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Concepts of Point estimators for the Stress-Strength Reliability R=P(X>Y) using the Generalized Length Biased Maxwell Distribution and Data obtained through Ranked Set Sampling

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Abstract: In this article, point and interval estimators are derived for the reliability model when both strength and stress variables are independent and follow generalized length-biased Maxwell distributions, utilizing ranked set sampling. Further, Monte Carlo simulations are performed to evaluate and compare the performance of the maximum likelihood estimator (MLE) based on RSS with that based on SRS. Finally, the proposed methodology is applied to real data to demonstrate its effectiveness. AIC and BIC values are used to determine the Goodness of fit of the fitted probability distributions to the real data sets. We concluded that maximum likelihood estimator of R based on RSS outperforms its SRS counterpart.

Keywords: Monte Carlo simulation, maximum likelihood estimator, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values.

1 Introduction

The stress-strength reliability model, represented as $R = P(X > Y)$, is commonly used to describe the life of a component that is subjected to a random stress variable Y and has a random strength variable X. The component fails when Y exceeds X. Inference for the stress-strength model is a widely studied problem in various fields such as medicine and quality control, with extensive bibliography available on the topic. Ranked set sampling (RSS) has gained popularity as a method for estimating R. RSS provides a more informative sample compared to simple random sampling (SRS) of the same size, as it incorporates information from the ranking process in addition to the quantified observations. This paper focuses on estimating R when the stress and strength variables follow two independent Generalized Length biased Maxwell distributions, with a common scale parameter and different shape parameters, using RSS. This paper focuses on the stress-strength model and its application in estimating the reliability parameter R. The stress-strength model has been extensively studied in the literature, with notable works by Kotz et al.(2003), Chaturvedi et al.(2016), and Hassan (2017). The concept of stress-strength modeling in a multi-component system was introduced by Bhattacharyya and Johnson in 1974, and more recent investigations have been carried out by Rao and Kantam (2010), Rao et al. (2017), Hassan (2017), and Hassan and Alohali (2018). The ranked set sampling (RSS) method is employed to estimate R in this study, as it has gained popularity in statistical inference due to its ability to yield a more informative sample of the underlying population compared to simple random sampling (SRS). The method leverages not only the quantified observations but also the ranking process, making it suitable for the stress-strength modeling scenario. We derive the point estimator of R based on the Generalized length biased Maxwell distribution and RSS using the maximum likelihood (ML) method.
The paper is structured as follows: In Section 2, advantages and benefits of Ranked Set Sampling have been discussed.

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In Section 3, an overview of the Generalized Length biased Maxwell distribution and the point estimator of R using the maximum likelihood method based on SRS is presented. Section 3 provides a brief description of RSS and the derivation of the point estimator for R using maximum likelihood based on RSS. In Section 4, we presented some numerical findings based on simulated as well as real data sets. Finally, in Section 5 We presented few remarks and conclusions.

2 Advantages and Benefits of Ranked Set Sampling

Ranked Set Sampling (RSS) is a statistical sampling technique that offers several advantages, particularly in reliability and life testing. Here are some key benefits:

2.1 Improved Efficiency

More accurate estimates: RSS can provide more precise estimates of population parameters (such as mean and variance) compared to simple random sampling (SRS), especially when the ranking process is effective.
Efficiency in small sample sizes: RSS is more efficient in estimating parameters even with smaller sample sizes, making it suitable for life testing where only a limited number of units are available.

2.2 Cost-effectiveness

Reduction in measurement cost: In some cases, the measurement of lifetimes or failure times can be expensive or destructive. With RSS, not all units need to be fully measured, reducing overall costs while still obtaining reliable results.
Partial rankings reduce effort: Since only the ranking of units is required in the preliminary stage, this reduces the need for full measurements on all sampled items.

2.3 Enhanced estimation in life testing

Better parameter estimation: In life testing and reliability analysis, parameters such as failure rates, hazard functions, and reliability functions can be estimated more accurately using RSS, leading to more reliable predictions of product lifetimes or failure times.
Handling censored data: In life testing, failure times may be censored (due to time constraints or non-failures within the test period). RSS can handle such censored data better, improving the reliability of the estimates.

2.4 Flexibility in sampling design

Adaptability to various distributions: RSS is not restricted to a particular distribution and can be applied to different types of populations, including those with skewed or heavy-tailed distributions, which are common in reliability and life testing data.
Use with different ranking methods: Expert judgment or auxiliary variables can be used to rank units, making RSS versatile in its application to various reliability scenarios. **e) Reduction in bias**
More representative sample: Since ranking helps select more representative units from the population, RSS reduces the bias that may occur in traditional random sampling techniques, providing better inference about the population.
Balanced data for extreme values: In life testing, extreme values (such as long survival times or early failures) are of significant interest. RSS helps ensure that such values are well-represented, leading to more balanced and insightful results.

2.5 Improved Inferences

Enhanced hypothesis testing: RSS improves the power of hypothesis tests in reliability studies. This means that tests for differences in failure rates, lifetimes, or reliability between groups are more sensitive, leading to more robust conclusions.
Better confidence intervals: The confidence intervals for life testing parameters (e.g., mean time to failure) are narrower and more precise using RSS compared to traditional sampling methods.

2.6 *Applications in Reliability Studies*

Failure-time analysis: In reliability studies, ranked set sampling allows for more efficient estimation of failure-time distributions, such as the exponential, Weibull, or lognormal distributions, which are often used to model lifetimes.
Estimation of reliability functions: RSS can be used to estimate the reliability function more effectively, particularly when the ranking process is informative and consistent with the underlying failure process.

3 Properties and Characteristics

The Generalized Length Biased Maxwell distribution, a continuous probability distribution, finds extensive utility across various fields such as demography, actuarial science, biology, survival analysis, and computer science. Its prevalence stems from its characteristic of possessing a rising hazard rate concerning the lifespan of systems.
The probability density function (PDF) and cumulative distribution function (CDF) of Generalized Length Biased Maxwell Distribution are given respectively as,

$$f(x;\theta,k)=\frac{2\theta^{(k+\frac{1}{2})}}{\Gamma(k+\frac{1}{2})}x^{2k}e^{-\theta x^2} \tag{1}$$

and

$$F(x;\theta,k)=\frac{\gamma(k+\frac{1}{2},\theta x^2)}{\Gamma(k+\frac{1}{2})} \tag{2}$$

respectively where $x>0,\theta>0,k>0$. $\Gamma(k+\frac{1}{2})$ is the gamma function and $\gamma(\cdot)$ is the incomplete gamma function.
The Hazard Rate of GLBM distribution is given by,

$$h(t)=\frac{f(t)}{R(t)}=\frac{2\theta^{(k+\frac{1}{2})}t^{2k}e^{-\theta t^2}}{\Gamma(k+\frac{1}{2})-\gamma(k+\frac{1}{2},\theta t^2)} \tag{3}$$

Whereas the reverse hazard rate is given by,

$$L(t)=\frac{f(t)}{F(t)}=\frac{2\theta^{(k+\frac{1}{2})}t^{2k}e^{-\theta t^2}}{\gamma(k+\frac{1}{2},\theta t^2)} \tag{4}$$

In Figure 1, we plot pdf of GLBM distribution for different scale parameter values. In figure 2 and 3, we plot Hazard rate and Reversed hazard rate of the distribution.

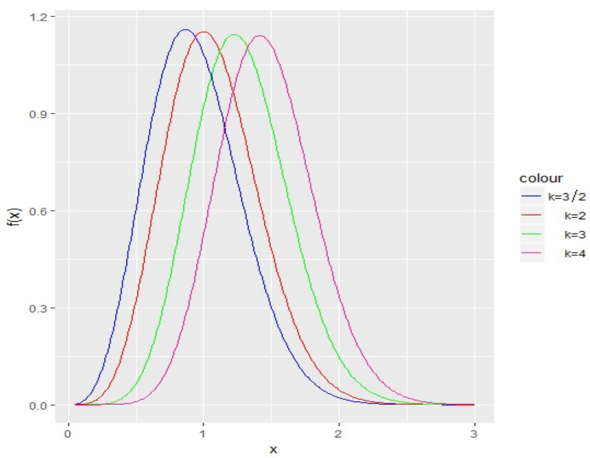

Fig. 1: PDF plot for $\theta=2$

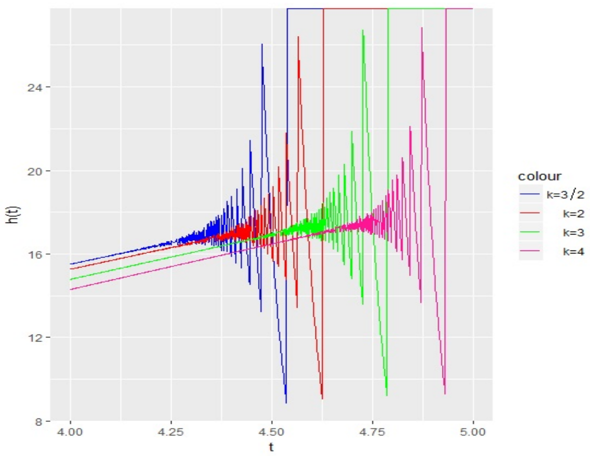

Fig. 2: Hazard plot for $\theta = 2$

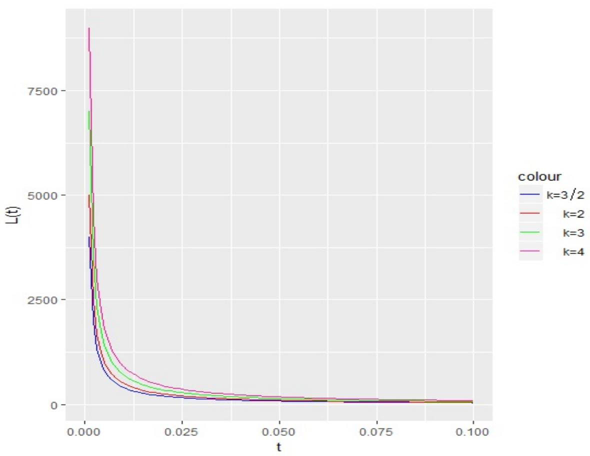

Fig. 3: Reverse hazard plot for $\theta = 2$

3.1 Point Estimator for $R = P[X > Y]$ based on SRS

In this section, we first derive expression of stress strength reliability function $\boldsymbol{R} = \boldsymbol{P}[\boldsymbol{Y} < \boldsymbol{X}]$. Let us consider X as strength variable and Y as stress variable both follow generalized length biased Maxwell distribution with known scale parameter k and different shape parameter θ_1 and θ_2 i.e.

$$X \sim GLBM(\theta_1, k) = \mathrm{f}\ (\theta_1, k)$$

and

$$Y \sim GLBM(\theta_2, k) = \mathrm{g}\ (\theta_2, k)$$

then

$$R = P(X > Y) = \int_{y=0}^{\infty}\int_{x=y}^{\infty} f(\theta_1, k)\, g\,(\theta_2, k)dx\, dy$$

$$= \frac{2\theta_2^{\,k+1/2}}{\Gamma\,(k+1/2)}\int_{y=0}^{\infty} y^{2k} e^{-\theta_2 y^2}\{1 - \frac{\Upsilon(k+1/2, \theta_1 y^2)}{\Gamma\,(k+1/2)}\}dy$$

This integral does not have closed form so it would be evaluated using numerical approximation methods.
Next we derive the ML estimators of θ_1 and θ_2. Let $x_1, x_2, ..., x_n$, be a random sample from GLBM$(theta_1, k)$ and $y_1, y_2, ..., y_n$, be a random sample from GLBM$(theta_2, k)$ respectively.
The joint pdf of $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ represented by $f_1(\cdot)$ and $g_1(\cdot)$ are given by,

$$f_1(x_1, x_2, \dots\dots\dots x_n, \theta_1, k) = \frac{2^n \theta_1^{\,n(k+1/2)}}{\Gamma\,(k+1/2)^n}.\exp\{-\theta_1 \sum_{i=1}^{n} x_i^{\,2} \prod_{i=}^{2n} x_i^{\,2k}\}$$

and

$$g_1(y_1, y_2, \dots\dots\dots y_m, \theta_2, k) = \frac{2^m \theta_2^{\,m(k+1/2)}}{\Gamma\,(k+1/2)^m}.\exp\{-\theta_2 \sum_{i=1}^{m} y_i^{\,2} \prod_{i=}^{2m} y_i^{\,2k}\}$$

The log likelihood functions are given by,

$$L_1\left(\frac{\theta_1}{k}, x\right) = nlog2 + n\,\left(k+1/2\right)log\theta_1 - nlog\Gamma\,(k+1/2) - \theta_1 \sum_{i=1}^{n} x_i^{\,2} + 2k\sum_{i=1}^{n} log\, x_i$$

and

$$L_2\left(\frac{\theta_2}{k}, y\right) = mlog2 + m\,\left(k+1/2\right)log\theta_2 - mlog\Gamma\,(k+1/2) - \theta_2 \sum_{i=1}^{m} y_i^{\,2} + 2k\sum_{i=1}^{m} log\, y_i$$

The ML estimators of θ_1 and θ_2 are obtained by solving the following questions.

$$\frac{n\,(k+1/2)}{\theta_1} - \sum_{i=1}^{n} x_i^{\,2} = 0$$

and

$$\frac{m\,(k+1/2)}{\theta_1} - \sum_{i=1}^{n} y_i^{\,2} = 0$$

Which gives the ML estimators of θ_1 and θ_2 respectively.

$$\widetilde{\theta_1} = \frac{n\left(k + {}^1/_2\right)}{\sum_{i=1}^{n} x_i^{\,2}}$$

$$\widetilde{\theta_2} = \frac{m\left(k + {}^1/_2\right)}{\sum_{i=1}^{m} y_i^{\,2}}$$

Using the invariance property of ML estimators, the ML estimators of R based on SRS is given as,

$$\hat{R} = \frac{2\theta_2^{\,(k+{}^1/_2)}}{\Gamma\left(k + {}^1/_2\right)} \int_{y=0}^{\infty} y^{2k} e^{-\theta_2 y^2} \{1 - \frac{\Upsilon(k + {}^1/_2\ ,\ \theta_1 y^2)}{\Gamma\left(k + {}^1/_2\right)}\} dy$$

3.2 Point estimator for $R = P(X > Y)$ based on RSS

We first generate ranked set sample of size $n_1 = r_1 m_{(1)}$ and $n_2 = r_2 m_{(2)}$ from GLBM distribution with parameters (θ_1, k) and (θ_2, k) respectively using the algorithm used to generate RSS.
Let $x_{ij}(i = 1, \ldots\ldots m_1, j = 1, \ldots\ldots r_1)$ denote the generated RSS of size $n_1 = r_1 m_{(1)}$ from GLBM(θ_1, k)
and
$y_{kl}(k = 1, \ldots\ldots m_2, l = 1, \ldots\ldots r_2)$ denote the generated RSS of size $n_2 = r_2 m_{(2)}$ from GLBM(θ_2, k)
The PDF of X_{ij} and Y_{kl} are given by,

$$f_i(x_{ij}) = \frac{m_1!}{(i-1)!(m_1-i)!} [F(x_{ij})]^{i-1}.\ \ [1 - F(x_{ij})^{m_1 - i}.f(x_{ij})]$$

and

$$g_k(\,y_{kl}) = \frac{m_2!}{(k-1)!(m_2-k)!} [\{F(y_{kl})^{k-1}\}.\ \{1 - F(y_{kl})\}^{m_2-k}.f(y_{kl})]$$

Likelihood function is given by,

$$L=\prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i(x_{ij}).\ \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k(y_{kl})$$

$$=\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\frac{m_1!}{(i-1)!(m_1-i)!}.\ \{(\frac{\Upsilon(k+1/2),\theta_1 x_{ij}}{\Gamma(k+1/2)})^{i-1}\}.$$

$$\{\left(1-\frac{\Upsilon(k+1/2),\ \theta_1 x_{ij}}{\Gamma(k+1/2)}\right)^{m_1-i}\}.\frac{2\theta_1^{(k+1/2)}}{\Gamma(k+1/2)}.\ x_{ij}^{2k}.e^{-\theta_1 x_{ij}^2}.$$

$$.\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\frac{m_2!}{(k-1)!(m_2-k)!}.\{(\frac{\Upsilon(k+1/2),\theta_2 y_{kl}}{\Gamma(k+1/2)})^{k-1}\}.\{\left(1-\frac{\Upsilon(k+1/2),\ \theta_2 y_{kl}}{\Gamma(k+1/2)}\right)^{m_2-k}\}.$$

$$(\frac{2\theta_2^{(k+1/2)}}{\Gamma(k+1/2)}.\ y_{kl}^{2k}.e^{-\theta_2 y_{kl}^2})$$

$$=W\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\{(\frac{\Upsilon(k+1/2),\theta_1 x_{ij}}{\Gamma(k+1/2)})^{i-1}.\frac{2\theta_1^{(k+1/2)}}{\Gamma(k+1/2)}.\ x_{ij}^{2k}.e^{-\theta_1 x_{ij}^2}.\ \{\left(1-\frac{\Upsilon(k+1/2),\ \theta_1 x_{ij}}{\Gamma(k+1/2)}\right)^{m_1-i}\}.$$

$$.\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\frac{\Upsilon(k+1/2),\theta_2 y_{kl}}{\Gamma(k+1/2)})^{k-1}\}.(\frac{2\theta_2^{(k+1/2)}}{\Gamma(k+1/2)}.\ y_{kl}^{2k}.e^{-\theta_2 y_{kl}^2}).\ \{\left(1-\frac{\Upsilon(k+1/2),\ \theta_2 y_{kl}}{\Gamma(k+1/2)}\right)^{m_2-k}\}$$

$$=\frac{2^{n_1+n_2}.W\theta_1^{n_1(k+1/2)}.\theta_2^{n_2(k+1/2)}}{\Gamma(k+1/2)^{n_1+n_2}}.\prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\{(\frac{\Upsilon(k+1/2),\theta_1 x_{ij}}{\Gamma(k+1/2)})^{i-1}.$$

$$\{\left(1-\frac{\Upsilon(k+1/2),\ \theta_1 x_{ij}}{\Gamma(k+1/2)}\right)^{m_1-i}\}.\ x_{ij}^{2k}.e^{-\theta_1 x_{ij}^2}.\prod_{k=1}^{r_2}\prod_{l=1}^{m_2}\{(\frac{\Upsilon(k+1/2),\theta_2 y_{kl}}{\Gamma(k+1/2)})^{k-1}\}.$$

$$\{\left(1-\frac{\Upsilon(k+1/2),\ \theta_2 y_{kl}}{\Gamma(k+1/2)}\right)^{m_2-k}\}..\ y_{kl}^{2k}.e^{-\theta_2 y_{kl}^2}$$

$$\Delta=\frac{2^{n_1+n_2}.W}{\Gamma(k+1/2)^{n_1+n_2}}$$

Log likelihood is given by,

$$logL=$$
$$log\Delta+n_1(k+1/2)log\theta_1++n_2(k+1/2)log\theta_2+2k\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}-\theta_1\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}^2+$$
$$2k\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}y_{lk}-\theta_2\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}y_{lk}^2+\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\{(i-1)logA+(m_1-i)\log(1-A)\}+$$
$$\sum_{k=1}^{r_2}\sum_{l=1}^{m_2}\{(k-1)logB+(m_2-k)+\log(1-B)\}$$

Differentiating the above equation (10) with respect to θ_1 and θ_2 and equating to 0, we have,

$$\frac{\partial logL}{\partial \theta_1} = 0$$

$$\frac{n_1(k+1/2)}{\theta_1} - \sum_{i=1}^{r_1}\sum_{j=1}^{m_1} x_{ij}{}^2 + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1} \frac{\{(i-1)\partial A}{A\partial \theta_1} - \frac{(m_1-i)}{(1-A)}\frac{\partial A}{\partial \theta_1}\} = 0$$

$$\frac{\partial logL}{\partial \theta_2} = 0$$

$$\frac{n_2(k+1/2)}{\theta_2} - \sum_{k=1}^{r_2}\sum_{l=1}^{m_2} y_{kl}{}^2 + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2} \frac{\{(k-1)\partial B}{B\partial \theta_2} - \frac{(m_2-k)}{(1-B)}\frac{\partial B}{\partial \theta_2}\} = 0$$

By solving equations 11 and 12 using numerical methods, we get the ML estimators of θ_1 and θ_2.
Using the invariance property of maximum likelihood estimators, we get the maximum likelihood estimator of $R = P(X > Y)$, under ranked set sampling as

$$R_{RSS}^{ML} = \frac{2\theta_2{}^{(k+1/2)}}{\Gamma\left(k+1/2\right)} \int_{y=0}^{\infty} y^{2k} e^{-\theta_2 y^2} \{1 - \frac{\Upsilon(k+1/2\,,\,\theta_1 y^2)}{\Gamma\left(k+1/2\right)}\} dy$$

4 Numerical Findings

4.1 Simulation Study

In this part, we first generated data from Generalized Length Biased Maxwell distribution. Two Types of Sampling Schemes namely Simple Random Sampling (SRS) and Ranked Set Sampling (RSS) are used to generate samples. To compare the estimators of Stress-Strength Reliability $R = P(X > Y)$ derived using the two sampling schemes, we calculated the Mean Square Error (MSE) and Bias using which we finally calculated Relative efficiency of the proposed estimators. Following are the steps of simulation study:

1. We first generate 1500 Simple Random Samples say $x_1, x_2, ..., x_{n_1}$ and $y_1, y_2, ..., y_{n_2}$ of sizes (n_1, n_2) = (10,10), (10,15), (15,15), (15,20) (20,20) (20,25) (25,25) (25,30), (30,30), (30,35), (35,40), (40,40) from Generalized Length Biased Maxwell distribution.
2. Next we generate 1500 Ranked set Samples from Generalized Length Biased Maxwell distribution say $x_{11}, \ldots\ldots, x_{m_1 r_1}$ and $y_{11}, \ldots\ldots, y_{m_2 r_2}$ with set sizes m_1=m_2= 2, 4, 6 and number of cycles r_1=r_2= 10, 15.
3. The value of scale parameter k is taken to be 1 and value of θ_1=2,θ_2=3,4,5.
4. Bias, MSE and Relative efficiencies are calculated. All the computations were performed in R. The results are summarized in Table 1.

4.2 Real Data Analysis

In this segment, we delve into an analysis of a dataset previously documented by Bader and Priest (1982), which encapsulates the tensile strength of individual carbon fibers, quantified in GPA, alongside impregnated 1000-carbon fiber tows. The individual fibers underwent testing under tension, utilizing gauge lengths of 20 mm and 10 mm. In their 2017 study, Chaudhary et al. fitted a Maxwell distribution to these datasets, extracting estimations for stress-strength reliability, denoted as P. Our analysis demonstrates that the Generalized Length Biased Maxwell distribution yields a superior fit to these datasets, evidenced by lower Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values. Table 2 showcases the fitted models, parameter estimates, and their corresponding AIC and BIC values.

Table 2: Fitted models to carbon fibers data, parameter estimates, AIC and BIC values

Data set	Model	Parameter estimates	AIC	BIC
Single fibers of 20mm	Maxwell	$\hat{\theta} = 0.2354(0.0223)$	162.2642	164.5683
Single fibers of 10mm	Maxwell	$\hat{\theta} = 0.1540(0.0158)$	165.1698	167.3129
Single fibers of 20mm	GLB Maxwell	$\hat{\alpha} = 0.2239(NaN)$	108.3512	115.2634
		$\hat{\theta} = 2.1828(NaN)$		
		$\hat{k} = 21.2904(NaN)$		
Single fibers of 10mm	GLB Maxwell	$\hat{\alpha} = 2.4739(3.0547)$	120.6971	127.1265
		$\hat{\theta} = 0.5999(0.0994)$		
		$\hat{k} = 3.315(3.5149)$		

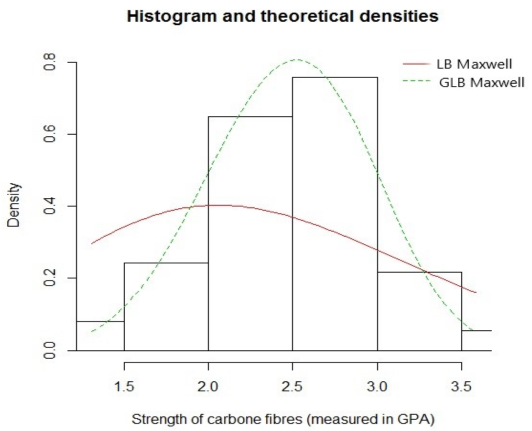

Fig. 4: Empirical and theoretical Densities for dataset 1

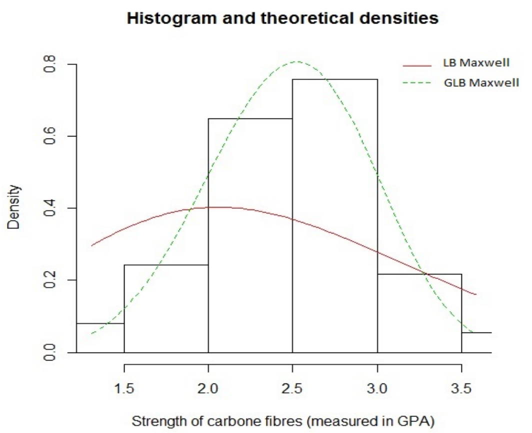

Fig. 5: Empirical and theoretical Densities for dataset 2

TABLE 1: Comparison of Estimators of R based on two samplings using Bias, MSE and Relative Efficiency

				SRS			RSS			
				$r_1 = r_2 = 10$						
θ_2	(n_1, n_2)	(m_1, m_2)	R_{True}	$R_{MLE, SRS}$	Bias	MSE	$R_{MLE, RSS}$	Bias	MSE	RE
3	(10,10)	(2,2)	0.3431	0.3571	0.0140	0.000196	0.3331	-0.0100	0.000100	1.96
	(10,15)	(2,4)		0.4271	0.0840	0.007056	0.2752	-0.0679	0.004610	1.53
	(20,20)	(4,4)		0.3542	0.0111	0.000123	0.3517	0.0086	0.000074	1.67
	(25,30)	(4,6)		0.4275	0.0844	0.007123	0.4215	0.0784	0.006147	1.16
	(35,40)	(6,6)		0.3581	0.0150	0.000225	0.3561	0.0130	0.000169	1.33
4	(10,10)	(2,2)	0.5195	0.5045	-0.0150	0.000225	0.5072	-0.0123	0.000151	1.49
	(10,15)	(2,4)		0.5187	-0.0008	0.000001	0.5202	0.0007	0.000000	1.31
	(20,20)	(4,4)		0.4997	-0.0198	0.000392	0.5054	-0.0141	0.000199	1.97
	(25,30)	(4,6)		0.5341	0.0146	0.000213	0.5074	-0.0121	0.000146	1.46
	(35,40)	(6,6)		0.5105	-0.0090	0.000081	0.5112	-0.0083	0.000069	1.18
5	(10,10)	(2,2)	0.7125	0.7012	-0.0113	0.000128	0.7222	0.0097	0.000094	1.36
	(10,15)	(2,4)		0.7215	0.0090	0.000081	0.7045	-0.0080	0.000064	1.27
	(20,20)	(4,4)		0.7096	-0.0029	0.000008	0.7151	0.0026	0.000007	1.24
	(25,30)	(4,6)		0.7618	0.0493	0.002430	0.7546	0.0421	0.001772	1.37
	(35,40)	(6,6)		0.6895	-0.0230	0.000529	0.7302	0.0177	0.000313	1.69

				$r_1 = r_2 = 15$						
3	(10,10)	(2,2)	0.3431	0.3594	0.0163	0.000266	0.3572	0.0141	0.000199	1.34
	(10,15)	(2,4)		0.4012	0.0581	0.003376	0.3904	0.0473	0.002237	1.51
	(20,20)	(4,4)		0.3502	0.0071	0.000050	0.3492	0.0061	0.000037	1.35
	(25,30)	(4,6)		0.4062	0.0631	0.003982	0.3902	0.0471	0.002218	1.79
	(35,40)	(6,6)		0.3604	0.0173	0.000299	0.3291	-0.0140	0.000196	1.53
4	(10,10)	(2,2)	0.5195	0.5165	-0.0030	0.000009	0.5173	-0.0022	0.000005	1.86
	(10,15)	(2,4)		0.5292	0.0097	0.000094	0.5125	-0.0070	0.000049	1.92
	(20,20)	(4,4)		0.5143	-0.0052	0.000027	0.5238	0.0043	0.000018	1.46
	(25,30)	(4,6)		0.5684	0.0489	0.002391	0.5661	0.0466	0.002172	1.10
	(35,40)	(6,6)		0.5181	-0.0014	0.000002	0.5184	-0.0011	0.000001	1.62
5	(10,10)	(2,2)	0.7125	0.7972	0.0847	0.007174	0.6437	-0.0688	0.004733	1.52
	(10,15)	(2,4)		0.7921	0.0796	0.006336	0.7705	0.0580	0.003364	1.88
	(20,20)	(4,4)		0.7101	-0.0024	0.000006	0.7144	0.0019	0.000004	1.60
	(25,30)	(4,6)		0.7591	0.0466	0.002172	0.7512	0.0387	0.001498	1.45
	(35,40)	(6,6)		0.6542	-0.0583	0.003399	0.6701	-0.0424	0.001798	1.89

Furthermore, Figures 4 and 5 visually underscore the enhanced fitting prowess of the Generalized Length Biased Maxwell distribution relative to the Maxwell distribution. We extracted the samples of size n=50 based on the two sampling schemes namely SRS and RSS. We computed MSE$[R_{RSS}^{ML}]$ and MSE $[R_{SRS}^{ML}]$. Based on these values we finally calculated RE= 1.98 which evidences the supremacy of RSS over SRS.

5 Remarks and Conclusion

In this investigation, we unveiled a generalized manifestation of the Length-biased Maxwell distribution. Our inquiry delved into the fundamental characteristics of this distribution, probing various methodologies for pinpointing parameters and assessing reliability functions. Furthermore, we elucidated that the previously introduced length-biased Maxwell distribution by Saghir et al. (2017) emerges as a specific instance of the GLBM distribution. Employing numerical techniques, we meticulously scrutinized the veracity of our findings through simulated data. Additionally, we applied the GLBM distribution to an authentic dataset, discerning that it offers a superior fit to real-world observations when juxtaposed with both Maxwell and Length-biased Maxwell distributions.
Furthermore, we tackle the intricate challenge of estimating the reliability system $R = P[Y < X]$, where the stress Y and strength X follow independent GLBM distribution, anchored within the realm of Reliability Stress-Strength (RSS) analysis. Our study unveils the derivation of the maximum likelihood estimator for R, considering both Simple Random Sampling (SRS) and ranked set sampling (RSS) scenarios.
We embark on a comprehensive Monte Carlo simulation study to juxtapose the efficacy of point and interval estimators for R under both SRS and RSS conditions. Employing the notion of relative efficiency, we discern that the maximum likelihood estimator of R based on RSS outperforms its SRS counterpart. Moreover, we subject our estimator to empirical validation using real-world data, affirming its practical utility. Looking ahead, our future research endeavors will endeavor to extend our methodology to the estimation of stress-strength models in multicomponent scenarios, building upon the foundation established within the RSS framework.

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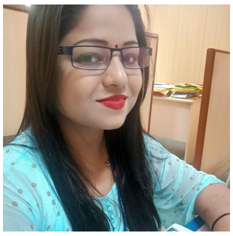

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