y)y + \Theta(x)\Theta(y)d(y), y] = 0, $$

(3.5)

for all $x, y \in I$. An application of (3.4) and (3.5) gives

$$ \Theta(x)\Theta(y)d(y), y = 0 $$

(2)

for all $x, y \in I$, i.e.

$$ \Theta(x)\Theta(y)[d(y), y] + \Theta(x)[\Theta(y), y]d(y) + [\Theta(x), y]\Theta(y)d(y) = 0 $$

(3)

for all $x, y \in I$. Replacing $x$ by $z$ in (3.6) and using (3.6), we find that

$$ [\Theta(z), y]\Theta(x)\Theta(y)d(y) = 0, $$

(3.7)

for all $x, y, z \in I$. Replacing $x$ by $x$ in (3.7), we get

$$ [\Theta(z), y]\Theta(x)\Theta(r)\Theta(y)d(y) = 0 $$

for all $x, y, z \in I$, i.e. $\Theta(z), y]\Theta(x)\Theta(r)\Theta(y)d(y) = 0$ for all $x, y, z \in I$. Thus the primeness of $R$ gives that for each $r \in I$, either

$$ [\Theta(z), y]\Theta(x) = 0 $$

for all $y \in I$. The sets $y \in I$ for which these two properties hold, are additive subgroups of $I$ whose union is $I$. Then either

$$ [\Theta(z), y]\Theta(x) = 0 $$

or $\Theta(y)d(y) = 0$, for all $x, y, z \in I$. If $\Theta(y)d(y) = 0$, for all $y \in I$, then linearity gives

$$ \Theta(x)d(y) + \Theta(y)d(x) = 0 $$

(3.8)

for all $x, y \in I$. Replace $y$ by $xy$ to get

$$ \Theta(x)d(z) + \Theta(x)\Theta(z)d(y) + \Theta(z)\Theta(y)d(x) = 0 $$

(3.9)

for all $x, y \in I$. Comparing (3.8) and (3.9), we get

$$ \Theta(x)d(z) + \Theta(x)\Theta(z)d(y) - \Theta(z)\Theta(x)d(y) = 0 $$

for all $x, y \in I$. That is,

$$ \Theta(x)d(z)yr + [\Theta(x), \Theta(z)]d(y)r + [\Theta(x), \Theta(z)]\Theta(y)d(r) = 0 $$

(3.10)

for all $x, y, z \in I$, $r \in R$. An application of (3.9) in (3.10) yields that

$$ \Theta(x), \Theta(z)[\Theta(y)d(r) = 0 $$

(3.11)

for all $x, y, z \in I$, $r \in R$.

Now replace $y$ by $y$ to get

$$ [\Theta(x), \Theta(z)]\Theta(y)sd(r) = 0 $$

for all $x, y, z \in I$, $r, s \in R$, i.e. $[\Theta(x), \Theta(z)]RIRd(r) = 0$ for all $x, z \in I$, $r \in R$. Thus the primeness of $R$ with noncommutative, lead to $RIRd(r) = 0$ for all $r \in R$. By using the primeness of $R$ and $I$ is nonzero ideal, we obtain $d(r) = 0$ for all $r \in R$. Thus, we have $d(R) = 0$. (ii) Since $F$ is a generalized $\Theta$-derivation with associated $\Theta$-derivation $d$, then by using the fact $\Theta$ is an automorphism of $R$ and result in (i), we get the required result.

Arguing as above we can prove the following.

**Theorem 3.4:** Let $R$ be a noncommutative prime ring and $I$ a nonzero ideal of $R$. Suppose $\Theta$ is an automorphism of $R$. If $F$ is a generalized $\Theta$-derivation with associated $\Theta$-derivation $d$ such that $F(xy) + yx \in Z(R)$ for all $x, y \in I$, then

$$ (ii) d(R) = 0 $$

(ii) $F$ is left centralizer of $I$.

**Theorem 3.5:**

Let $R$ be a noncommutative prime ring and $I$ a nonzero ideal of $R$. Suppose $\Theta$ is an automorphism of $R$. If $R$ admits a generalized $\Theta$-derivation $F$ with associated nonzero $\Theta$-derivation $d$ such that $F(xy) - yx \in Z(R)$ for all $x, y \in I$, then either $d(R) = 0$ or $F$ is commuting of $I$.

**Proof:** By assumption we have $F(xy) - yx \in Z(R)$ for all $x, y \in I$. Replacing $y$ by $yr$, we find that $F(xy) - yx + F(xy)d(r) \in Z(R)$ for all $x, y \in I$, and then either $d(R) = 0$ or $F$ is commuting of $I$.

Thus the primeness of $R$ with noncommutative, lead to $RIRd(r) = 0$ for all $r \in R$. By using the primeness of $R$ and $I$ is nonzero ideal, we obtain $d(r) = 0$ for all $r \in R$. Thus, we have $d(R) = 0$. (ii) Since $F$ is a generalized $\Theta$-derivation with associated $\Theta$-derivation $d$, then by using the fact $\Theta$ is an automorphism of $R$ and result in (i), we get the required result.

Arguing as above we can prove the following.
Theorem 3.6: Let $R$ be a noncommutative prime ring and $I$ a nonzero ideal of $R$. Suppose $\Theta$ is an automorphism of $R$. If $R$ admits a generalized $\Theta$-derivation $F$ with associated $\Theta$-derivation $d$ such that $F(xy) + \Theta(xy)d(y) - \Theta(xy)y\Theta(x)d(z) - xy\Theta(xy)z \in Z(R)$ for all $x, y \in I$, then $R$ contains nonzero central ideal.

Proof: By assumption, we have $F(xy) - xoy \in Z(R)$ for all $x, y \in I$. This can be written as $F(xy) + \Theta(xy)d(y) - \Theta(xy)y\Theta(x)d(z) - xy\Theta(xy)z \in Z(R)$. Replacing $y$ by $y z$, we obtain

$$F(xy)yz + \Theta(xy)d(y)z + \Theta(xy)\Theta(y)d(z) - xoyz \in Z(R),$$

for all $x, y, z \in I$. Replacing $x$ by $xy$ in (3.16), we obtain

$$F(xy)yz + \Theta(xy)d(y)z + \Theta(xy)\Theta(y)d(z) - xoyz \in Z(R),$$

for all $x, y, z \in I$. Then from (3.17), we get

$$(F(xy) - xoy)yz + \Theta(xy)d(y)z + \Theta(xy)\Theta(y)d(z) - xy\Theta(xy)z \in Z(R),$$

for all $x, y, z \in I$. Since $F(xy) - xoy \in Z(R)$ for all $x, y \in I$, then from above equation (3.18), we arrived at

$$yz(F(xy) - xoy) + xz + \Theta(xy)d(y)z + \Theta(xy)\Theta(y)d(z) - xy\Theta(xy)z \in Z(R),$$

for all $x, y, z \in I$. Subtracting (3.19) and (3.18), we obtain

$$|F(xy) - xoy|, |x|z = 0 \text{ for all } x, y, z \in I.$$ Then

$$F(xy) = xoy \text{ for all } x, y, z \in I.$$ If $xy + \Theta(xy)d(y)z + \Theta(xy)\Theta(y)d(z) - xy\Theta(xy)z \in Z(R),$$

for all $x, y, z \in I$. The relation $F(xy) - xoy \in Z(R)$ for all $x, y \in I$, after replacing $r$ and $y$ by $x$, we get, $F(x^2), x^2 = 0$ for all $x \in I$. Replacing $x$ by $x^2$ in above equation (3.20), with using the relation $F(x^2), x^2 = 0$ for all $x \in I$, we obtain

$$x^2[d(x^2), x^2] = 0$$

for all $x \in I$.

Right-multiplying (3.21) by $[s, t]$, we arrived to

$$x^2[d(x^2), x^4][s, t] = 0 \text{ for all } x \in I, s, t \in R.$$ We set $a = x^2[d(x^2), x^4], a \in R$; then $a[s, t] = 0$ for all $y, z \in R$. Apply Lemma 2.1, we get $R$ contains nonzero central ideal. This meaning, we get the required result. By using the result in Theorem (3.7) with apply Lemma (2.2), we can prove the following theorem.

Theorem 3.8: Let $R$ be a noncommutative prime ring and $I$ a nonzero ideal of $R$. Suppose that $\Theta$ is an automorphism of $R$. If $R$ admits a generalized $\Theta$-derivation $F$ with associated $\Theta$-derivation $d$ such that $F(xy) - xoy \in Z(R)$ for all $x, y \in I$, then $R$ is commutative.

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