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A Fitted Finite Difference Scheme for solving Singularly Perturbed Two Point Boundary Value Problems

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Abstract: The present study addresses an efficient exponentially fitted method to obtain the solution of singularly perturbed two point boundary value problems(BVPs) on uniform mesh. A fitting factor is introduced in a Taylor series based derived scheme using the theory of singular perturbations. Thomas algorithm is used to solve the resulting tri-diagonal system of equations. Stability and convergence of the method are investigated. The applicability of the method is shown with numerical experiments performed on the three model test example problems. The computational results are compared with the results obtained by other methods. The study showed that the present method approximates the exact/approximate solution very well.

Keywords: Singular perturbations, boundary value problem, stability and convergence of numerical methods, exponential fitting factor

1 Introduction

A differential equation becomes singularly perturbed when the highest derivative in the equation is multiplied by a small positive parameter $\varepsilon\,(0 < \varepsilon << 1)$ and such singularly perturbed equations with initial and/or boundary conditions is termed as singular perturbation problems. The small parameter ε is called the singular perturbation parameter. The capriciousness of such problems can easily be detected in the various fields of applied sciences. They often arise in electrical networks, quantum mechanics, fluid dynamics, chemical reactions, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, etc. [1,2,3,4,5,6]. Some examples are boundary layer problems, Wentzel, Kramers and Brillouin (WKB) problems, the modelling of steady and unsteady viscous flow problems with large Reynolds numbers [7, 8], convective heat transport problems with large Peclet numbers and the magneto-hydrodynamics duct problems at high Hartman numbers [9], etc. The solution of such problems exhibits sharp boundary and/or interior layers when ε is very small. The solution varies most rapidly in some parts and slowly in other parts of the domain. Thus, the treatment of singularly perturbed problems (SPPs) poses severe difficulties that have to be addressed to ensure accurate numerical solutions. There are a variety of asymptotic and numerical methods for the solution of the singular perturbation problems. For a good discussion, we may refer to the momographs: Bellman [10], Doolan et al. [11,12,13], Verhulst [14], Kevorkian and Cole [15], Nayfeh [16], O'Malley [17], Bender and Orszag [18], Farrell et al. [19], Miller et al. [20], Kevorkian and Cole [21], Protter and Weinberger [22], Roos et al. [8], Shishkin et al. [23], and, the survey papers by Kadalbajoo et al. [24,25,26,27].

Recently, Reddy and Reddy [28] have presented a numerical integration method which finds the approximate solution of a general singularly perturbed BVP with left, right and interior layers. The authors in [29,30] have suggested exponentially fitted finite difference schemes on a uniform mesh for solving model equation of the form (1). Reddy and Mohapatra [31] have presented an efficient numerical method with exponentially fitted factor to obtain the solution of singularly perturbed two point BVPs where the boundary layer is present at one end point (either left or right). Gbsl Soujanya et al. [32] have developed a non-symmetric finite difference method with exponential fitting factor to solve SPPs exhibiting layer behaviour using Numerov's method. The articles [3,33,34,35,36,37] propose different numerical approaches which combine fitted mesh methods and fitted operator methods for solving SPPs.

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The present study aims to present a simple, but computationally efficient fitted finite difference scheme for the solution of a class of singularly perturbed two-point BVPs which exhibits boundary layer at left or right end point of the interval considered. The computational results show the capability of the present method in producing accurate results with minimal computational effort when perturbation parameter $\varepsilon \to 0$.

The paper is arranged as follows: Section 2 addresses the statement of the continuous problem with the properties of the solution. Sections 3 and 4 present the description of the proposed method when the boundary layer is present at left and right end points of the underlying interval respectively. Stability and convergence of the method are analysed in Section 5. In Section 6, the efficiency and applicability of the proposed method are illustrated with some numerical experiments. Conclusion is presented in Section 7 followed by the references.

2 Statement of the Problem :

We consider the following class/form of Singularly Perturbed BVP:

$$L_\tau u(t) \equiv \varepsilon u''(t) + f(t)u'(t) + g(t)u(t) = r(t); \quad (1)$$

subject to the boundary conditions:

$$u(0) = \eta \text{ and } u(1) = \gamma \quad (2)$$

on $I = [0,1]$, where $\varepsilon\,(0 < \varepsilon << 1)$ is a perturbation parameter and η, γ are known finite constants. We assume that the functions $f(t), g(t), r(t)$ are sufficiently smooth and bounded functions on I, $g(t) \le 0$ on the entire interval I, f^* is postive constant and g_* is negative constant such that

$$|f(t)| \le f^*, g(t) \le g_* < 0, \; t \in I.$$

If it is assumed that $f(t) \ge W^* > 0$ on the entire interval $[0,1]$, where $W^* > 0$ refers to a positive constant, the equation (1) along with (2) possesses a unique solution with the boundary layer in the neighbourhood of $t = 0$, i.e. at left end point of the interval for small values of ε, while the position of the boundary layer is in the neighbourhood of $t = 1$ if $f(t) \le W^* < 0$ on the entire interval $[0,1]$, where $W^* > 0$ denotes a negative constant. The operator $L_\tau = \varepsilon \frac{d^2}{dt^2} + f(t)\frac{d}{dt} + g(t)$ in (1) satisfies the following minimum principle [17].

Lemma 2.1. Suppose $\omega(t)$ represents a smooth function satisfying the conditions $\omega(0) \ge 0, \omega(1) \ge 0$. Then, $L_\tau \omega(t) \le 0, \forall t \in (0,1)$ implies $\omega(t) \ge 0, \forall t \in [0,1]$.

proof : Let $m \in [0,1]$ be such that $\omega(m) < 0$ and $\omega(m) = \min_{t\in[0,1]} \omega(t)$. Since $m \notin \{0,1\}$, therefore, $\omega'(m) = 0$ and $\omega''(m) \ge 0$. Hence, we obtain

$$L_\tau \omega(m) = \varepsilon \omega''(m) + f(m)\omega'(m) + g(m)\omega(m) > 0,$$

which contradicts our assumption. Hence, $\omega(m) \ge 0$ and so $\omega(t) \ge 0 \forall t \in [0,1]$.

Lemma 2.2. Let $u(t)$ be the solution of the problem (1) and (2). Then, we have

$$\|u\| \le a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|).$$

where $\|\cdot\|$ is the L_∞ norm given by $\|u\| = \max_{0\le t\le 1} |u(t)|$.

proof : Let $\omega^\pm(t)$ be two barrier functions defined by

$$\omega^\pm(t) = a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|) \pm u(t)$$

Then, this implies

$$\begin{aligned}\omega^\pm(0) &= a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|) \pm u(0)\\ &= a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|) \pm \eta_0 \text{ since}, u(0) = \eta(0) = \eta_0\\ &\ge 0\end{aligned}$$

$$\begin{aligned}\omega^\pm(1) &= a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|) \pm u(1)\\ &= a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|) \pm \gamma \text{ since}, u(1) = \gamma\\ &\ge 0\end{aligned}$$

$$\begin{aligned}\Rightarrow L_\tau \omega^\pm(t) &= \varepsilon(\omega^\pm(t))'' + f(t)(\omega^\pm(t))' + g(t)\omega^\pm(t)\\ &= g(t)\left[a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|)\right] \pm L_\tau u(t)\\ &= g(t)\left[a_2^{-1}\|r\| + \max(|\eta_0|, |\gamma|)\right] \pm r(x) \text{ using } (1)\end{aligned}$$

As $g(t) \le -a_2 < 0$ implies $g(t)a_2^{-1} \le -1$ and since $\|r\| \ge r(t)$, we have

$$\Rightarrow L_\tau \omega^\pm(t) \le (-\|r\| \pm r(t)) + g(t)\max(|\eta_0|, |\gamma|) \le 0, \forall t \in [0,1].$$

Thus, using the minimum principle, $\omega^\pm(t) \ge 0$, $\forall t \in [0,1]$. Now, for computing the error that has occurred in our numerical approximations, the derivative of the solution $u(t)$ should possess a boundedness which remains valid for all $t \in (0, 1]$.

Using Lemma 2.1, the required estimate is obtained.

3 Description of the method for left-End Boundary Layer Problems:

In this section, we will describe the proposed method for the solution of the problem (1) with (2) having boundary layer at left end point of the interval considered.

The solution of (1) with (2) is of the following form (cf. [17], pp.22-26):

$$u(t) = u_0(t) + \frac{f(0)}{f(t)}(\eta - u_0(0))\,e^{-\int_0^t \left(\frac{f(t)}{\varepsilon} - \frac{g(t)}{f(t)}\right)dt} + o(\varepsilon) \quad (3)$$

where $u_0(t)$ denotes the solution of the following problem:

$$f(t)u_0'(t) + g(t)u_0(t) = r(t); \; u_0(1) = \gamma \quad (4)$$

Under the consideration of Taylor's series expansions for $f(t)$ and $g(t)$ about the point $'t=0'$ upto their first terms only, the equation (3) becomes:

$$u(t)=u_0(t)+(\eta-u_0(0))\,e^{-\left(\frac{f(0)}{\varepsilon}-\frac{g(0)}{f(0)}\right)t}+o(\varepsilon) \quad (5)$$

Furthermore, considering equation (5) at the point $t=t_l=lh,\ l=0,1,2,...,N$ and taking the limit as $h\to 0$, we obtain

$$\lim_{h\to 0} u(lh)=u_0(0)+(\eta-u_0(0))\,e^{-\left(\frac{f^2(0)-\varepsilon g(0)}{f(0)}\right)l\rho}+o(\varepsilon) \quad (6)$$

where $\rho=h/\varepsilon$.
Now, applying Taylor's series expansion procedure, we have:

$$u(t_{l+1})=u_{l+1}=u_l+hu'_l+\frac{h^2}{2!}u''_l+\frac{h^3}{3!}u'''_l+\frac{h^4}{4!}u_l^{(4)}+\frac{h^5}{5!}u_l^{(5)}+\frac{h^6}{6!}u_l^{(6)}+\frac{h^7}{7!}u_l^{(7)}+\frac{h^8}{8!}u_l^{(8)}+O(h^9) \quad (7)$$

$$u(t_{l-1})=u_{l-1}=u_l-hu'_l+\frac{h^2}{2!}u''_l-\frac{h^3}{3!}u'''_l+\frac{h^4}{4!}u_l^{(4)}-\frac{h^5}{5!}u_l^{(5)}+\frac{h^6}{6!}u_l^{(6)}-\frac{h^7}{7!}u_l^{(7)}+\frac{h^8}{8!}u_l^{(8)}-O(h^9) \quad (8)$$

From finite differences, we get

$$u_{l-1}-2u_l+u_{l+1}=\frac{2h^2}{2!}u''_l+\frac{2h^4}{4!}u_l^{(4)}+\frac{2h^6}{6!}u_l^{(6)}+\frac{2h^8}{8!}u_l^{(8)}+O(h^{10}) \quad (9)$$

and

$$u''_{l-1}-2u''_l+u''_{l+1}=\frac{2h^2}{2!}u_l^{(4)}+\frac{2h^4}{4!}u_l^{(6)}+\frac{2h^6}{6!}u_l^{(8)}+\frac{2h^8}{8!}u_l^{(10)}+O(h^{12})$$

Substituting $\frac{h^4}{12}u_l^{(6)}$ from the above equation in (9), we get

$$u_{l-1}-2u_l+u_{l+1}=h^2u''_l+\frac{h^2}{30}\left(u''_{l-1}-2u''_l+u''_{l+1}-h^2u_l^{(4)}-\frac{h^6}{360}u_l^{(8)}\right)+\frac{h^4}{12}u_l^{(4)}+\frac{2h^8}{8!}u_l^{(8)}+O(h^{10})$$

$$u_{l-1}-2u_l+u_{l+1}=\frac{h^2}{30}\left(u''_{l-1}+28u''_l+u''_{l+1}\right)+R \quad (10)$$

where $R=\frac{h^4}{20}u_l^{(4)}-\frac{13h^6}{302400}u_l^{(8)}+O(h^{10})$.
Now, using equation (1) we get

$$\varepsilon u''_{l+1}=-f_{l+1}u'_{l+1}-g_{l+1}u_{l+1}+r_{l+1} \quad (11)$$

$$\varepsilon u''_l=-f_lu'_l-g_lu_l+r_l \quad (12)$$

$$\varepsilon u''_{l-1}=-f_{l-1}u'_{l-1}-g_{l-1}u_{l-1}+r_{l-1} \quad (13)$$

We know that three point approximations for first order derivative are as follows:

$$u'_l=\frac{u_{l+1}-u_{l-1}}{2h} \quad (14)$$

$$u'_{l+1}=\frac{3u_{l+1}-4u_l+u_{l-1}}{2h} \quad (15)$$

$$u'_{l-1}=\frac{-u_{l+1}+4u_l-3u_{l-1}}{2h} \quad (16)$$

Substituting (14), (15) and (16) in (11), (12) and (13) respectively, and simplifying the equation (10), we get

$$\varepsilon\left(\frac{u_{l-1}-2u_l+u_{l+1}}{h^2}\right)+\frac{f_{l-1}}{60h}\left(-3u_{l-1}+4u_l-u_{l+1}\right)+\frac{28f_l}{60h}\left(u_{l+1}-u_{l-1}\right)+\frac{f_{l+1}}{60h}\left(u_{l-1}-4u_l+3u_{l+1}\right)+\frac{g_{l-1}}{30}u_{l-1}+\frac{28g_l}{30}u_l+\frac{g_{l+1}}{30}u_{l+1}=\frac{1}{30}\left(r_{l-1}+28r_l+r_{l+1}\right) \quad (17)$$

Now introducing the fitting factor $\sigma(\rho)$ in the above scheme (17), we have

$$\sigma(\rho)\,\varepsilon\left(\frac{u_{l-1}-2u_l+u_{l+1}}{h^2}\right)+\frac{f_{l-1}}{60h}\left(-3u_{l-1}+4u_l-u_{l+1}\right)+\frac{28f_l}{60h}\left(u_{l+1}-u_{l-1}\right)+\frac{f_{l+1}}{60h}\left(u_{l-1}-4u_l+3u_{l+1}\right)+\frac{g_{l-1}}{30}u_{l-1}+\frac{28g_l}{30}u_l+\frac{g_{l+1}}{30}u_{l+1}=\frac{1}{30}\left(r_{l-1}+28r_l+r_{l+1}\right) \quad (18)$$

The fitting factor $\sigma(\rho)$ is to be determined so as to make the solution of difference scheme (18) converges uniformly to the solution of (1) - (2).
Clearly under the limit when $h\to 0$, the scheme (18) becomes

$$\sigma(\rho)\,\varepsilon\left(\frac{u_{l-1}-2u_l+u_{l+1}}{h}\right)+\frac{f(0)}{60}\left(-3u_{l-1}+4u_l-u_{l+1}\right)+\frac{28f(0)}{60}\left(u_{l+1}-u_{l-1}\right)+\frac{f(0)}{60}\left(u_{l-1}-4u_l+3u_{l+1}\right)=0 \quad (19)$$

Let $F=\frac{f^2(0)-\varepsilon g(0)}{g(0)}$.
By using (6), we get

$$\lim_{h\to 0}\left(u(lh-h)-2u(lh)+u(lh+h)\right)=(\eta-u_0(0))*e^{-Fl\rho}\left(e^{F\rho}+e^{-F\rho}-2\right)$$

$$\lim_{h\to 0}\left(-3u(lh-h)+4u(lh)-u(lh+h)\right)=(\eta-u_0(0))*e^{-Fl\rho}*\left(-3e^{F\rho}-e^{-F\rho}+4\right)$$

$$\lim_{h\to 0}\left(u(lh-h)-4u(lh)+3u(lh+h)\right)=(\eta-u_0(0))*e^{-Fl\rho}*\left(e^{F\rho}+3e^{-F\rho}-4\right)$$

$$\lim_{h\to 0}\left(u(lh+h)-u(lh-h)\right)=(\eta-u_0(0))\,e^{-Fl\rho}\left(e^{-F\rho}-e^{F\rho}\right)$$

Using the above equations in the equation (19) we get:

$$\frac{\sigma(\rho)}{\rho}\left(e^{F\rho}+e^{-F\rho}-2\right)=-\frac{f(0)}{60}\left(-30e^{F\rho}+30e^{-F\rho}\right) \quad (20)$$

Simplifying (20) we get,

$$\sigma(\rho)=\frac{\rho f(0)}{2}\coth\left(\frac{\left(f^2(0)-\varepsilon g(0)\right)\rho}{2f(0)}\right) \quad (21)$$

Which is the required constant fitting factor $\sigma(\rho)$ in this left end boundary layer problem case.

Finally, from the equation (18) with the value of $\sigma(\rho)$ given by equation (21), we obtain the following three-term recurrence relationship:

$$P_l u_{l-1} - Q_l u_l + R_l u_{l+1} = H_l, \ (l = 1,2,3,\ldots,N-1) \quad (22)$$

where

$$P_l = \frac{\sigma\varepsilon}{h^2} - \frac{3f_{l-1}}{60h} + \frac{g_{l-1}}{30} - \frac{28f_l}{60h} + \frac{f_{l+1}}{60h}$$

$$Q_l = \frac{2\sigma\varepsilon}{h^2} - \frac{4f_{l-1}}{60h} - \frac{28g_l}{30} + \frac{4f_{l+1}}{60h}$$

$$R_l = \frac{\sigma\varepsilon}{h^2} - \frac{f_{l-1}}{60h} + \frac{g_{l+1}}{30} + \frac{28f_l}{60h} + \frac{3f_{l+1}}{60h}$$

$$H_l = \frac{1}{30}\left(r_{l-1} + 28r_l + r_{l+1}\right)$$

The equation (22) produces a system of $(N-1)$ equations with $(N-1)$ unknowns u_1 to u_{N-1}. These $(N-1)$ equations together with the boundary conditions equation (2) are sufficient to solve the obtained tri-diagonal system with the help of an efficient solver called Thomas Algorithm which is also known as 'Discrete Invariant Imbedding algorithm' [28,30,31].

Note that the method of LU decomposition (or Gaussian elimination) which is equivalent to the "Thomas algorithm" provides a numerically stable technique for solving the system when the coefficient matrix of the system is diagonally dominant or irreducibly diagonally dominant and so non singular.

4 Description of the method for Right-End Boundary Layer Problems:

In this section, we describe the proposed method for the solution of the problem (1) with (2) having boundary layer at right end point of the interval considered.

The solution of (1) with (2) is of the following form (cf. [17], pp.22-26):

$$u(t) = u_0(t) + \frac{f(1)}{f(t)}\left(\gamma - u_0(1)\right) e^{\int_t^1 \left(\frac{f(t)}{\varepsilon} - \frac{g(t)}{f(t)}\right) dt} + o(\varepsilon) \quad (23)$$

where $u_0(t)$ represents the solution of the reduced problem:

$$f(t)u_0'(t) + g(t)u_0(t) = r(t); \ u_0(0) = \eta \quad (24)$$

Expanding $f(t)$ and $g(t)$ in (23) with the help of the Taylor's series about the point $'t = 1'$ and restricting to their first terms, we obtain:

$$u(t) = u_0(t) + \left(\gamma - u_0(1)\right) e^{\left(\frac{f(1)}{\varepsilon} - \frac{g(1)}{f(1)}\right)(1-t)} + o(\varepsilon) \quad (25)$$

Moreover, considering equation (25) at the point $t = t_l = lh,\ l = 0,1,2,\ldots,N$ and taking the limit as $h \to 0$ we obtain

$$\lim_{h\to 0} u(lh) = u_0(0) + \left(\gamma - u_0(1)\right) e^{\left(\frac{f^2(1)-\varepsilon g(1)}{f(1)}\right)\left(\frac{1}{\varepsilon} - l\rho\right)} + o(\varepsilon) \quad (26)$$

where $\rho = h/\varepsilon$.
Let $\hat{F} = \frac{f^2(1)-\varepsilon g(1)}{f(1)}$.
Using (26), we get

$$\lim_{h\to 0}\left(u(lh-h) - 2u(lh) + u(lh+h)\right) = \left(\gamma - u_0(1)\right) * e^{\hat{F}\left(\frac{1}{\varepsilon} - l\rho\right)} * \left(e^{\hat{F}\rho} + e^{-\hat{F}\rho} - 2\right)$$

$$\lim_{h\to 0}\left(-3u(lh-h) + 4u(lh) - u(lh+h)\right) = \left(\gamma - u_0(1)\right) * ,e^{\hat{F}\left(\frac{1}{\varepsilon} - l\rho\right)} * \left(-3e^{\hat{F}\rho} - e^{-\hat{F}\rho} + 4\right)$$

$$\lim_{h\to 0}\left(u(lh-h) - 4u(lh) + 3u(lh+h)\right) = \left(\gamma - u_0(0)\right) * e^{\hat{F}\left(\frac{1}{\varepsilon} - l\rho\right)} * \left(e^{\hat{F}\rho} + 3e^{-\hat{F}\rho} - 4\right)$$

$$\lim_{h\to 0}\left(u(lh+h) - u(lh-h)\right) = \left(\gamma - u_0(0)\right) e^{\hat{F}\left(\frac{1}{\varepsilon} - l\rho\right)} \left(e^{-\hat{F}\rho} - e^{\hat{F}\rho}\right)$$

Using the above equations in the equation (19), we get

$$\frac{\sigma(\rho)}{\rho}\left(e^{\hat{F}\rho} + e^{-\hat{F}\rho} - 2\right) = -\frac{f(0)}{60}\left(-30e^{\hat{F}\rho} + 30e^{-\hat{F}\rho}\right) \quad (27)$$

Simplifying (27), we get

$$\sigma(\rho) = \frac{\rho f(0)}{2} \coth\left(\frac{\left(f^2(1) - \varepsilon g(1)\right)\rho}{2f(1)}\right) \quad (28)$$

which is a required constant fitting factor $\sigma(\rho)$ in this right end boundary layer problem case.
Finally, from the equation (18) with the value of $\sigma(\rho)$ given by equation (28), we obtain the following three-term recurrence relationship of the form:

$$P_l u_{l-1} - Q_l u_l + R_l u_{l+1} = H_l, \ (l = 1,2,3,\ldots,N-1) \quad (29)$$

where

$$P_l = \frac{\sigma\varepsilon}{h^2} - \frac{3f_{l-1}}{60h} + \frac{g_{l-1}}{30} - \frac{28f_l}{60h} + \frac{f_{l+1}}{60h}$$

$$Q_l = \frac{2\sigma\varepsilon}{h^2} - \frac{4f_{l-1}}{60h} - \frac{28g_l}{30} + \frac{4f_{l+1}}{60h}$$

$$R_l = \frac{\sigma\varepsilon}{h^2} - \frac{f_{l-1}}{60h} + \frac{g_{l+1}}{30} + \frac{28f_l}{60h} + \frac{3f_{l+1}}{60h}$$

$$H_l = \frac{1}{30}\left(r_{l-1} + 28r_l + r_{l+1}\right)$$

The equation (29) produces a system of $(N-1)$ equations with $(N-1)$ unknowns u_1 to u_{N-1}. These $(N-1)$ equations together with the boundary conditions equation (2) are sufficient to solve the obtained tri-diagonal system with the help of an efficient solver called Thomas

Algorithm which is also known as 'Discrete Invariant Imbedding algorithm' [28,30,31].

Remark: When $f(0) = f(1)$ and $g(0) = g(1)$, both the fitting factors become equal and the constant fitting factor is

$$\sigma(\rho) = \frac{\rho f(0)}{2} \coth\left(\frac{\left(f^2(0) - \varepsilon g(0)\right)\rho}{2f(0)}\right) \quad (30)$$

5 Stability and Convergence Analysis:

In this section, we analyze stability and convergence of our proposed scheme (22) with (2). Similarly, we can analyze stability and convergence of our proposed scheme (29) with (2).

Theorem 5.1 The matrix A is strictly diagonally dominant if the condition $h < \frac{2\varepsilon\sigma}{f^* K}$, where $K = \left(1 - \frac{hg_*}{15f^*}\right)$ is satisfied.

proof : If $h < \frac{2\varepsilon\sigma}{f^* K}$, where $K = \left(1 - \frac{hg_*}{15f^*}\right)$ then

$$|P_l| = 1 - \frac{3hf_{l-1}}{60\varepsilon\sigma} + \frac{h^2 g_{l-1}}{30\varepsilon\sigma} - \frac{28hf_l}{60\varepsilon\sigma} + \frac{hf_{l+1}}{60\varepsilon\sigma}$$

$$|R_l| = 1 - \frac{hf_{l-1}}{60\varepsilon\sigma} + \frac{h^2 g_{l+1}}{30\varepsilon\sigma} + \frac{28hf_l}{60\varepsilon\sigma} + \frac{3hf_{l+1}}{60\varepsilon\sigma}$$

and

$$|P_l| + |R_l| = 2 - \frac{4hf_{l-1}}{60\varepsilon\sigma} + \frac{h^2 g_{l-1}}{30\varepsilon\sigma} + \frac{h^2 g_{l+1}}{30\varepsilon\sigma} + \frac{4hf_{l+1}}{60\varepsilon\sigma} < F_i = 2 - \frac{4hf_{l-1}}{60\varepsilon\sigma} - \frac{28h^2 g_l}{30\varepsilon\sigma} + \frac{4hf_{l+1}}{60\varepsilon\sigma};\ l = 2,3,\ldots,N-1$$

Also, $|R_1| < Q_{1,}\ |P_N| < Q_N$, which completes the proof.

Corollary 5.1 If the condition $f = 0$ holds, the matrix A becomes positive definite with no restriction on the mesh spacing h.

proof : Under the condition $f = 0$, the matrix A is strictly diagonally dominant with no restriction on the mesh spacing h. In addition, A is a symmetric matrix (as $g(x)$ is constant) and the diagonal entries are positive, so A is positive definite.

Definition 5.1. (Consistency): Let

$$\tau_{l,\pi}[w] \equiv L_h w(t_l) - Lw(t_l),\ l = 1,2,\ldots,N,$$

where w represents a smooth function on I. Then, the difference problem (22)-(2) bear consistency with the differential problem (1)-(2) if

$$\left|\tau_{l,\pi}[w]\right| \to 0 \ as\ h \to 0,$$

where the quantities $\tau_{l,\pi}[w]$, for $l = 1,2,\ldots,N$, are local truncation (or local discretization) errors.

Definition 5.2. The difference problem (22)-(2) has local $p^{th}-$ order accuracy if, for sufficiently smooth data, a positive constant C exists independent of h and ε such that

$$\max_{1\le l\le N} \left|\tau_{l,\pi}[w]\right| \le Ch^p.$$

The consistency of the difference problem (22)-(2) with (1)-(2) and its locally second-order accuracy is demonstrated by the following lemma.

Lemma 3.1. If $w \in C^4(I)$, then

$$\tau_{l,\pi}[w] = h^2\left[\frac{\varepsilon\sigma}{12} w^4(\nu_l) + \frac{2f(t_l)}{15} w^3(\theta_l)\right],$$

where $\nu_l$and θ_l lie in the interval $\left(t_{l-1,}\, t_{l+1}\right)$.

proof : By definition

$$\tau_l(h) = \sigma\varepsilon\left\{\frac{w_{l+1} - 2w_l + w_{l-1}}{h^2} - w''_l\right\} + \left\{\left(\frac{-3w_{l-1} + 4w_l - w_{l+1}}{2h}\right) - w'_{l-1}\right\}\frac{f_{l-1}}{30} + \frac{28f_l}{30}\left\{\left(\frac{w_{l+1} - w_{l-1}}{2h}\right) - w'_l\right\} + \frac{f_{l+1}}{30}\left\{\left(\frac{w_{l-1} - 4w_l + 3w_{l+1}}{2h}\right) - w'_{l+1}\right\},\ l = 1,2,\ldots,N \quad (31)$$

Using Taylor's theorem, it can be easily shown that

$$\left(\frac{-3w_{l-1} + 4w_l - w_{l+1}}{2h}\right) - w'_{l-1} = \frac{-h^2}{3} w^{(3)}(\theta_l),\quad \theta_l \in (t_{l-1}, t_{l+1}),$$

$$\left(\frac{w_{l+1} - w_{l-1}}{2h}\right) - w'_l = \frac{h^2}{6} w^{(3)}(\theta_l),\quad \theta_l \in (t_{l-1}, t_{l+1}) \quad (32)$$

$$\left(\frac{w_{l-1} - 4w_l + 3w_{l+1}}{2h}\right) - w'_{l+1} = \frac{-h^2}{3} w^{(3)}(\theta_l),\quad \theta_l \in (t_{l-1}, t_{l+1}).$$

$$\text{Also, } \frac{w_{l+1} - 2w_l + w_{l-1}}{h^2} - w''_l = \frac{h^2}{12} w^{(4)}(\nu_l),\quad \nu_l \in (t_{l-1}, t_{l+1}). \quad (33)$$

Substituting (32) and (33) in (31), we obtain our desired result.

Definition 5.3. (Stability): The linear difference operator L_h is stable if, for sufficiently small h, a constant K exists independent of satisfying the condition

$$|v_l| \le \kappa\left\{\max(|v_0|, |v_{N+1}|) + \max_{1\le l\le N} |L_h v_l|\right\},\ l = 0,1,\ldots,N+1,$$

for any mesh function $\{v_l\}_{l=0}^{N+1}$.

Now, we aim to prove that, for sufficiently small h, the difference operator L_h given in (22), is stable.

Theorem 3.2. The difference operator L_h given in (22), is stable if the functions f and g satisfy (22) and the condition $h < \frac{2\varepsilon\sigma}{f^* K}$ is satisfied, where $K = \left(1 - \frac{hg_*}{15f^*}\right)$ with $\kappa = \max\{1,\, 1/g_*\}$.

proof : If $|v_{l*}| = \max_{0\le l\le N+1} |v_l|,\ 1 \le l* \le N$, then, from (21), we obtain

$$Q_{l*} v_{l*} = P_{l*} v_{l*-1} + R_{l*} v_{l*+1} + hL_h v_{l*}.$$

Thus, $Q_{l*}|v_{l*}| \le (|P_{l*}| + |R_{l*}|)|v_{l*}| + h \max_{1\le l\le N} |L_h v_l|$.

If $h < \frac{2\varepsilon\sigma}{f^* K}$, where $K = \left(1 - \frac{hg_*}{15 f^*}\right)$, then

$$Q_{l*} = |P_{l*}| + |R_{l*}| + \left(\frac{28 g_{l*}}{30} - (g_{l*-1} + g_{l*+1})\right)\frac{h^2}{\varepsilon\sigma},$$

and it implies that

$$\frac{h^2}{\varepsilon\sigma}\left(\frac{28 g_{l*}}{30} - (g_{l*-1} + g_{l*+1})\right)|v_{l*}| \le h \max_{1\le l\le N} |L_h v_l|,$$

$$or,\ |v_{l*}| \le \frac{\varepsilon\sigma}{h\left(\frac{28 g_{l*}}{30} - (g_{l*-1} + g_{l*+1})\right)} \max_{1\le l\le N} |L_h v_l|.$$

Thus, if $\max_{0\le l\le N+1} |v_l|$ occurs for $1 \le l \le N$, then

$$\max_{0\le l\le N+1} |v_l| \le \frac{\varepsilon\sigma}{h\left(\frac{28 g_{l*}}{30} - (g_{l*-1} + g_{l*+1})\right)} \max_{1\le l\le N} |L_h v_l|,$$

and clearly

$$\max_{0\le l\le N+1} |v_l| \le \kappa\left\{\max(|v_0|, |v_{N+1}|) + \max_{1\le l\le N} |L_h v_l|\right\}$$
$$\text{with } \kappa = \max\{1, 1/g_*\}. \tag{34}$$

If $\max_{0\le l\le N+1} |v_l| \le \max\{|v_0|, |v_{N+1}|\}$, (34) follows instantly. An immediate consequence of stability is the uniqueness (and existence since the problem is linear) of the difference approximation $\{u_l\}_{l=0}^{N+1}$ (which was proved earlier). If two solutions exist, their difference $\{v_l\}_{l=0}^{N+1}$ will satisfy

$$L_h v_l = 0, 1 \le l \le N,$$
$$v_0 = v_{N+1} = 0.$$

As a result of stability, we get $v_l = 0, 0 \le l \le N+1$.

Definition 5.4. (Convergence): Let the solution of the boundary value problem (1)-(2) be represented by u and $\{u_l\}_{l=0}^{N+1}$ denotes the difference approximation defined in (22)-(2). The difference approximation is convergent to u if

$$\max_{1\le l\le N} |u_l - u(t_l)| \to 0 \text{ as } h \to 0.$$

The difference $u_l - u(t_l)$ denotes the global truncation (or discretization) error at the point t_l, $l = 1, 2, \ldots, N$.

Definition 5.5. The difference approximation denoted by $\{u_l\}_{l=0}^{N+1}$ is a $p^{th}-$ order approximation to the solution u of (1)-(2) if, for a sufficiently small h and ε, a constant C exists independent of h, and ε such that

$$\max_{0\le l\le N+1} \{|u_l - u(t_l)|\} \le Ch^p.$$

Theorem 5.3. Suppose $u \in C^4(I)$ and $h < \frac{2\varepsilon\sigma}{f^* K}$, where $K = \left(1 - \frac{hg_*}{15 f^*}\right)$. Then, the difference solution $\{u_l\}_{l=0}^{N+1}$ of (22)-(2) converges to u, the solution of (1)-(2). Moreover,

$$\max_{0\le l\le N+1} \{|u_l - u(t_l)|\} \le Ch.$$

It implies that the convergence order of difference scheme (22) is $O(h)$.

proof: Under the given assumptions, the difference problem (22)-(2) bears consistency with the boundary value problem (1)-(2) and the operator L_h is stable.

$$\text{Since } L_h[u_l - u(t_l)] = r(t_l) - L_h u(t_l) = Lu(t_l) - L_h u(t_l) = -\tau_{l,\pi}[u],$$

and $u_0 - u(t_0) = u_{N+1} - u(t_{N+1}) = 0$, it follows from the stability of L_h that

$$|u_l - u(t_l)| \le \frac{\varepsilon\sigma}{h\left(\frac{28 g_*}{30} - (g_{l*+1} + g_{l*-1})\right)} \max_{1\le l\le N} |\tau_{l,\pi}[u]|.$$

The desired result follows from Lemma (5.1). This theorem establishes that $\{u_l\}_{l=0}^{N+1}$ is a first-order approximation to the solution u of (1).

6 Numerical Experiments :

The effectiveness of the present method has been demonstrated by implementing it on the three test problems. These problems have been chosen becouse of their wide discussion in pieces of literature and the availability of their approximate solutions for comparison. The computational results have been presented in Tables 1 to 6 and are compared with the existing results. Comparisons of the solutions with some existing results are presented in Tables 1,3 and 5 for the considered example problems: 1 to 3. These comparisons show that the capability of the proposed scheme in achieving slightly improved results from the results of the papers [29,30,31,32].

For different values of grid point N and perturbation parameter ε, the maximum absolute errors (MAE) E_ε^N are defined by $E_\varepsilon^N = \max_{0\le l\le N} [|u(t_l) - u_l|]$, where $u(t_l)$ and u_l denote the exact and approximate solution respectively. The double mesh principle[8] is used to calculate the rate of convergence defined as $r_\varepsilon^N = \log_2\left(E_\varepsilon^N / E_\varepsilon^{2N}\right)$. The addressed method is capable of achieving uniform results, when perturbation parameter $\varepsilon \to 0$ for any fixed value of the mesh size h.

6.1 Numerical Example Problems with left-end boundary layer:

The applicability of the proposed method for left-end boundary layer problems is demonstrated computationally by considering one linear and one non-linear model test problems given below.

Example 1 *Consider the following variable coefficient homogeneous singular perturbation problem from Kevorkian and Cole[10]:*

$$\varepsilon u''(t) + \left(1 - \frac{t}{2}\right) u'(t) - \frac{1}{2}u(t) = 0; t \in [0,1]$$

with boundary conditions $u(0) = 0$ and $u(1) = 1$.
The uniformly valid approximation to the exact solution of this example as given by Nayfeh [11] is

$$u(t) = \frac{1}{2-t} - \frac{1}{2}e^{-(t-t^2/4)/\varepsilon}$$

where the boundary layer is present at the left side of the domain near $t = 0$.

Table 1 compares the results (Maximum absolute errors) with the existing results [32] for example problem-1, for various values of ε and grid point N. It is clear that the presented scheme is able to produce slightly improved results from the results in [32]. The computational results(MAE) and rates of convergence presented in Table 2 for example problem-1 show that the present scheme is capable of producing almost first order accurate uniformly convergent solution.

Table 1: Comparison of computational results(MAE) with existing results [with fitting factor(w.f.f.) and without fitting factor(w.o.f.f.)] for various values of ε and N for example problem-1.

	$\varepsilon = 10^{-3}$				$\varepsilon = 10^{-5}$			
	Soujanya[27]		Our Results		Soujanya[27]		Our Results	
N ↓	w.f.f.	w.o.f.f.	w.f.f.	w.o.f.f	w.f.f.	w.o.f.f	w.f.f.	w.o.f.f
8	4.48e-02	1.05	3.71E-02	7.76E-01	4.48e-02	1.88	3.71E-02	0.99
16	2.44e-02	6.01e-01	2.10E-02	5.47E-01	2.44e-02	1.89	2.10E-02	0.98
32	1.28e-02	4.39e-01	1.12E-02	4.38E-01	1.28e-02	1.77	1.12E-02	0.95
64	6.62e-03	3.84e-01	5.79E-03	3.86E-01	6.62e-03	1.32	5.79E-03	0.85
128	3.77e-03	2.94e-01	2.95E-03	2.96E-01	3.36e-03	0.75	2.95E-03	0.65

Example 2 *Consider the following non-linear singular perturbation problem from Bender and Orszag [13,p. 463, Eq.(9.7.1)]:*

$$\varepsilon u''(t) + u'(t) + e^{(u(t))} = 0; t \in [0, 1]$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$.
The linear problem concerned to this example is:

$$\varepsilon u''(t) + 2u'(t) + \frac{2}{t+1}u(t) = \frac{2}{t+1}\left[\ln\left(\frac{2}{t+1}\right) - 1\right]; t \in [0, 1]$$

The uniform valid approximation of Bender and Orszag [13, P. 463, Eq. (9.7.6)] is

$$u(t) = \ln\left(\frac{2}{t+1}\right) - \ln(2)e^{-2t/\varepsilon},$$

which possesses a boundary layer of thickness $o(\varepsilon)$ near $t = 0$ (cf. Bender and Orszag [3]).

Table 2: Computational results in terms of Maximum absolute errors for different values of N and ε and the Rate of Convergence r_ε^N for example problem- 1.

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-3}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.55E-03	9.83E-04
r_ε^N	0.819657	0.910842	0.950282	0.970874	0.928494	0.658229	
10^{-4}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-6}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-8}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-10}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-12}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-15}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-18}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-20}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-25}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	
10^{-30}	3.71E-02	2.10E-02	1.12E-02	5.79E-03	2.95E-03	1.49E-03	7.39E-04
r_ε^N	0.819657	0.910842	0.950282	0.971363	0.984924	1.013220	

Table 3 compares the results (Maximum absolute errors) with the existing results for example problem-2, for various values of ε and grid point N. It is clear that the presented scheme is able to produce slightly improved results from the results in [30,31]. The computational results(MAE) and rates of convergence presented in Table 4 for example problem-2 again show that the present scheme is capable of producing almost first order accurate uniformly convergent solution.

Table 3: Comparison of computational results(MAE) with existing results for various values of ε and N for example problem-2.

	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
N ↓	Mohapatra [26]	Mohapatra [25]	Our Results	Mohapatra [26]	Mohapatra [25]	Our Results
16	1.962e-2	1.9628e-2	1.817E-02	1.962e-2	1.9623e-2	1.817E-02
32	1.031e-2	1.0315e-2	9.588E-03	1.031e-2	1.0311e-2	9.588E-03
64	5.284e-3	5.2847e-3	5.094E-03	5.284e-3	5.2842e-3	5.094E-03
128	2.675e-3	2.6759e-3	2.626E-03	2.675e-3	2.6755e-3	2.626E-03
256	1.344e-3	1.3444e-3	1.334E-03	1.344e-3	1.3440e-3	1.334E-03
512	6.754e-4	6.7549e-4	6.728E-04	6.754e-4	6.7547e-4	6.728E-04

6.2 Numerical Example Problems with right-end boundary layer:

To demonstrate the applicability of proposed method computationally for right-end boundary layer problem, we have considered the following one linear model test problems:

Example 3 *Consider the following homogeneous linear singular perturbation problem from Mohapatra et. al. [24,25] and Soujanya et. al. [32]:*

$$\varepsilon u''(t) - u'(t) - (1+\varepsilon)u(t) = 0; t \in [0, 1]$$

Table 4: Computational results in terms of Maximum absolute errors for different values of N and ε and the Rate of Convergence r_ε^N for example problem- 2.

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-3}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.34E-03	6.96E-04
r_ε^N	0.839769	0.922257	0.912431	0.955383	0.976577	0.938852	
10^{-4}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-6}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-8}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-10}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-12}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-15}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-18}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-20}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-25}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	
10^{-30}	3.25E-02	1.82E-02	9.59E-03	5.09E-03	2.63E-03	1.33E-03	6.73E-04
r_ε^N	0.839769	0.922257	0.912431	0.955932	0.977108	0.987509	

with boundary conditions
$u(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon)$ and $u(1) = 1 + 1/e$.
The exact solution is given by:

$$u(t) = \exp(-t) + \exp\left[(1+\varepsilon)(t-1)/\varepsilon\right],$$

which has a boundary layer at the right side of the domain near $t = 1$.

Table 5 compares the results (Maximum absolute errors) with the existing results for example problem-3, for various values of ε and grid point N. It is clear that the presented scheme is able to produce slightly improved results from the results in the article [29,30]. As like the results for the left-end boundary layer problems-1 and 2, the computational results(MAE) and rates of convergence presented in Table 6 for this right-end boundary layer problem-3 clearly show that the present scheme is capable of producing almost first order accurate uniformly convergent solution.

Table 5: Comparison of computational results(MAE) with existing results for various values of ε and N for example problem-3.

	$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
N ↓	Mohapatra [24]	Mohapatra [25]	Our Results	Mohapatra [24]	Mohapatra [25]	Our Results
16	1.1143e-2	1.1143e-2	1.102E-02	1.1141e-2	1.1141e-2	1.102E-02
32	5.6345e-3	5.6345e-2	5.614E-03	5.6343e-3	5.6343e-2	5.614E-03
64	2.8197e-3	2.8197e-3	2.815E-03	2.8192e-3	2.8192e-3	2.815E-03
128	1.3958e-3	1.3958e-3	1.394E-03	1.3955e-3	1.3955e-3	1.394E-03
256	6.8346e-4	6.8346e-4	6.738E-04	6.8342e-4	6.8342e-4	6.738E-04
512	3.2758e-4	3.2758e-4	3.326E-04	3.2754e-4	3.2754e-4	3.326E-04

7 Conclusion :

We have derived an exponentially fitted tridiagonal scheme for solving singularly perturbed two-point boundary value problems with boundary layer at one end points(left or right). We have carried out stability and convergence analysis for the proposed scheme and performed the numerical experiments on two linear and one nonlinear model example problems for different values of $N = 1/h$ and perturbation parameter ε which show that the scheme is of almost first order accurate. The computational results are presented in tables and compared with the existing results. Comparisons show that the proposed scheme is comparable with the schemes presented in the articles [29,30,31,32]. Furthermore, one can easily observe from the Tables 2,4 and 6 that the presented fitted scheme is capable of producing first order accurate uniformly convergent solution for any fixed value of step size $h = 1/N > \varepsilon$, when perturbation parameter $\varepsilon \to 0$. The main feature of the proposed fitted scheme is that it neither depends on the very fine mess size [37] nor on deviating argument [28]. Finally, it is concluded that the present method appears to be one of the best alternatives for solving singularly perturbed boundary value problems numerically with a small amount of computational time.

Table 6: Computational results in terms of Maximum absolute errors for different values of N and ε and the Rate of Convergence r_ε^N for example problem- 3.

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^{-3}	2.04E-02	1.07E-02	5.29E-03	2.49E-03	1.07E-03	3.78E-04	2.24E-04
r_ε^N	0.933784	1.016544	1.088589	1.223553	1.493251	0.754745	
10^{-4}	2.07E-02	1.10E-02	5.61E-03	2.82E-03	1.39E-03	6.74E-04	3.33E-04
r_ε^N	0.911596	0.973023	0.995894	1.048838	1.018532	1.012934	
10^{-6}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-8}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-10}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-12}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-15}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-18}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-20}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-25}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	
10^{-30}	2.08E-02	1.11E-02	5.65E-03	2.85E-03	1.43E-03	7.20E-04	3.56E-04
r_ε^N	0.910455	0.967724	0.986783	0.994444	0.991155	1.017136	

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References

[1] E. J. Hinch, *Perturbation Methods,* Cambridge University Press, Cambridge, 1991.
[2] M. H. Holmes, *Introduction to Perturbation Methods,* Springer, Berlin 1995.

[3] J. Vigo-Aguiar and S. Natesan, An efficient numerical method for singular perturbation problems, J. Comput. Appl. Math., **192(1)**, 132—141, 2006.

[4] S. Owyed, M. Abdou, A.-H. Abdel-Aty, W. Alharbi and R. Nekhili, Numerical and approximate solutions for coupled time fractional nonlinear evolutions equations via reduced differential transform method, Chaos, Solitons & Fractals, **131** 109474, 2020.

[5] M. M. Khater, R. A. Attia, A.-H. Abdel-Aty, M. Abdou, H. Eleuch and D. Lu, Analytical and semi-analytical ample solutions of the higher-order nonlinear Schrödinger equation with the non-Kerr nonlinear term, Results in Physics, **16** 103000, 2020.

[6] M. M. A. Khater, R. A. M. Attia, A.-H. Abdel-Aty, S. Abdel-Khalek, Y. Al-Hadeethi and D. Lu, On the computational and numerical solutions of the transmission of nerve impulses of an excitable system (the neuron system), Journal of Intelligent & Fuzzy Systems, **38** 2603–2610, 2020.

[7] J. J. H. Miller, R. E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems, Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions,* World Scientific, River Edge NJ, 1996.

[8] H. G. Roos, M.Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations,* Springer, Berlin, 1996.

[9] R. R. Gold, Magnetohydrodynamic pipe flow Part I, J. Fluid Mech., **13(4)**, 505–512, 1962.

[10] R. Bellman, *Perturbation Techniques in Mathematics,* Physics and Engineering, New York Winston, 1964.

[11] E. P. Doolan, J. J. H. Miller and W. H. A. Schilders, *Uniform Numerical methods for problems with initial and boundary layers,* Boole, Dublin Ireland, 1980.

[12] D. Kumar, A parameter-uniform method for singularly perturbed turning point problems exhibiting interior or twin boundary layers, *International Journal of Computer Mathematics* **96** (2019) 865-882.

[13] A. Das and S. Natesan, Parameter-uniform numerical method for singularly perturbed 2D delay parabolic convection–diffusion problems on Shishkin mesh. *J. Appl. Math. Comput.* **59** (2019) 207–225.

[14] F. Verhulst, *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Timescale Dynamics,* Springer, Berlin, 2005.

[15] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics,* Springer-Verlag, New York, 1981.

[16] A. H. Nayfeh, *Perturbation Methods,* Wiley, New York, 1979.

[17] R. E. O'Malley, *Introduction to Singular Perturbations,* Academic Press, New York, 1974.

[18] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* McGraw-Hill, New York, 1978.

[19] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Robust Computational Techniques for Boundary Layers,* Chapman & Hall/CRC Press, Boca Raton FL, 2000.

[20] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems (revised edition),* World Scientific, Singapore, 2012.

[21] J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods,* Springer-Verlag, New York 1996.

[22] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations,* Springer-Verlag, New York Berlin Heidelberg Tokyo, 1967.

[23] G. I. Shishkin and L. P. Shishkina, *Difference Methods For Singular Perturbation Problems,* CRC Press, Boca Raton, 2009.

[24] M. K. Kadalbajoo and V. Gupta, A brief survey on numerical methods for solving singularly perturbed problems, Appl. Math. Comput., **217(8)**, 3641–3716, 2010.

[25] M. K. Kadalbajoo and K. C. Patidar, A survey of numerical techniques for solving singularly perturbed ordinary differential equations, Appl. Math. Comput., **130(2)**, 457-510, 2002.

[26] M. K. Kadalbajoo and D. Kumar, Initial value technique for singularly perturbed two point boundary value problems using an exponentially fitted finite difference scheme, Comput. Math. Appl., **57(7)**, 1147—1156, 2009.

[27] M. K. Kadalbajoo and Y. N. Reddy, Asymptotic and Numerical analysis of Singular Perturbation Problems: A Survey, Appl. Math. Comput., **30(3)**, 223–259, 1989.

[28] Y. N. Reddy and K. A. Reddy, Numerical integration method for general singularly perturbed two point boundary value problems, Appl. Math. Comput., **133(2-3)**, 351—373, 2002.

[29] J. Mohapatra and M. K. Mahalik, An efficient numerical method for singularly perturbed second order ordinary differential equation, J. Math. Model., **3(1)**, 33–48, 2015.

[30] J. Mohapatra and N. R. Reddy, Exponentially Fitted Finite Difference Scheme For Singularly Perturbed Two Point Boundary Value Problems, Int. J. Appl. Comput. Math.**1(2)**, 267–278, 2014.

[31] N. R. Reddy and J. Mohapatra, An Efficient Numerical Method for Singularly Perturbed Two Point Boundary Value Problems Exhibiting Boundary Layers, Natl. Acad. Sci. Lett., **38**(4), 355—359, 2015.

[32] GBSL Soujanya, K. Phaneendra and Y. N. Reddy, An Exponentially Fitted Non Symmetric Finite Difference Method for Singular Perturbation Problems, Wseas Transactions on Mathematics, **12(7)**, 767–776, 2013.

[33] U. Ascher and R. Weis, Collocation for singular perturbation problem III: Non-linear problems without turning points, SIAM J. Sci. Stat. Comput. , **5(4)**, 811-829, 1984.

[34] P. P. Chakravarthy and Y. N. Reddy, Exponentially fitted modified upwind scheme for singular perturbation problems, Int. J. Fluid Mech. Res., **33(2)**, 119—136, 2006.

[35] J. Jayakumar and N. Ramanujam, A numerical method for singular perturbation problems arising in chemical reactor theory, Comput. Math. Appl., **27(5)**, 83—99, 1994.

[36] H. M. Habib and E. R. El-Zahar, An algorithm for solving singular perturbation problems with mechanization, Appl. Math. Comput., **188(1)**, 286—302, 2007.

[37] M. K. Kadalbajoo and D. Kumar, Variable mesh finite difference method for self adjoint singularly perturbed two point boundary value problems, J. Comput. Math., **28(5)**, 711–724, 2010.