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On Groups Acting on Trees of Finite Extensions of Free Groups

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Abstract: A group G has the property P if G is finitely generated and is of a finite extension of a free group. In this paper we prove that if the group G has the property P and H is a subgroup of G thenIf H is of finite index, then H has the property P or H is finite and normal, then the quotient group G/H has the property P.
Furthermore, we prove that if G is a group acting on a tree X without inversions such that the stabilize G_v of each vertex v of X has the property P, $G_v \neq G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then G has the property P.
We have applications to tree product of the groups and HNN extension groups.

Keywords: Groups acting on trees, Finite extensions of free groups, Tree product of groups and HNN extension groups.

1 Introduction

For the structures of group acting on trees without inversions we refer the readers to [1], [7], [8] and [12]. In [2, Th.1.3], Gregorac, proved that if $G = *_H A_i, i \in I$ free product of the groups A_i, $i \in I$ with amalgamated subgroup H such that A_i, $i \in I$ are finite extensions of free groups, and, H and I are finite, then G is an extension of free group. In [3, Th.1], Karrass, Pietrowski and Solitar proved that a G is a finite extension of a free group if and only if G is an HNN group of the form $G = \langle gen(K), t_1, \ldots, t_n \mid rel(K),$ $t_i L_i t_i^{-1} = M_i, i = 1, \ldots, n \rangle$ only if G is an HNN group where *K* is a tree product of a finite number of finite groups (the vertices of K), and each (associated) subgroup L_i and M_i are subgroups of a vertex of *K*.

In this paper we generalize such results to groups acting on trees without inversions in a way that the stabilizers of the vertices of the tree have the property P, the stabilizers of the edges are finite, and the quotient graph for the action of the group on the tree is finite. As applications we show that the subgroups of finite index and the quotients of groups having the property P have the property P. We end the paper with examples of groups acting on trees without inversions having the property P. We begin a general background of groups acting on trees without inversions introduced in [1], [7], [8] and [10] as follows. A Graph X consists of two disjoint sets $V(X)$ (the set of vertices of X) and $E(X)$ (the set of edges of X) with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and $\eta : E(X) \to E(X)$ is an involution satisfying the conditions $\partial_0 \eta = \partial_1$ and $\partial_1 \eta = \partial_0$. For simplicity, if $e \in E(X)$, then we write $\partial_0(e) = o(e)$, $\partial_1(e) = t(e)$, and $\eta(e) = \bar{e}$. This implies that $o(\bar{e}) = t(e)$, $t(\bar{e}) = o(e)$, and $\bar{\bar{e}} = e$. We say that a group G acts on a graph X without inversions if there is a group homomorphism $\phi : G \to Aut(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$, then we write $g(x)$ for $(\phi(g))(x)$. Thus, if $g \in G$, and $y \in E(X)$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, $g(\bar{y}) = \overline{g(y)}$. If the group G acts on the graph X and $x \in X$ (x is a vertex or edge), then

A. The stabilizer of x, denoted G_x is defined to be the set $G_x = \{g \in G : g(x) = x\}$. It is clear that $G_x \leq G$, and if $x \in E(X)$ and $u \in \{o(x), t(x)\}$, then $G_{\bar{x}} = G_x$ and $G_x \leq G_u$,

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B. The orbit of x denoted $G(x)$ is defined to be the set $G(x) = \{g(x) : g \in G\}$. It is clear that G acts on the graph X without inversions if and only if $G(\bar{e}) \neq G(e)$ for any $e \in E(X)$,

C. the set of the orbits G/X of the action of G on X is defined as $G/X = \{G(x) : x \in X\}$. G/X forms a graph called the quotient graph for the action of G on X, where $V(G/X) = \{G(v) : v \in V(X)\}$, $E(G/X) = \{G(e) : e \in E(X)\}$, and if $e \in E(X)$, then $o(G(e)) = G(o(e))$, $t(G(e)) = G(t(e))$, and $\overline{G(e)} = G(\bar{e})$.

Definition 1. Let G be a group acting on a tree X without inversions, and let T and Y be two sub trees of X such that $T \subseteq Y$, and each edge of Y has at least one end in T. Assume that T and Y are satisfying the following:

(i) T contains exactly one vertex from each vertex orbit.

(ii) Y contains exactly one edge y (say) from edge orbit. The pair $(T; Y)$ is called a fundamental domain for the action of G on X. For the existence of fundamental domains, we refer the readers to [5]. We note that the set of vertices V(T) of T is in one to one correspondence with the set of vertices V(X/G) of X/G and the set of edges E(Y) of Y is in one to one correspondence with the set of edges E (X/G) of X/G.

For the rest of this section, G is a group acting on a tree X without inversions, and $(T; Y)$ is the fundamental domain for the action of G on X. We have the following definitions.

Definition 2. For any vertex $v \in V(X)$, there exist a unique vertex denoted v^* of T and an element g (not necessarily unique) of G such that $g(v^*) = v$. That is, $G(v^*) = G(v)$. Moreover, if $v \in V(T)$, then v* = v, and for each edge $y \in E(Y)$, let $[y]$ be any element of G satisfying the following:

(a) if $o(y) \in V(T)$, then $[y]((t(y))^*) = t(y)$, and $[y] = 1$ in case $y \in E(T)$,

(b) if $t(y) \in V(T)$, then $[y](o(y)) = (o(y))^*$, and $[y] = [\bar{y}]^{-1}$. Furthermore, let $+y$ be the edge $+y = y$ if $o(y) \in V(T)$, and $+y = [y](y)$ if $t(y) \in V(T)$. It is clear that $o(+y) = (o(y))^*$, and $G_{+y} \leq G_{(o(y))^*}$. If $y \in E(T)$, then $G_{+y} = G_y$.

Definition 3. If g∈G is an element of G and e∈E(Y) is an edge of Y, define [g, e] to the pair [g, e] = (gG_{+e}, e). Define $\hat{X}$ to be the set $\hat{X} = \{[g; e] : g \in G, e \in E(Y)\}$.

2 The Main Result

Theorem 1 of [3] can be stated as follows. A group G is a finite extension of a free group if and only if G is an HNN group of the form $G = \langle gen(K), t_1, ..., t_n \mid rel(K), t_i L_i t_i^{-1} = M_i, i = 1, ..., n \rangle$ Where K and I are finite. We note that if G is an HNN extension group of presentation $G = \langle gen(K), t_i \mid rel(K), t_i L_i t_i^{-1} = M_i, i \in I \rangle$ of base K and of associated pairs (L_i, M_i) of isomorphic subgroups of K, $i \in I$, where $\langle gen(K) | rel(K) \rangle$ stands for any presentation of K, and $t_i L_i t_i^{-1} = M_i$, $i \in I$ stands for the relations $t_i x t_i^{-1} = \phi_i(x)$, $x \in A_i$. Then G acts on the tree X without inversions defined as follow: $V(X) = \{gK : g \in G\}$, and $E(X) = \{(gM_i, t_i), (gL_i, t_i^{-1})\}$, where $g \in G$ and $i \in I$. For the edges (gM_i, t_i) and (gL_i, t_i^{-1}), $i \in I$, define $o(gM_i, t_i) = o(gL_i, t_i^{-1}) = gK$, $t(gM_i, t_i) = gt_i K$,

$t(gL_i, t_i^{-1}) = gt_i^{-1} K$, and $\overline{(gM_i, t_i)} = (gt_i L_i, t_i^{-1})$, and $\overline{(gL_i, t_i^{-1})} = (gt_i^{-1} M_i, t_i)$. Let $f \in G$. Then for the vertex gG and the edges (gM_i, t_i) and (gL_i, t_i^{-1}) of X, define $f(gK) = fgK$, $f(gM_i, t_i) = (fgM_i, t_i)$, and $f(gL_i, t_i^{-1}) = (fgL_i, t_i^{-1})$. The stabilizer of the vertex $v = gK$ is $K_v = gKg^{-1}$, a conjugate of G, the stabilizers of the edges (gB_i, t_i), and (gA_i, t_i^{-1}) are $gM_i g^{-1}$, a conjugate of M_i, and $gL_i g^{-1}$, a conjugate of L_i are finite for all $i \in I$.

The orbits of gG, (gB_i, t_i) and (gL_i, t_i^{-1}) are $\{fK : f \in G\}$ and $\{(fM_i, t_i) : f \in G\}$.

If the group G acts on a tree X without inversions such that $G_v \neq G$ for any vertex $v \in V(X)$ of X, then by [5, Th.4], there exists a fundamental domain (T;Y) for the action of G on X, and [7, Th. 5.1], G has the presentation G = $\left\langle gen(G_v), y \middle| rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y].y^{-1} = G_y \right\rangle$, where , m and y stand for edges of E(Y) such that $m \in E(T)$, $o(y) \in V(T)$, $t(y) \notin V(T)$.

Let $K = \prod^{*}_{v \in V(X)} (G_v; G_m = G_m)$

$L_m = [m]^{-1} G_m [m]$ and $M_m = G_m$. It is clear that K is a tree product of the groups G_v with amalgamation subgroups L_m, M_m.This implies that $\left\langle gen(K), y \middle| rel(K),\ y.[y]^{-1}G_y[y].y^{-1} = G_y \right\rangle$ is an HNN extension group of base K, associated isomorphic subgroups L_m, M_m, and stable letters the edges $y \in E(X)$ ox X such that $o(y) \in V(T)$, $t(y) \notin V(T)$. This leads the following lemma.

Lemma 2.1. A group G has the property P if and only if there exists a tree X on which G acts on X without inversions such that the stabilize G_v of each vertex v of X is finite, $G_v \neq G$, and the quotient graph G/X for the action of G on X is finite.

Proposition 2.1. the group G has the property P and H is a subgroup of G then

(i) If H is of finite index, then H has the property P.

(ii) If H is finite and normal, then the quotient group G/H has the property P.

Proof. Since G has the property P, G is finitely generated, andby Lemma 2.1, exists a tree X such that G acts on X without inversions, the stabilize G_v for each vertex v of X is finite and the quotient graph G/X

for the action of G on X is finite

(i) Since G is finitely generated and H is of finite index in G, by the Reidemeister-Schreier subgroup theorem [6,Corollary 2.8, page 93], H is finitely generated. Then H acts on X by restriction. It is clear that the vertex stabilize H_v of the vertex v of X satisfies $H_v = H \cap G_v$. Since G_v is finite, H_v is finite. Since H is of finite index in G, and G/X isfinite, therefore by Lemma 7 of [7], the quotient graph H/X for the action of H on X is finite. Thus, H has the property P.

(ii) Since G is finitely generated, it is clear that G/H is finitely generated. Let X^H be the set $X^H = \{x \in X : H \leq G_x\}$. Then by Proposition 4.3 of [8], X^H is a subtree of X and G/H acts on X^H without inversions, where if $g \in G$, and $x \in E(X^H)$ then such that gH(x) = g(x). It is clear that the stabilizer of $x \in X^H$ under the action of G/H on X^H is $(G/H)_x = G_x/H$, where G_x is the stabilizer of x under the action of G on X. Since stabilizer of each element $x \in X$ of X under the action of G on X is finite, therefore stabilizer of each $x \in X^H$ under the action of G/H on X^H is finite. If $x \in V(V)$ and $(G/H)_x = G_x/H$ = G/H, then G_x = G. Contradiction. Hence $(G/H)_x = G_x/H \neq$ G/H for any vertex $x \in V(V)$. It is clear that if $x \in X^H$, where x is a vertex or an edge, then the orbit (G/H)(x) of x under the action of G/H on X^H is given by (G/H)(x) = G(x) where G(x) is the orbit of x under the action of G on X. This implies that the quotient graph $G/H/X^H$ for the action of G/H on X^H is = $\{(G/H)(x) = G(x) : x \in X\} \subseteq G/X$. Since G/X is finite, therefore $G/H/X^H$ is finite. Consequently, the quotient group G/H has the property P. This completes the proof.

Before we prove the main result of this paper, we introduce the following concept taken from [1, page 78]. Let H be a subgroup of a group G and let H act on a set X. Define ≡ to be the relation on $G \times X$ defined as $(f, u) \equiv (g, v)$, if there exists $h \in H$ such that $f = gh$ and $u = h^{-1}(v)$. It is easy to show that ≡ is an equivalence relation on $G \times X$. The equivalence class containing $(f; u)$ is denoted by $f \otimes_H u$. Thus, $f \otimes_H u = \{(fh; h^{-1}(u)) : h \in H\}$.

Define $G \otimes_H X$ to be the set $G \otimes_H X = \{g \otimes_H x : g \in G, x \in X\}$.the main result of this section is the following theorem.

Theorem 2.1. If G is a group acting on a tree X without inversions such that the stabilize G_v for each vertex v of X has the property P, $G_v \neq G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then the group G has the property P.

Proof. Let $v \in V(X)$. SinceG_v has the property P, G_v is finitely generated. Since the quotient graph G/X for the action of G on X is finite, by Lemma 4.4 of [11], G is finitely generated. Furthermore, by Lemma 2.1, exists a tree X_v such that G_v acts on X_v without inversions, the stabilizer $(G_v)_u$ for each vertex u of X_v is finite $(G_v)_u \neq G_v$, andthe quotient graph G_v/X_v for the action of G_v on X_v is finite. Let $(T; Y)$ be a fundamental domain for the action of G on

X. Since the quotient graph G/X for the action of G on X is finite, T and Y are finite. By Lemma 4.4 of [6], G is generated by the generators of G_v, $v \in V(T)$ and by the elements $[y]$, $y \in E(Y)$. By Theorem 3.4 of [7], there exists a tree denoted as

$$\tilde{X} = \hat{X} \cup \left(\bigcup_{v \in V(T)} (G \otimes_{G_v} X_v) \right),$$

Where $\hat{X} = \{[g; e] : g \in G,\ e \in E(Y)\}$, and $[g; e] = (gG_{+e}, e)$, $V(\tilde{X}) = \bigcup_{v \in V(T)} (G \otimes_{G_v} V(X_v))$

and $E(\tilde{X}) = \hat{X} \cup \left(\bigcup_{v \in V(T)} (G \otimes_{G_v} E(X_v)) \right)$.

The ends and the inverse of the edge $g \otimes_{G_v} e$ are defined as follows: $t(g \otimes_{G_v} e) = g \otimes_{G_v} t(e)$, $o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e)$ and $\overline{g \otimes_{G_v} e} = g \otimes_{G_v} \bar{e}$, where $t(e)$, $o(e)$, and $\bar{e}$ are the ends and the inverse of the edge e in X_v. G acts on $\tilde{X}$ as follows: if $f,\ g \in G,\ y \in E(Y),\ v \in V(T),\ e \in E(X_v)$, and $u \in V(X_v)$, then $f[g; y] = [fg; y]$, $f(g \otimes_{G_v} e) = fg \otimes_{G_v} e$, and $f(g \otimes_{G_v} u) = fg \otimes_{G_v} u$. If $g \in G$ and $e \in E(Y)$ such that $g(1 \otimes_{G_v} e) = \overline{1 \otimes_{G_v} e} = 1 \otimes_{G_v} \bar{e}$, then $g \in G_v$ and $e \in E(X_v)$, $g(e) = \bar{e}$. Hence, G_v acts on X_v with inversions. This is a contradiction because G_v has the property P. This implies that G acts on $\tilde{X}$ without inversions. Now for $g \in G$ and $x \in X_v$, it is clear that the stabilizer $G_{g \otimes_{G_v} x}$ of the vertex $g \otimes_{G_v} x$ is $G_{g \otimes_{G_v} x} = g(G_v)_x g^{-1}$, where $(G_v)_x$ is the stabilizer of x under the action of G_v on X_v. Since $(G_v)_x$ is finite, therefore, $G_{g \otimes_{G_v} x}$ is finite. So the stabilizer of each vertex of $\tilde{X}$ under the action of G on $\tilde{X}$ is finite. Now we show that the quotient graph $G/\tilde{X}$ for the action of G on $\tilde{X}$ is finite. The fundamental domain $(T; Y)$ for the action of G on X induces a fundamental domain $(T_v; Y_v)$ for the action of G_v on X_v for every vertex v of T.

For each edge e∈E(Y), let v_e be a vertex $v_e \in V(T_{(o(e))^*})$ such that $G_{+e} \leq (G_{(o(e))^*})_{v_e}$, where $(G_{(o(e))^*})_{v_e}$ is the vertex stabilizer of the vertex v_e under the action of $G_{(o(e))^*}$ on $X_{(o(e))^*}$. Let $\hat{T} = \{[1; e]$: e∈E(T)}, $\hat{Y} = \{[1; e]$, [[e];e]: e∈E(Y)}, $\tilde{T} = \hat{T} \cup (\bigcup_{v \in V(T)} (1 \otimes_{G_v} T_v))$, and $\tilde{Y} = \hat{Y} \cup (\bigcup_{v \in V(T)} (1 \otimes_{G_v} Y_v))$, where [e] is the value of the edge e defined as in Definition 1 and [[e], e] is defined as in Definition 3. Then by Theorem 3.2 of [9], $(\tilde{T}; \tilde{Y})$ forms a fundamental domain for the action of G on $\tilde{X}$. Since for each edge e∈E(X) of X the stabilizer G_e is finite and (T;Y) and $(T_v; Y_v)$, v∈V(T) are finite, then the fundamental domain $(\tilde{T}; \tilde{Y})$ is finite. This shows that the quotient graph $G/\tilde{X}$ for the action of G on $\tilde{X}$ is finite. Then Lemma 2.1 implies that the group G has the property P. This completes the proof.

3 Application

Now we apply Theorem 2.1 to tree product of groups and HNN groups introduced in [4]. Tree product of groups and HNN groups are examples of groups acting on trees without inversions.

If $A = \prod^*_{i \in I} (A_i; U_{ij} = U_{ji})$ is a tree product of the groups A_i, $i \in I$ with amalgamation subgroups U_{ij}, $i,\ j \in I$, then A acts on the tree X without inversions defined as follow: $V(X) = \{(gA_i, i) : g \in A,\ i \in I\}$ and $E(X) = \{(gU_{ij}, ij) : g \in A,\ i,\ j \in I\}$. If y is the edge $y = (gU_{ij}, ij)$, then $o(y) = (gA_i, i)$, $t(y) = (gA_j, j)$, and $\bar{y} = (gU_{ji}, ji)$. The group A acts on X as follows: let $f \in A$. Then $f((gA_i, i)) = (fgA_i, i)$ and $f((gU_{ij}, ij)) = (fgU_{ij}, ij)$. If v is the vertex $v = (gA_i, i) \in V(X)$ and y is the edge $y = (gU_{ij}, ij) \in E(X)$, then the stabilizer of v is $A_v = gA_i g^{-1} \cong A_i$, a conjugate of A_i and the stabilizer of y is $A_y = gU_{ij}g^{-1} \cong U_{ij}$, a conjugate of U_{ij}. The orbit of v is the set $A(v) = \{(agA_i, i) : a \in A,\ i \in I\}$ and the orbit of y is $A(y) = \{(agU_{ij}, ij) : a \in I,\ i,\ j \in I\}$. Now, we turn to the definition of an HNN group. Let G be a group and let I be an index set. Let $\{A_i : i \in I\}$ and

$\{B_i : i \in I\}$ be two families of subgroups of G. For each $i \in I$, let $\phi_i : A_i \to B_i$ is isomorphism. The group G^* of the presentation

$G^* = \langle gen(G),\, t_i \,|\, rel(G),\, t_i A_i t_i^{-1} = B_i,\, i \in I\rangle$ is called an HNN group of base G and of associated pairs (A_i, B_i) of isomorphic subgroups of G, $i \in I$, where $\langle gen(G) | rel(G)\rangle$ stands for any presentationof G, and $t_i A_i t_i^{-1} = B_i, i \in I$ stands for the relations $t_i a_i t_i^{-1} = \phi_i(a_i),\, a_i \in A_i$. The HNN group G^* acts on the tree X without inversions defined as follow: $V(X) = \{gG : g \in G^*\}$, and $E(X) = \{(gB_i, t_i), (gA_i, t_i^{-1})\}$, where $g \in G^*$ and $i \in I$. For the edges (gB_i, t_i) and $(gA_i, t_i^{-1}), i \in I$, define $o(gB_i, t_i) = o(gA_i, t_i^{-1}) = gG$, $t(gB_i, t_i) = gt_iG,\, t(gA_i, t_i^{-1}) = gt_i^{-1}G$, and $\overline{(gB_i, t_i)} = (gt_iA_i, t_i^{-1})$, and $\overline{(gA_i, t_i^{-1})} = (gt_i^{-1}B_i, t_i)$. Let $f \in G^*$. Then for the vertex gG and the edges (gB_i, t_i) and (gA_i, t_i^{-1}) of X, define $f(gG) = fgG$, $f(gB_i, t_i) = (fgB_i, t_i)$, and $f(gA_i, t_i^{-1}) = (fgA_i, t_i^{-1})$. The stabilizer of the vertex $v = gG$ is $G_v = gGg^{-1}$, a conjugate of G, the stabilizers of the edges (gB_i, t_i), and (gA_i, t_i^{-1}) are gB_ig^{-1}, a conjugate of B_i, and gA_ig^{-1}, a conjugate of A_i are finite for all $i \in I$. The orbits of gG, (gB_i, t_i) and (gA_i, t_i^{-1}) are $\{fG : f \in G^*\}$

and $\{(fB_i, t_i) : f \in G^*\}$. We have the following propositions as applications of Theorem 2.1.

Proposition 3.1. Let $A = \prod^*_{i\in I}(A_i; U_{ij} = U_{ji})$ be a tree product of the groups A_i, $i \in I$ with amalgamation subgroups U_{ij}, $i, j \in I$ such that the group A_i has the property P for all i∈I, I is finite, and U_{ij} is finite for all i,j∈I. Then A has the property P.

Corollary 3.1. Let $A = *_C A_i, i \in I$ be the free product of the groups A_i, $i \in I$ with amalgamation subgroup C such that the group A_i has the property P for all i∈I, I is finite, and C is finite. Then the group A has the property P.

Proposition 3.2. Let G^* be the HNN group $G^* = \langle gen(G), t_i | rel(G),\, t_iA_it_i^{-1} = B_i, i \in I\rangle$ of base G and of associated pairs (A_i, B_i) of isomorphic subgroups of G such that the group G has the property P, A_i and B_i are finite for all i∈I, and I is finite. Then the group G* has the property P.

4 Conclusions

In this paper, we proved that if G is a group acting on a tree X without inversions such that the stabilize G_v of each vertex v of X has the property P, $G_v \neq G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then G has the property P. On the other hand, we have applications to tree product of the groups and HNN extension groups.

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