Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100414

On a Practical Methodology for Solving BVP Problems by Using a Modified Version of Picard Method

U. Filobello-Nino[1], *H. Vazquez-Leal*[1*], *A. Perez-Sesma*[1], *J. Cervantes-Perez*[1], *L. Hernandez-Martinez*[2], *A. Herrera-May*[3], *V. M. Jimenez-Fernandez*[1], *A. Marin-Hernandez*[4], *C. Hoyos-Reyes*[1], *A. Diaz-Sanchez*[2] *and J. Huerta Chua*[5]

[1] Electronic Instrumentation and Atmospheric Sciences School, Universidad Veracruzana, Cto. Gonzalo Aguirre Beltrán S/N, 91000, Xalapa, Veracruz, México.
[2] National Institute for Astrophysics, Optics and Electronics, Luis Enrique Erro #1, Sta. María Tonantzintla 72840, Puebla, México.
[3] Micro and Nanotechnology Research Center, University of Veracruz, Calzada, Ruiz Cortines 455, Boca del Rio 94292, Veracruz, México.
[4] Department of Artificial Intelligence, Universidad Veracruzana Sebastián Camacho 5 Centro, 91000, Xalapa, Veracruz, México.
[5] Civil Engineering School, Universidad Veracruzana, Venustiano Carranza S/N, Col. Revolucion, 93390, Poza Rica, Veracruz, México.

Received: 27 Mar. 2016, Revised: 13 May 2016, Accepted: 14 May 2016
Published online: 1 Jul. 2016

Abstract: The aim of this paper is to propose a modified version of Picard Method, the Boundary Value Problems Picard Method (BVPP), which allows the solution of BVP problems, with few iterations. What is more, as case study, BVPP is employed to get approximate solutions to four differential equations; two linear and two nonlinear. Comparing figures between approximate and exact solutions, it is shown that BVPP method can generate handy approximate solutions with the desired degree of accuracy.

Keywords: Linear Differential Equation, Nonlinear Differential Equation, Picard Method, Approximate Solutions, Boundary Value Problems.

1 Introduction

Solving nonlinear differential equations is relevant because phenomena on the frontiers of modern sciences are often nonlinear in nature. On the engineering and science fields, physical phenomena are frequently modeled using nonlinear differential equations. Scientists who work in such disciplines constantly face the problems of solving linear and nonlinear ordinary differential equations, partial differential equations, and systems of nonlinear ordinary differential equations. Recently a wide variety of methods focused to find approximate solutions to nonlinear differential equations, as an alternative to classical methods, have been reported. Such as those based on: variational approaches [1,2,3,4], tanh method [5], exp-function [6,7], Adomian's decomposition method [8,9,10,11,12,13], parameter expansion [14], homotopy perturbation method [15,16,13,17,18,19,20,21,22,23,24,25,26,27,4,28,29,30,31,32,33,34,35,36,37,38,39,40,41], homotopy analysis method [42,43,44,45,46], homotopy asymptotic method [47], perturbation method [48,49], modified Taylor method [50], generalized homotopy method [51], differential transformation method [52], among many others. Also, a few exact solutions to nonlinear differential equations have been reported occasionally [53].

As it is well known, boundary value problems of ordinary differential equations have many applications in sciences. The case of BVP for nonlinear ODES includes, Michaelis Menten equation [54], that describes the kinetics of enzyme-catalyzed reactions, Gelfand's differential equation [35,49] which governing combustible gas dynamics (see below, this study proposes an approximate solution for this equation), Troesch's equation [55,56,57,58,59,60], arising in the investigation of confinement of a plasma column by a radiation pressure, among many others. On the other hand, the theory of BVP for linear ODES, is a well established branch of mathematics, with many applications. Between problems of interest, related to these equations, are found: The one-dimensional quantum problem, of a particle of

* Corresponding author e-mail: hvazquez@uv.mx

mass m confined in a region of zero potential by an infinite potential at two points x=a and x=b [61], Heat transfer equation [61], Wave equation which describes for instance, transverse vibrations of a uniform stretched string between two fixed points, let us say x = a and x = b [62], The Laplace equation, which governs the temperature field corresponding to the steady state in a plate [62], and so on. Generally, many problems expressed in terms of partial differential equations, give rise through method of separation of variables, to BVP for linear ODES [61,62].

The Picard Iteration Method (PIM) [63,64,65] is a well established iterative method; although it has been employed above all as a formal procedure for establishing the existence and uniqueness theorems of differential equations, their usefulness in practice is relatively small. This is mainly due to the convergence of the method is slow, and also because the integration process involved, rapidly becomes very long and tedious. Nevertheless, the technique has several significant advantages. Unlike other known methods, Picard's method applies to linear and nonlinear problems, with identical ease. Also, based on well-established criteria and theorems, PIM allows to predict from the beginning, if the iterative process involved will converge to the solution of the problem, even if such solution is unique, which many methods for nonlinear differential equations cannot guarantee.

Our main goal in this study is take advantage of the fortress of the method, and try to solve its drawbacks, with the end to employ it, as a useful tool to obtain approximate solutions of BVP problems for linear and nonlinear ordinary second order differential equations. This paper is organized as follows. In Section 2, a brief review of the basic idea for Picard iteration method is provided. In Section 3, we will present, the Boundary Value Problems Picard Method (BVPP) as a modified version of Picard method. Additionally, Section 4 presents four cases study, including a comparison of BVPP with other methods to show its precision and versatility. Besides a discussion on the results is presented in Section 5. Finally, a brief conclusion is given in Section .6

2 Picard Iteration Method.

We begin reformulating the initial value problem

$$y''(t) = f(t,y(t),y'(t)); \qquad y(t_0) = A, \qquad y'(t_0) = B \tag{1}$$

as the following equivalent integral equation

$$y(t) = A + Bt + \int_{t_0}^{t}\int_{t_0}^{t} f\left(t',y(t'),y'(t')\right) dt'dt, \tag{2}$$

The solution for (2) can be expressed as the limit of a sequence of functions $y_n(t)$, in the limit $n \to \infty$, in accordance with the recurrence formula

$$y_n(t) = A + Bt + \int_{t_0}^{t}\int_{t_0}^{t} f\left(t',y_{n-1}(t'),y'_{n-1}(t')\right) dt'dt, \quad n = 1,2,3.. \tag{3}$$

When the right hand side of (1), $f(t,y(t),y'(t))$ is a continuous function for all its arguments, and having continuous first partial derivatives with respect to y and y' in a neighborhood of the initial conditions of (1) then, it is well known that regardless of the choice of the initial function $y_0(t)$, the sequence $\{y_n(t)\}$, generated by the iterative process given by (3), converges to a solution of the problem (1) [62,66,67,68].

In the same way, assuming that $f(t,y(t),y'(t))$ satisfies the Lipschitz condition, it would be possible to establish, a more strong but usually difficult to apply criterion. For the purposes of this study, it is sufficient to ensure that starting of the iterative process (3), we will get a solution for (2) [66,67].

3 Basic Idea of Boundary Value Problems Picard Method (BVPP).

Next, we will find highly accurate approximate solutions for boundary value problems, following a method, which incorporates the boundary values of the original problem, to the classical version of Picard method with initial conditions.

An important case of BVP problems, is one where the values of the sought solution are given at two points t_0 and t_1, but not the derivatives (Dirichlet boundary conditions), i.e.

$$y''(t) = f(t,y(t),y'(t)); \qquad y(t_0) = A, \qquad y(t_1) = C, \tag{4}$$

therefore the value for the derivative at t_0, will be denoted by $y'(t_0) = \beta$. We will approach our BVP problem, assuming for the time being, that the value of β is known (although it is initially unknown) and right hand side of (4) is a continuous function. Besides, we will use the freedom to choose the trial function $y_0(t)$, in order to include both boundary values and accelerate the convergence of the method. Thus, we exploit the virtues of PIM, remedying its defects.

It should be noted that, the unknown value of $y'(t_0)$, will be determined, as part of the proposed procedure.

The method begins proposing as trial function a polynomial function $P(t)$ which contains one or more parameters $D,E,F,..$ to be determined.

$$y_0(t) = P(t,D,E,F,..). \tag{5}$$

According to the above, BVPP method employ instead of (2), the following integral equation

$$y(t) = A + \beta t + \int_{t_0}^{t}\int_{t_0}^{t} f\left(t', y(t'), y'(t')\right) dt' dt, \qquad (6)$$

where, the value of β is unknown for the time being .

The solution for (6) can be expressed as the limit of a sequence of functions $y_n(t)$, in the limit $n \to \infty$, in accordance with the recurrence formula

$$y_n(t,\beta,D,E,F,..) = A + \beta t + \int_{t_0}^{t}\int_{t_0}^{t} f\left(t', y_{n-1}(t',D,E,F,..), y'_{n-1}(t',D,E,F,..)\right) dt' dt,$$
$$n = 1,2,3.. \qquad (7)$$

Since it has been assumed the continuity of $f(t', y(t'), y'(t'))$ then, irrespective of (5), the successive approximations $\{y_n(t)\}$ (7), converge to the solution of the following problem, resembling (4) (see section 2).

$$y''(t) = f\left(t, y(t), y'(t)\right); \qquad y(t_0) = A, \qquad y'(t_0) = \beta. \qquad (8)$$

Next, in order to ensure that the n-th iteration of BVPP (7), is also an approximate solution for (4), the values of $\beta, D, E, F, ..$, are chosen to ensure that approximate solutions satisfy $y(t_1) = C$ also, and therefore (4). It will be seen that although (4) and (8) are related in this way in order to motivate BVPP convergence, in practice is not required to consider explicitly the auxiliary problem (8). Finally we indicate three ways to calculate optimally the values of the above parameters, in order to accelerate the convergence and obtain highly accurate approximate solutions.

Method 1.

Assuming that the nth approximation is sufficient, then from (7) we can write symbolically

$$y_n = H(t,\beta,D,E,F,..), \qquad (9)$$

where $H(t,\beta,D,E,F,..)$, is a certain function obtained from the iterative process above mentioned.

This method assume known the numerical solution of (4), so that (9) is evaluated as many points within the interval $[t_0,t_1]$ as parameters to be determined, that is to say

$$\begin{aligned} y_n(t_0) &= H(t_0,\beta,D,E,F,..),\\ y_n(t_1) &= H(t_1,\beta,D,E,F,..),\\ y_n(t_2) &= H(t_2,\beta,D,E,F,..),\\ y_n(t_3) &= H(t_3,\beta,D,E,F,..),\\ y_n(t_4) &= H(t_4,\beta,D,E,F,..), \end{aligned} \qquad (10)$$

.... and so on,

where $t_2, t_3, t_4 \in (t_0, t_1)$ and the values of $y_n(t_0), y_n(t_1), y_n(t_2), y_n(t_3), y_n(t_4), ..$, are known.

(10) is a system of algebraic equations, whose solution allows to determine the value of the parameters $\beta, D, E, F, ..$

It $'$ is expected that it result in a good fit, considering several inner points, even the first iteration of BVPP can be highly accurate and sufficient (see cases study).

Method 2.

This proposal follows again the steps that led to (9), but unlike the previous method, we will use a software like the Nonlinear Fit built-in command from Maple 15, to identify optimally the parameters $\beta, D, E, F, ..$

Method 3.

This method aims to determine the above parameters, by using the least squares method.

To get that (9) corresponds to a good approximate analytic solution of (4), we have to optimize the values of $\beta, D, E, F, ..$ For it, after substituting (9) into (4), results the following residual.

$$\begin{aligned} R(t,\beta,D,E,F,..) &= y''(t,\beta,D,E,F)\\ &- f\left(t, y(t,\beta,D,E,F), y'(t,\beta,D,E,F)\right), \end{aligned} \qquad (11)$$

Next, it is applied the least square method, minimizing the square residual error [47]

$$I(\beta,D,E,F,..) = \int_{t_0}^{t_1} R^2(t,\beta,D,E,F,..)dt, \qquad (12)$$

identifying the values of $\beta, D, E, F, ..$ from the conditions

$$\frac{\partial I}{\partial \beta} = 0, \quad \frac{\partial I}{\partial D} = 0, \quad \frac{\partial I}{\partial E} = 0, \quad \frac{\partial I}{\partial F} = 0, .. \qquad (13)$$

Note that unlike the previous methods, the procedure outlined by equations (11)-(13) does not require in advance the knowledge of the numerical solution.

Once BVPP method optimize the values of $\beta, D, E, F, ..$ we will obtain good approximations requiring only a few iterations. It is expected that by choosing other values for these parameters, it would generate an iterative process slower and cumbersome.

4 Cases Study.

In what follows, we will assume equations satisfying the continuity requirements mentioned.

Example 1.

We will employ method 1, in order to find an approximate Solution of Gelfand's Equation.

As it is known, Gelfand's equation [35,49] models the chaotic dynamics in combustible gas thermal ignition. Therefore it is important to search for accurate solutions for this equation.

The equation to solve is

$$\frac{d^2y(x)}{dx^2}+\varepsilon e^{y(x)}=0,$$
$$0\le x\le 1,\qquad y(0)=0,\qquad y(1)=0,\quad (14)$$

where ε is a positive parameter.

It is possible to find a handy approximate solution for (14) by applying the BVPP method.

Thus, first we expand the exponential term of Gelfand's problem, resulting

$$y''+\varepsilon\left(1+y+\frac{1}{2}y^2+\frac{1}{6}y^3+..\right)=0,$$
$$0\le x\le 1,\qquad y(0)=0,\qquad y(1)=0,\quad (15)$$

Equation (6) for this case is given by

$$y(x)=\beta x-\varepsilon\int_0^x\int_0^x\left(1+y+\frac{1}{2}y^2\right)dx'dx,\qquad (16)$$

after keeping the second power of y variable.

We choose as trial function $y_0(x)$

$$y_0(x)=Bx+Cx^2+Dx^3.\qquad (17)$$

The corresponding recurrence formula is given by

$$\begin{aligned}y_n(x,\beta,B,C,D)&=\beta x\\&-\varepsilon\int_0^x\int_0^x\left(1+y_{n-1}(x,\beta,B,C,D)+\right.\\&\left.+\frac{1}{2}y_{n-1}^2(x,\beta,B,C,D)\right)dx'dx.\\&n=1,2,3..\qquad (18)\end{aligned}$$

The first iteration of BVPP (n=1),

$$\begin{aligned}y_1(x,\beta,B,C,D)&=\beta x\\&-\varepsilon\int_0^x\int_0^x\left(1+y_0(x,\beta,B,C,D)+\right.\\&\left.+\frac{1}{2}y_0^2(x,\beta,B,C,D)\right)dx'dx.\qquad (19)\end{aligned}$$

Assuming that first iteration is sufficient, after substituting (17) into (19), we obtain

$$\begin{aligned}y_1(x,\beta,B,C,D)&=\beta x-\tfrac{\varepsilon}{2}x^2-\tfrac{\varepsilon B}{6}x^3\\&+\left[-\tfrac{\varepsilon C}{12}+\tfrac{B}{24}\right]x^4+\left[\tfrac{D}{20}+\tfrac{BC}{20}\right]x^5\qquad (20)\\&+\left[\tfrac{C^2}{60}+\tfrac{BD}{30}\right]x^6+\tfrac{CD}{42}x^7+\tfrac{D^2}{112}x^8.\end{aligned}$$

In accordance with first method, we generate four algebraic equations, by substituting the boundary value of $x=1$, also the values: $x=0.1$,$x=0.5$, and $x=0.7$, which belong to $[0,1]$, in order to calculate β,B,C,D. Note that in this case, the condition $y_1(0,\beta,B,C,D)=0$, is automatically satisfied.

In order to test the effectiveness of the method, we will consider the values of $\varepsilon=3$ and $\varepsilon=3.5$, despite of the solutions corresponding to values of the parameter $\varepsilon\ge 1$, are considered the most difficult to model [47,69,70].

The solution for the described system of algebraic equations for $\varepsilon=3$, results in

$$\beta=2.321014274,\qquad B=2.725506295,$$
$$C=-0.9993962770,\qquad D=2.181675403.$$

By substituting these values into (20) is obtained the following approximate solution for (14).

$$\begin{aligned}y_1(x)&=2.321014274x-1.5x^2-1.362753148x^3\\&+0.3634118315x^4-0.027109272x^5\\&+0.2148522168x^6-0.05191329226x^7\\&+0.04249738896x^8.\qquad (21)\end{aligned}$$

In the same way, the parameters for $\varepsilon=3.5$, are given by

$$\beta=3.707678793,\qquad B=4.533738715,$$
$$C=2.030713696,\qquad D=2.093200793,$$

and the corresponding approximate solution for (14).

$$\begin{aligned}y_1(x)&=3.707678793x-1.75x^2-2.644680917x^3\\&-0.4033857148x^4+0.5649963048x^5\\&+0.385064151x^6+0.1012069409x^7\\&+0.03912044250x^8.\qquad (22)\end{aligned}$$

Example 2.

This example shows the use of method 2. As a case study we propose the following nonlinear differential equation [61].

$$\frac{d^2y(x)}{dx^2}-\varepsilon y^2(x)=0,$$
$$0\le x\le 1,\qquad y(0)=0,\qquad y(1)=1,\quad (23)$$

where ε, is a positive parameter.

Equation (6) for this case is given by

$$y(x)=\beta x+\varepsilon\int_0^x\int_0^x y^2dx'dx,\qquad (24)$$

next, we select as trial function $y_0(x)$

$$y_0(x) = Bx + Cx^2. \tag{25}$$

The recurrence formula for this case is given by

$$y_n(x,\beta,B,C) = \beta x + \varepsilon \int_0^x \int_0^x \left(y_{n-1}^2(x,\beta,B,C)\right) dx' dx. \qquad n = 1,2,3.. \tag{26}$$

The first iteration of BVPP results in

$$y_1(x,\beta,B,C) = \beta x + \varepsilon \int_0^x \int_0^x \left(y_0^2(x,\beta,B,C)\right) dx' dx. \tag{27}$$

By substituting (25) into (27), we obtain

$$y_1(x,\beta,B,C) = \beta x + \varepsilon \left[\frac{B^2x^2}{2} + \frac{BCx^3}{3} + \frac{C^2x^4}{12}\right], \tag{28}$$

from the condition $y_1(1) = 1$, we obtain the value

$$\beta = 1 - \varepsilon \left[\frac{B^2}{2} + \frac{BC}{3} + \frac{C^2}{12}\right], \tag{29}$$

therefore

$$y_1(x,B,C) = x + \frac{\varepsilon B^2}{2}\left[x^2 - x\right] + \frac{\varepsilon BC}{3}\left[x^3 - x\right] + \frac{\varepsilon C^2}{12}\left[x^4 - x\right]. \tag{30}$$

In order to show the effectiveness of the method, we consider as case study, a large value of the parameter $\varepsilon = 30$. As it is well known, the solutions corresponding to values of the parameter $\varepsilon \geq 1$, are considered the most difficult to model, for equations depending on a parameter as (23) (for instance, classical perturbation method (CPM), provides in general, better results for small perturbation parameters $\varepsilon << 1$, see below) [47,69,70].

Thus, (30) adopts the form

$$y_1(x,B,C) = x + 15B^2\left[x^2 - x\right] + 10BC\left[x^3 - x\right] + \frac{5C^2}{2}\left[x^4 - x\right]. \tag{31}$$

To get that (31) corresponds to an accurate analytical approximate solution of (23), we identify optimally the constants C and B by using the Nonlinear Fit built-in command from Maple 15, which results in $B = -0.30848851834098$, and $C = 1.00432567368877$ (from (29) $\beta = 0.1490767537$).

Substituting these values into (31), we get

$$y_1(x) = 0.149076753781481x + 1.42747748922320x^2 - 3.09822939008034x^3 + 2.52167514707567x^4. \tag{32}$$

In order to show the accuracy of BVPP, we will compare our approximation (32) with the following third order approximate solution for (23) deduced, employing CPM

$$y(x) = 6.6845238095238x + 13.065476190476x^4 - 23.214285714286x^7 + 4.462857142857x^{10} \tag{33}$$

(see discussion and Figure 3).

Example 3.

This example shows the use of method 3. As a case study we propose the following linear differential equation [62].

$$\frac{d^2y(x)}{dx^2} - x^2y(x) = 0, \qquad 0 \leq x \leq 1, \qquad y(0) = 0, \qquad y(1) = 1, \tag{34}$$

Equation (6) for this case is given by

$$y(x) = \beta x + \int_0^x \int_0^x x^2 y dx' dx, \tag{35}$$

next, we select as trial function $y_0(x)$

$$y_0(x) = B + Cx. \tag{36}$$

The recurrence formula for this case is given by

$$y_n(x,\beta,B,C) = \beta x + \int_0^x \int_0^x \left(x^2 y_{n-1}(x,\beta,B,C)\right) dx' dx. \qquad n = 1,2,3.. \tag{37}$$

The first iteration of BVPP results in

$$y_1(x,\beta,B,C) = \beta x + \int_0^x \int_0^x \left(x^2 y_0(x,\beta,B,C)\right) dx' dx. \tag{38}$$

By substituting (36) into (38), we obtain

$$y_1(x,\beta,B,C) = \beta x + \frac{Bx^4}{12} + \frac{Cx^5}{20}. \tag{39}$$

Next, in order to obtain a better approximate solution, it is obtained the second iteration.

Evaluating (37), for n = 2, it is obtained

$$y_2(x,\beta,B,C) = \beta x + \int_0^x \int_0^x \left(x^2 y_1(x,\beta,B,C)\right) dx' dx, \tag{40}$$

Substituting (39) into (40), we get

$$y_2(x,B,\beta) = x^9 + \beta\left[x - x^9\right] + \frac{\beta}{20}\left[x^5 - x^9\right] + \frac{B}{672}\left[x^8 - x^9\right], \tag{41}$$

where, we employed the value

$$C = 1440\left(1 - \frac{21}{20}\beta - \frac{B}{672}\right), \tag{42}$$

obtained from condition $y_2(1) = 1$.

From the steps outlined by equations (11) - (13), we get the following algebraic system for the unknown quantities β and B.

$$\frac{259694131}{1453636800}\beta + \frac{3642882473}{75558877954560}B = \frac{24716233}{145363680},$$
$$\frac{122321146}{163875}\beta + \frac{259694131}{1453636800}B = \frac{7761232}{10925}. \tag{43}$$

Thus, the values of β and B, which minimize the square residual error, are given by $\beta = 0.9517485996$ and $B = -0.01705817486$.

Finally, substituting these values into (41), we get

$$y_2(x) = 0.0006893544x^9 - 0.00002538418878x^8 + 0.04758742998x^5 + 0.9517485996x. \tag{44}$$

Example 4.

This case study, shows the comparison among method 2, method 3, and variation of parameters for linear differential equations [61,62].

We will find an approximate solution for the differential equation.

$$\frac{d^2y(x)}{dx^2} + y(x) = x^{10}, \quad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 0, \tag{45}$$

This equation has an exact solution as follows, using the method of variation of parameters (VP) for linear differential equations.

$$y(x) = A\sin x + \int_0^x s^{10} sin(x-s)ds, \tag{46}$$

where $A = -\frac{1}{\sin 1}\int_0^1 s^{10} sin(x-s)ds$.

A disadvantage of VP is the big effort that must be done to solve the integrals in (46) (each requires 10 integrations by parts, in this case). Instead, we will see that BVPP provides a solution very accurate and handy for applications.

Equation (6) for this case is given by

$$y(x) = \beta x + \int_0^x \int_0^x (x^{10} - y) dx' dx, \tag{47}$$

next, we select as trial function $y_0(x)$

$$y_0(x) = B + Cx. \tag{48}$$

The recurrence formula for this case is given by

$$y_n(x,\beta,B,C) = \beta x + \int_0^x \int_0^x \left(x^{10} - y_{n-1}(x,\beta,B,C)\right) dx' dx. \quad n = 1,2,3.. \tag{49}$$

The first iteration of BVPP results in

$$y_1(x,\beta,B,C) = \beta x + \int_0^x \int_0^x \left(x^{10} - y_0(x,\beta,B,C)\right) dx' dx. \tag{50}$$

Substituting (48) into (50), we get

$$y_1(x,B,C) = \frac{1}{132}\left[x^{12} - x\right] + \frac{B}{2}\left[x - x^2\right] + \frac{C}{6}\left[x - x^3\right]. \tag{51}$$

where, we used the condition $y_1(1) = 0$, to obtain

$$\beta = -\frac{1}{132} + \frac{B}{2} + \frac{C}{6}. \tag{52}$$

Next, we will get two approximate solutions for (45) by using aforementioned method 2 and method 3.

Method 2

To get that (51) corresponds to a precise approximate analytic solution for (45), we identify optimally the constants$B$and$C$ by using the Nonlinear Fit built-in command from Maple 15, which results in $B = -0.00085048287500623$, and $C = -0.0062854559150948$ (from (52) $\beta = -0.009048575$).

Substituting these values into (51), we get

$$y_1(x) = 0.03125x^{12} - 0.00904857499910982x + 0.000425241437503110x^2 + 0.00104757598584913x^3 \tag{53}$$

Method 3

In this case, we will identify the constants B and C following the steps outlined by equations (11) - (13), so that

$$\frac{110}{189}C+\frac{101}{120}B=-\frac{221}{60480},$$
$$\frac{101}{120}C+\frac{101}{60}B=-\frac{127}{21840}. \tag{54}$$

By solving the above algebraic system, we find that the values of B and C, minimizing the square residual error, are given by $B=-0.0011384093831064$ and $C=-0.0046321138852940$.

Substituting these values into (51), we get

$$y_1(x)=0.03125x^{12}$$
$$-0.0089169812481931x+0.00056920469155320x^2$$
$$+0.00077201898088233x^3, \tag{55}$$

from (52) results the value $\beta=-0.008916981249$.

5 Discussion

This paper proposed a modified version of Picard Method, the Boundary Value Problems Picard Method (BVPP), in order to find approximate solutions for BVP problems. One of the main results, which follows from the accuracy of the approximate results obtained by BVPP, is that the slow convergence of PIM, is consequence of a inadequate choice of the trial function (even, many authors suggest starting Picard iterative process, by using as trial function, the initial condition of the differential equation to solve). The procedure followed by BVPP relies on the auxiliary initial value problem (8), where the value of $y'(t_0)=\beta$, is unknown. Assuming that the right hand side $f(t,y(t),y'(t))$ and its partial derivatives; satisfy certain continuity conditions (as it was explained) then irrespective of trial function, the successive approximations $\{y_n(t)\}$ (7) converge to the solution of (8).

In order to get that $\{y_n(t)\}$ also become in a solution for (4), we employed in our examples, as trial functions, some polynomial functions of different degrees containing some parameters, which were determined so that, the approximate solution, satisfies both boundary conditions of (4) and also contributed to improve the process of getting adequate trial functions. With this purpose, we proposed three methods to calculate optimally the mentioned parameters. As a matter of fact, we obtained highly accurate analytical approximate solutions.

The first method assumes as known the numerical solution for (4), so that the approximate solution given by the BVPP method, is evaluated at many points within the interval of interest as parameters to be determined. The above procedure gives rise to a system of algebraic equations, whose solution let determines the value of the parameters.

The second method proposed a software like the Nonlinear Fit built-in command from Maple 15, to identify optimally the constants.

The third method determines the adjusting parameters, by using the least squares method.

Next, we applied BVPP method to find approximate solutions to four differential equations, from which two were nonlinear and the others linear.

To exemplify Method 1 we obtained an approximate solution for Gelfand's equation. In accordance with this method, we generated four algebraic equations, by substituting the boundary value of the interval$x=1$, also the values: $x=0.1$,$x=0.5$, and $x=0.7$, which belong to $[0,1]$, with the purpose to calculate β,B,C,D. In order to test the effectiveness of the method, we considered the values of $\varepsilon=3$ and $\varepsilon=3.5$. For both cases study, we obtained highly accurate approximate solutions (21) and (22) for (14) as depicted in Figure 1 and Figure 2. A relevant fact is that, it was employed just the first iteration of BVPP (n=1) and were considered large values of Gelfand's parameters, whose solutions are difficult to model.

In principle it is possible to improve the precision of approximate solutions, by introducing more adjusting parameters in trial function (17), although would be necessary solving larger algebraic system of equations.

Method 2, was discussed, considering the solution of the second order nonlinear differential equation (23), depending on a parameter. In order to show the effectiveness of the method, we considered as case study, the first iteration of BVPP (n=1) for a large value of the parameter $\varepsilon=30$ (32), and its comparison with (33), which is an approximation for (23) corresponding to third order approximation of classical perturbation method CPM.

Figure 3 shows that BVPP is efficient in comparison to CPM, although we considered only the first iteration, even for large values of the parameter, where other methods fail.

Method 3, was exemplified by solving the linear differential equation (34). In order to obtain a better approximate solution, it was considered the second iteration of BVPP method (n=2). In this case the parameters of the trial function (36) were determined by using the least squares method. Figure 4 shows that, as occurred with the above examples, the approximate solution (44) is highly accurate and handy.

Finally, we employed BVPP, to find an approximate solution for (45), although this linear equation has exact solution, by using variation of parameters (VP). A disadvantage of VP is the big effort that must be done to solve the integrals in (46) (each requires 10 integrations by parts, in this case). Instead, BVPP provided solutions very accurate and useful, adequate for practical applications. The adjusting parameters in (51) were calculated, following the algorithms from the method 2 and method 3, from where the approximations (53) and (55) arise respectively.

Figure 5 shows that (53), (55) and exact solution are in good agreement. Although in this case method (2) was more precise than method (3), this latter and therefore the process of getting (55), did not require in advance the knowledge of the numerical solution. In this sense, Method 3 has more analytical basis than the other methods.

Finally, this work introduced a modification of Piccard method, valid for boundary value problems. As long as we know, there is not in the literature, antecedents of a method like BVPP, which applies Piccard method to BVP, by proposing a suitable trial function that optimizes the approximation, in order to accelerate the convergence for the process of obtaining analytical approximate solutions for linear and nonlinear ODES. We noted that in general, the classical procedure of PIM involves the use of the initial conditions (of the differential equation to be solved) as the starting trial function; however, such criteria often leads to poor convergence, excessive iterations and large/cumbersome approximations. Therefore, inappropriate for practical applications.

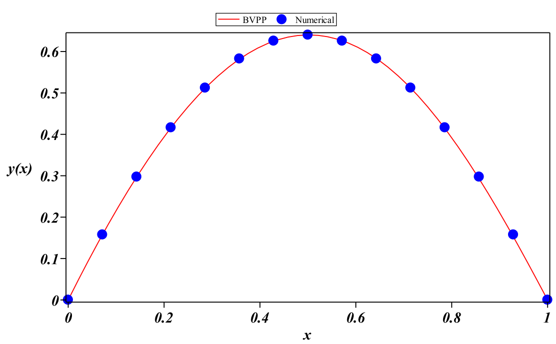

Fig. 1: Comparison of approximate solution (21) (solid line) for (14) (dots) considering $\varepsilon = 3$

6 Conclusions

This work introduced the Boundary Value Problems Picard Method (BVPP), as a useful tool with high potential, in order to find approximate solutions for BVP. Based on mathematical assumptions, which ensure that Picard iterative method, described in Section 2, converges to the solution of a problem as (8), independently of the chosen trial function and initial conditions. BVPP builds a valid solution to the auxiliary problem (8), and then

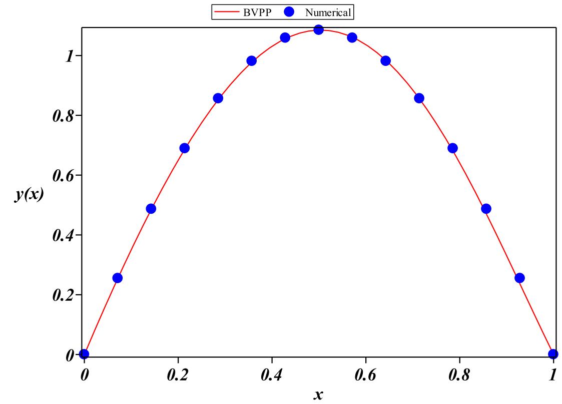

Fig. 2: Comparison of approximate solution (22) (solid line) for (14) (dots) considering $\varepsilon = 3.5$

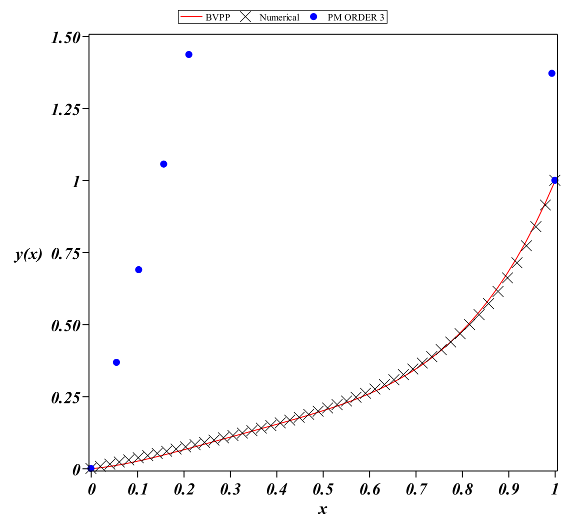

Fig. 3: Comparison of approximations BVPP (32) (solid line) and PM (33) (dots) for (23) (diagonal cross).

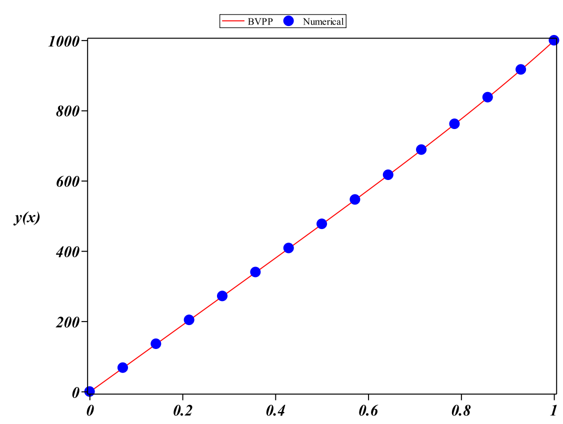

Fig. 4: Comparison of approximate solution (44) (solid line) for (34) (dots).

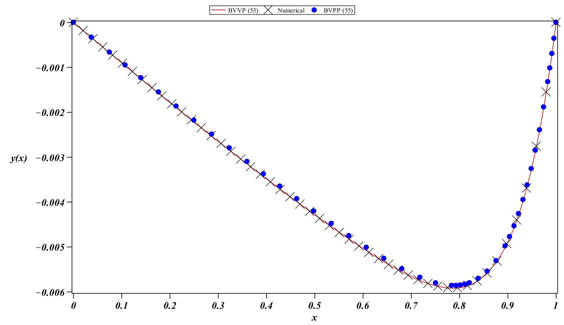

Fig. 5: Comparison of BVPP approximations (53) (solid line) and (55) (dots) for (45) (diagonal cross).

adjusts it to become, also in a solution for the problem to be solved (4).

The procedure assumed, that it is possible to obtain a good approximate solution for (4), based on a polynomial trial function, provided with certain adjusting parameters, which were employed to link problems (4) and (8) mentioned before, and optimize the approximate solutions proposed by BVPP. May be expected to improve the precision of approximate solutions, by introducing more adjusting parameters in trial functions. Furthermore, from the procedure studied, it is deduced that the method provides an adequate criterion to start the iterative process, with good results, and few iterations.

Acknowledgement

We gratefully acknowledge the financial support from the National Council for Science and Technology of México (CONACyT) through grant CB-2010-01 #157024. The authors would like to thank Roberto Castaneda-Sheissa, Rogelio-Alejandro Callejas-Molina, and Roberto Ruiz-Gomez for their contribution to this project.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] L.M.B. Assas. Approximate solutions for the generalized k-dv- burgers'equation by he's variational iteration method. *Phys. Scr.*, 76:161–164, 2007.
[2] J.H. He. Variational approach for nonlinear oscillators. *Chaos, Solitons and Fractals*, 34:1430–1439, 2007.
[3] Kazemnia M., S.A. Zahedi, M. Vaezi, and N. Tolou. Assessment of modified variational iteration method in bvps high-order differential equations. *Journal of Applied Sciences*, 8:4192–4197, 2008.
[4] Reza Noorzad, A Tahmasebi Poor, and Mehdi Omidvar. Variational iteration method and homotopy-perturbation method for solving burgers equation in fluid dynamics. *J Appl Sci*, 8:369–373, 2008.
[5] D.J. Evans and K.R. Raslan. The tanh function method for solving some important nonlinear partial differential. *Int. J. Computat. Math.*, 82(897-905), 2005.
[6] F. Xu. A generalized soliton solution of the konopelchenko-dubrovsky equation using exp-function method. *ZeitschriftNaturforschung - Section A Journal of Physical Sciences*, 62(12):685–688, 2007.
[7] Mahmoudi J., N. Tolou, I. Khatami, A. Barari, and D.D. Ganji. Explicit solution of nonlinear zk-bbm wave equation using exp-function method. *Journal of Applied Sciences*, 8:358–363, 2008.
[8] G. Adomian. A review of decomposition method in applied mathematics. *Mathematical Analysis and Applications*, 135:501–544, 1988.
[9] Babolian E. and J. Biazar. On the order of convergence of adomian method. *Applied Mathematics and Computation*, 130(2):383–387, 2002.
[10] A. Kooch and M. Abadyan. Efficiency of modified adomian decomposition for simulating the instability of nano-electromechanical switches: comparison with the conventional decomposition method. *Trends in Applied Sciences Research*, 7:57–67, 2012.
[11] A. Kooch and M. Abadyan. Evaluating the ability of modified adomian decomposition method to simulate the instability of freestanding carbon nanotube: comparison with conventional decomposition method. *Journal of Applied Sciences*, 11:3421–3428, 2011.
[12] S. Heidari Vanani, S. K. and M. Avaji. A low-cost numerical algorithm for the solution of nonlinear delay boundary integral equations. *Journal of Applied Sciences*, 11:3504–3509, 2011.
[13] S. H. Chowdhury. A comparison between the modified homotopy perturbation method and adomian decomposition method for solving nonlinear heat transfer equations. *Journal of Applied Sciences*, 11:1416–1420, 2011.
[14] L.-N. Zhang and L. Xu. Determination of the limit cycle by he's parameter expansion for oscillators in a potential. *ZeitschriftfrNaturforschung - Section A Journal of Physical Sciences*, 62(7-8):396–398, 2007.
[15] J.H. He. A coupling method of a homotopy technique and a perturbation technique for nonlinear problems. *Int. J. Non-Linear Mech.*, 351:37–43, 1998.
[16] J.H. He. Homotopy perturbation technique. *Comput. Methods Applied Mech. Eng.*, 178:257–262, 1999.
[17] Ji-Huan He. Homotopy perturbation method for solving boundary value problems. *Physics Letters A*, 350(12):87 – 88, 2006.
[18] Ji-Huan He. Recent development of the homotopy perturbation method. *Topological Methods in Nonlinear Analysis,*, 31(2):205–209, 2008.
[19] A Belendez, C Pascual, ML Alvarez, DI Méndez, MS Yebra, and A Hernández. Higher order analytical approximate solutions to the nonlinear pendulum by he's homotopy method. *Physica Scripta*, 79(1):015009, 2009.
[20] Ji-Huan He. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *International Journal of Non-Linear Mechanics*, 35(1):37–43, 2000.
[21] Moustafa El-Shahed. Application of he's homotopy perturbation method to volterra's integro-differential equation. *International Journal of Nonlinear Sciences and Numerical Simulation*, 6(2):163–168, 2005.

[22] Ji-Huan He. Some asymptotic methods for strongly nonlinear equations. *International Journal of Modern Physics B*, 20(10):1141–1199, 2006.

[23] DD Ganji, H Babazadeh, F Noori, MM Pirouz, and M Janipour. An application of homotopy perturbation method for non linear blasius equation to boundary layer flow over a flat plate. *International Journal of Nonlinear Science*, 7(4):399–404, 2009.

[24] DD Ganji, H Mirgolbabaei, Me Miansari, and Mo Miansari. Application of homotopy perturbation method to solve linear and non-linear systems of ordinary differential equations and differential equation of order three. *Journal of Applied Sciences*, 8:1256–1261, 2008.

[25] A Fereidoon, Y Rostamiyan, M Akbarzade, and Davood Domiri Ganji. Application of hes homotopy perturbation method to nonlinear shock damper dynamics. *Archive of Applied Mechanics*, 80(6):641–649, 2010.

[26] P.R. Sharma and Giriraj Methi. Applications of homotopy perturbation method to partial differential equations. *Asian Journal of Mathematics & Statistics*, 4:140–150, 2011.

[27] Hossein Aminikhah. Analytical approximation to the solution of nonlinear blasius viscous flow equation by ltnhpm. *ISRN Mathematical Analysis*, 2012, 2012.

[28] Hector Vazquez-Leal, Uriel Filobello-Nino, Roberto Castañeda-Sheissa, Luis Hernández-Martínez, and Arturo Sarmiento-Reyes. Modified hpms inspired by homotopy continuation methods. *Mathematical Problems in Engineering*, 2012, 2012.

[29] Hector Vazquez-Leal, Roberto Castaneda-Sheissa, Uriel Filobello-Nino, Arturo Sarmiento-Reyes, and Jesus Sanchez Orea. High accurate simple approximation of normal distribution integral. *Mathematical problems in engineering*, 2012, 2012.

[30] U Filobello-Nino, Hector Vazquez-Leal, R Castaneda-Sheissa, A Yildirim, L Hernandez-Martinez, D Pereyra-Diaz, A Perez-Sesma, and C Hoyos-Reyes. An approximate solution of blasius equation by using hpm method. *Asian Journal of Mathematics & Statistics*, 5(2):50, 2012.

[31] Jafar Biazar and Hossein Aminikhah. Study of convergence of homotopy perturbation method for systems of partial differential equations. *Computers & Mathematics with Applications*, 58(11):2221–2230, 2009.

[32] Jafar Biazar and Hosein Ghazvini. Convergence of the homotopy perturbation method for partial differential equations. *Nonlinear Analysis: Real World Applications*, 10(5):2633–2640, 2009.

[33] U Filobello-Nino, H Vazquez-Leal, Y Khan, R Castaneda-Sheissa, A Yildirim, L Hernandez-Martinez, J Sanchez-Orea, R Castaneda-Sheissa, and F Rabago Bernal. Hpm applied to solve nonlinear circuits: a study case. *Appl Math Sci*, 6(85-88):4331–4344, 2012.

[34] DD Ganji, AR Sahouli, and M Famouri. A new modification of hes homotopy perturbation method for rapid convergence of nonlinear undamped oscillators. *Journal of Applied Mathematics and Computing*, 30(1-2):181–192, 2009.

[35] U Filobello-Nino, H Vazquez-Leal, Y Khan, A Perez-Sesma, A Diaz-Sanchez, VM Jimenez-Fernandez, A Herrera-May, D Pereyra-Diaz, JM Mendez-Perez, and J Sanchez-Orea. Laplace transform-homotopy perturbation method as a powerful tool to solve nonlinear problems with boundary conditions defined on finite intervals. *Computational and Applied Mathematics*, 34(1):1–16, 2013.

[36] MA Fariborzi Araghi and B Rezapour. Application of homotopy perturbation method to solve multidimensional schrodinger's equations. *International Journal of Mathematical Archive (IJMA) ISSN 2229-5046*, 2(11):1–6, 2011.

[37] MF Araghi and M Sotoodeh. An enhanced modified homotopy perturbation method for solving nonlinear volterra and fredholm integro-differential equation. *World Applied Sciences Journal*, 20(12):1646–1655, 2012.

[38] Mahdi Bayat, Mahmoud Bayat, and Iman Pakar. Nonlinear vibration of an electrostatically actuated microbeam. *Latin American Journal of Solids and Structures*, 11(3):534–544, 2014.

[39] M Bayat, I Pakar, and A Emadi. Vibration of electrostatically actuated microbeam by means of homotopy perturbation method. *Structural Engineering and Mechanics*, 48(6):823–831, 2013.

[40] Jafar Biazar and Behzad Ghanbari. The homotopy perturbation method for solving neutral functional–differential equations with proportional delays. *Journal of King Saud University-Science*, 24(1):33–37, 2012.

[41] Jafar Biazar and Mostafa Eslami. A new homotopy perturbation method for solving systems of partial differential equations. *Computers & Mathematics with Applications*, 62(1):225–234, 2011.

[42] Twinkle Patel, MN Mehta, and VH Pradhan. The numerical solution of burgers equation arising into the irradiation of tumour tissue in biological diffusing system by homotopy analysis method. *Asian J Appl Sci*, 5:60–66, 2012.

[43] A Basiri Parsa, MM Rashidi, O Anwar Bég, and SM Sadri. Semi-computational simulation of magneto-hemodynamic flow in a semi-porous channel using optimal homotopy and differential transform methods. *Computers in biology and medicine*, 43(9):1142–1153, 2013.

[44] MM Rashidi, SA Mohimanian Pour, T Hayat, and S Obaidat. Analytic approximate solutions for steady flow over a rotating disk in porous medium with heat transfer by homotopy analysis method. *Computers & Fluids*, 54:1–9, 2012.

[45] MM Rashidi, MT Rastegari, M Asadi, and O Anwar Bég. A study of non-newtonian flow and heat transfer over a non-isothermal wedge using the homotopy analysis method. *Chemical Engineering Communications*, 199(2):231–256, 2012.

[46] MM Rashidi, E Momoniat, M Ferdows, and A Basiriparsa. Lie group solution for free convective flow of a nanofluid past a chemically reacting horizontal plate in a porous media. *Mathematical Problems in Engineering*, 2014:21 pages.

[47] Vasile Marinca and Nicolae Herisanu. Nonlinear dynamical systems in engineering. *first edition. Springer-Verlag Berlin Heidelberg*, 2011.

[48] U Filobello-Nino, H Vazquez-Leal, Y Khan, A Yildirim, V. M Jimenez-Fernandez, A L Herrera-May, R Castaneda-Sheissa, and J Cervantes-Perez. Using perturbation methods and laplace-padé approximation to solve nonlinear problems. *Miskolc Mathematical Notes*, 14(1):89–101, 2013.

[49] U Filobello-Nino, H Vazquez-Leal, K Boubaker, Y Khan, A Perez-Sesma, A Sarmiento-Reyes, VM Jimenez-Fernandez, A Diaz-Sanchez, A Herrera-May, J Sanchez-Orea, et al. Perturbation method as a powerful tool to solve highly nonlinear problems: the case of gelfand's equation. *Asian Journal of Mathematics & Statistics*, 6(2):76, 2013.

[50] Hector Vazquez-Leal, Brahim Benhammouda, Uriel Antonio Filobello-Nino, Arturo Sarmiento-Reyes, Victor Manuel Jimenez-Fernandez, Antonio Marin-Hernandez, Agustin Leobardo Herrera-May, Alejandro Diaz-Sanchez, and Jesus Huerta-Chua. Modified taylor series method for solving nonlinear differential equations with mixed boundary conditions defined on finite intervals. *SpringerPlus*, 3(1):160, 2014.

[51] Hector Vazquez-Leal. Generalized homotopy method for solving nonlinear differential equations. *Computational and Applied Mathematics*, 33(1):275–288, 2014.

[52] Aydin Kurnaz, Galip Oturanç, and Mehmet E Kiris. n-dimensional differential transformation method for solving pdes. *International Journal of Computer Mathematics*, 82(3):369–380, 2005.

[53] U Filobello-Niño, H Vazquez-Leal, Y Khan, A Perez-Sesma, A Diaz-Sanchez, A Herrera-May, D Pereyra-Diaz, R Castañeda-Sheissa, VM Jimenez-Fernandez, and J Cervantes-Perez. A handy exact solution for flow due to a stretching boundary with partial slip. *Revista mexicana de física E*, 59(2013):51–55, 2013.

[54] J.D Murray. Mathematical biology: I. an introduction. 3rd. Edition, Springer, USA, 2002.

[55] Utku Erdogan and Turgut Ozis. A smart nonstandard finite difference scheme for second order nonlinear boundary value problems. *Journal of Computational Physics*, 230(17):6464–6474, 2011.

[56] Elias Deeba, SA Khuri, and Shishen Xie. An algorithm for solving boundary value problems. *Journal of Computational Physics*, 159(2):125–138, 2000.

[57] Xinlong Feng, Liquan Mei, and Guoliang He. An efficient algorithm for solving troeschs problem. *Applied Mathematics and Computation*, 189(1):500–507, 2007.

[58] SH Mirmoradia, I Hosseinpoura, S Ghanbarpour, and A Barari. Application of an approximate analytical method to nonlinear troeschs problem. *Applied Mathematical Sciences*, 3(29-32):1579–1585, 2009.

[59] Hany N Hassan and Magdy A El-Tawil. An efficient analytic approach for solving two-point nonlinear boundary value problems by homotopy analysis method. *Mathematical methods in the applied sciences*, 34(8):977–989, 2011.

[60] Hector Vazquez-Leal, Yasir Khan, Guillermo Fernandez-Anaya, Agustin Herrera-May, Arturo Sarmiento-Reyes, Uriel Filobello-Nino, Victor-M Jimenez-Fernandez, and Domitilo Pereyra-Diaz. A general solution for troesch's problem. *Mathematical Problems in Engineering*, 2012.

[61] Andy C King, John Billingham, and Stephen Robert Otto. *Differential equations: linear, nonlinear, ordinary, partial*. Cambridge University Press, 2003.

[62] Dennis Zill. *A first course in differential equations with modeling applications*, volume 10th Edition. Brooks /Cole Cengage Learning, 2012.

[63] JI Ramos. Picards iterative method for nonlinear advection–reaction–diffusion equations. *Applied Mathematics and Computation*, 215(4):1526–1536, 2009.

[64] Xiaoyan Deng, Bangju Wang, and Guangqing Long. The picard contraction mapping method for the parameter inversion of reaction-diffusion systems. *Computers & Mathematics with Applications*, 56(9):2347–2355, 2008.

[65] IK Youssef and HA El-Arabawy. Picard iteration algorithm combined with gauss–seidel technique for initial value problems. *Applied Mathematics and computation*, 190(1):345–355, 2007.

[66] L Elsgoltz. *Ecuaciones Diferenciales y Clculo Variacional*, volume Tercera Edición. Editorial MIR Moscu, 1983.

[67] William E Boyce, Richard C DiPrima, and Charles W Haines. *Elementary differential equations and boundary value problems*, volume Seventh Edition. John Wiley & Sons Inc., New York, 2001.

[68] Kaplan Wilfred. *Ordinary Differential Equations*, volume Seventh Edition. Addison-Wesley Company, Inc., 1958.

[69] T.L. Chow. Classical mechanics. *John Wiley and Sons Inc., USA.*, 1995.

[70] Mark H Holmes. *Introduction to perturbation methods*. Springer-Verlag, New York, 1995.

Uriel Filobello-Nino received his B.Sc. at Facultad de Física from Universidad Veracruzana and received the M.Sc., and Ph.D. degrees in physics from Universidad Nacional Autónoma de México. He is a full time professor and researcher associate at the Facultad de Instrumentación Electrónica at Universidad Veracruzana. He has been engaged in research on cosmology, but currently his research mainly covers applied mathematics, especially analytical approximate solutions for linear and non-linear differential equations. Prof. Filobello-Nio is author and coauthor of several research articles published in different prestigious JCR-ISI THOMSON journals.

Hector Vazquez-Leal Received the B.Sc. degree in Electronic Instrumentation Engineering in 1999 from University of Veracruz (UV), M.Sc. and Ph.D. degrees in Electronic Sciences in 2001/2005 from National Institute of Astrophysics, Optics and Electronics (INAOE), México. His current research mainly covers analytical-numerical methods, nonlinear circuits, robotics, applied mathematics, and wireless energy transfer. He is editor of one International peer-reviewed Journal and regular invited reviewer of more than 25 journals. Prof. Vazquez-Leal is author or coauthor of 87 research articles published in several prestigious journals.

Jose Antonio Agustin Perez-Sesma received his B.Sc. at Facultad de Física from Universidad Veracruzana, with specialty in atmospheric sciences. He obtained his MS degree in environmental geography with specialty in Hydrology from the Universidad Nacional Autónoma de México. He is a full time professor and research associate in Hydrology at the Facultad de Instrumentación Electrónica at Universidad Veracruzana. In the last five years Professor Sesma has been an author and co-author of five book chapters in the area of atmospheric sciences. As co-author, he has published several research articles in reputed international JCR-ISI THOMSON journals of mathematical and engineering sciences. He has been actively participating in projects sponsored by international organizations such as CONACYT, PROMEP, CONAGUA and International Atomic Energy Agency.

Juan Cervantes-Perez received his B.Sc. at Facultad de Física from Universidad Veracruzana and received his M.Sc., and Ph.D. degrees from Universidad Nacional Autónoma de México. His research interests is in the area of Bioclimatology. He has been actively participating in projects sponsored by International Atomic Energy Agency.

Luis Hernandez-Martinez received the Ph.D. degree in Electronic Engineering from the National Institute for Astrophysics, Optics and Electronics (INAOE) in 2001. Since 2001 he is a Full Researcher at the Electronics Department of the National Institute for Astrophysics, Optics and Electronics (INAOE), México. His topics of interest are design automation, nonlinear circuits and cellular neural networks.

Agustin L. Herrera-May received the Doctor of Engineering degree from Guanajuato University, México. He is a Research Scientist at the Micro and Nanotechnology Research Center (MICRONA) from Veracruzana University. He has served as a Reviewer of the Journal of Micromechanics and Microengineering, Sensors and Actuators A, Medical Engineering & Physics, Recent Patents on Nanomedicine, IEEE Transactions on Magnetic, Sensors, Journal of Microelectromechanical Systems, Measurement Science and Technology, IEEE Transactions on Magnetics, Applied Physics Research, Journal of Basic and Applied Physics, Applied Bionics and Biomechanics, Nanotechnology, IEEE Electron Device Letters, IEEE Transactions on Industrial Electronics, Journal of Physics D: Applied Physics, Micro and Nanosystems, EIA, Nanoscience and Nanotechnology Letters, Journal of Mechanical Science and Technology, Journal of Applied Research and Technology, Engineering Science and Technology: an International Journal, Micromachines, and IEEE International Midwest Symposium on Circuits and Systems. He is author or coauthor of more than 40 papers in technical journals. His research interests include microelectromechanical and nanoelectromechanical systems, mechanical vibrations, fracture, mechanical design, and finite-element method.

Victor Manuel Jimenez-Fernandez received the PhD degree in Electronics Science from the National Institute of Astrophysics, Optics and Electronics (INAOE), Puebla, México in 2006. From 2006 to 2007 he was visiting researcher in the Department of Electrical and Computer Engineering in the National University of the South in Bahía Blanca, Argentina. Since February 2009 he has been researcher at the University of Veracruz, México. His research interests include nonlinear system modeling and scientific computing.

Antonio Marin-Hernandez is a professor at Research Center on Artificial intelligence of the Universidad Veracruzana. He received the PhD degree in Robotics and Artificial Intelligence in 2004 by the INPT-LAAS-CNRS (France). His main research topics include: Mobile robotics, autonomous navigation, human-robot interaction, dynamic systems and optimization. He is referee of diverse journals on computer science, mathematics and engineering; and he has published research articles in journals on the same domains.

Claudio Hoyos-Reyes received his B.Sc. at Facultad de Instrumentación Electrónica from Universidad Veracruzana, with specialty in atmospheric sciences. He obtained his MS degree in environmental engineering from the Universidad Nacional Autónoma de México. Currently He is a Doctoral student at Centro de Investigaciones Atmosféricas y Ecológicas de la Universidad Autónoma de Veracruz and his area of research is environmental management. He has co-authored several research articles in reputed international JCR-ISI THOMSON journals of mathematical and engineering sciences. He has been actively participating in projects sponsored by international organizations such as PRONACOSE, CONAGUA and International Atomic Energy Agency.

Alejandro Diaz-Sanchez received the B.E. from the Madero Technical Institute and the M.Sc. from the National Institute for Astrophysics, Optics and Electronics, both in México, and the Ph.D. in Electrical Engineering from New México State University at Las Cruces, NM. He is actually working as Full Professor at the Instituto Nacional de Astrofísica, Optica y Electrónica (INAOE), in Tonantzintla, México. His research concerns analog and digital integrated circuits, high performance computer architectures and signal processing.

Jesus Huerta-Chua received the B.E. degree by honors in electronics and communication en-gineering from Veracruzana University, Veracruz, México, in 1998, the M.S. and PhD. degree from INAOE (National Institute of Astrophysics, Optics and Electronics), Puebla, México, in 2002 and 2008. From 2009 to 2013. he was metrologist in CE-NAM (The National Laboratory of Metrology from México) during this period he developed a National Measurement Reference System for calibration of Articial Mains Networks. He is currently a full time academic engineer in the Veracruzana University. His main research interests are in microelectronics devices, high frequency modeling and characterization of electronic devices, high frequency metrology.