Application Of Fox-Wright Generalized Hypergeometric Functions to Multivalent Functions

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Abstract: In the present paper, we introduce a new class of multivalent analytic functions by using Fox-Wright generalized Hypergeometric functions and we obtain the coefficient bounds, extreme points, integral representations, distortion bounds, radii of starlikeness and convexity and neighbourhood.

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1 Introduction

Consider the subclass $A(n, p)$ of functions $f(z) \in \mathfrak{A}$ of the form

$$f(z) = z^p - \sum_{k=n+p}^\infty a_k z^k, \quad (n, p \in \mathbb{N}) \quad (1)$$

analytic and multivalent functions in the unit disk $\mathbb{D} = \{ z : |z| < 1 \}$. The function $f(z)$ is said to be starlike of order $\delta(0 \leq \delta < p)$ if and only if

$$Re \left( \frac{zf'(z)}{f(z)} \right) > \delta, \quad (z \in \mathbb{D}). \quad (2)$$

On the other hand $f(z)$ is said to be convex of order $\delta(0 \leq \delta < p)$ if and only if

$$Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \delta, \quad (z \in \mathbb{D}) \quad (3)$$

Definition: Let $\mathbb{H}$ denote a Hilbert space on the complex plane. Let $\mathcal{V}$ denote an operator on $\mathbb{H}$. For a complex-valued function $f$ analytic on $\mathbb{D}$, let $f(\mathcal{V})$ denote the operator on $\mathbb{H}$ defined by the Riesz-Dunford Integral [2]

$$f(\mathcal{V}) = \frac{1}{2\pi i} \int_C (zI - \mathcal{V})^{-1} f(z)dz \quad (4)$$

where $I$ is the identity operator on $\mathbb{H}$, $C$ is positively oriented rectifiable Jordan contour in $\mathbb{D}$ (1) and contain the spectrum of interior domain. The operator $f(\mathcal{V})$ has series representation $f(\mathcal{V}) = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} \mathcal{V}^k$, which converges in the norm topology [4]. For complex parameters

$$\alpha_1, \cdots, \alpha_q \left( \frac{\alpha_j}{A_j} \neq 0, -1, -2, \cdots ; j = 1, \cdots, q \right)$$

and

$$\beta_1, \cdots, \beta_s \left( \frac{\beta_j}{B_j} \neq 0, -1, -2, \cdots ; j = 1, \cdots, s \right)$$

we define the Fox-Wright generalized hypergeometric function [5] (see also [7], [9], [13], [14]),

$$q \psi_1((\alpha_1, A_1), \cdots, (\alpha_q, A_q); (\beta_1, B_1), \cdots, (\beta_s, B_s); z)$$

$$= q \psi_1((\alpha_1, A_1)_{1, q}; (\beta_1, B_1)_{1, s}; z)$$

$$= \sum_{k=0}^\infty \left\{ \prod_{j=1}^q \Gamma(\alpha_j + A_j k) \right\} \left\{ \prod_{j=1}^s \Gamma(\beta_j + B_j k) \right\}^{-1} \frac{z^k}{k!} \quad (5)$$

$$(A_j > 0 \ (j = 1, \cdots, q); B_j > 0 \ (j = 1, \cdots, s); 1 + \sum_{j=1}^q B_j - \sum_{j=1}^s A_j \geq 0).$$

If $A_j = 1 \ (j = 1, \cdots, q)$ and $B_j = 1 \ (j = 1, \cdots, s)$, we have the relationship

$$w q \psi_1((\alpha_1, 1)_{1, q}; (\beta_1, 1)_{1, s}; z) = q F_s((\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s); z)$$

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where \( q F_{\alpha} \) is generalized hypergeometric function and
\[
w = \frac{r \Gamma(b) - r \Gamma(b)}{r \Gamma(a) - r \Gamma(a)}, \quad (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{C}) (1).
\]

Now let \( q, s \in \mathbb{N} \) and suppose that \( \alpha_1, \cdots, \alpha_q \) and \( \beta_1, \cdots, \beta_s \) are also positive real numbers. Then, we define the function
\[
\phi((\alpha_j, A_j), (\beta_j, B_j), 1, 1, z) = w q \psi((\alpha_j, A_j), (\beta_j, B_j), 1, 1, z)
\]
and consider the linear operator \([L]_{F_{\alpha}}(\beta, B_j)_{1, 1, 1} : \mathcal{A} \rightarrow \mathcal{A} \) defined by
\[
L((\alpha_j, A_j), (\beta_j, B_j), 1, 1, f(z)) = \phi((\alpha_j, A_j), (\beta_j, B_j), 1, 1, z) \cdot f(z).
\]

For simplicity, we write
\[
L((\alpha_j, A_j), (\beta_j, B_j), 1, 1, f(z)) = L_{(\alpha_j, A_j), (\beta_j, B_j), 1, 1, 1} f(z).
\]

We note that special cases of this operator were investigated by Dzio and Srivastava [3], by letting \( A_j = 1 \) for \( j = 1, \cdots, q \) and \( B_j = 1 \) for \( j = 1, \cdots, s \) in 7, and includes the Noor Integral operator [8].

Now we define the class of functions \( \mathcal{M}_{p}^{(\alpha, \beta)} \) consisting of functions \( f \) defined by 1 which satisfy the condition
\[
\text{Re} \left\{ \frac{L(\alpha J_f x^q)}{x^q} \right\} > \alpha \left( \frac{L(\alpha J_f x^q)}{x^q} - p \right) + \beta, \quad (9)
\]
for \( \alpha \geq 0, 0 \leq q < p \) and for all operator \( \mathcal{V} \) such that \( \mathcal{V} \neq 0 \) and \( \| \mathcal{V} \| < 1, 0 \) being the null operator on \( \mathbb{H} \).

We need the following.

Let \( f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \) and \( g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \) then the Hadamard product \( f \ast g \) is defined as
\[
(f \ast g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k.
\]

**Coefficient bounds**

At first, we prove necessary and sufficient condition for the function \( f(z) \) as defined by 1 to belong to the class \( \mathcal{M}_{p}^{(\alpha, \beta)} \).

**Theorem 2.1** : Let a function \( f(z) \in T(n, p) \). Then \( f(z) \in \mathcal{M}_{p}^{(\alpha, \beta)} \) if and only if
\[
\sum_{k=n+p}^{\infty} \sigma_k a_k \left( \frac{1 + \alpha}{1 + p} - k(\alpha p + \beta) \right) < 1, \quad (11)
\]

\( 0 < p, \alpha \geq 0 \) and where
\[
\sigma_k = \frac{\Gamma(\alpha + A_1(k-n)) \cdots \Gamma(\alpha_q + A_q(k-n))}{\Gamma(1 + B_1(k-n)) \cdots \Gamma(1 + B_s(k-n))(k-n)!}, \quad k \in \mathbb{N}.
\]

**Proof** : Let \( f \in \mathcal{M}_{p}^{(\alpha, \beta)} \). Using the fact for real \( \gamma \)
\[
\text{Re}(w) > \alpha |w - p| + \alpha \text{Re}(w(1 + \alpha e^{\gamma}) - p \alpha e^{\gamma}) > \beta
\]
and letting \( w = \frac{L(\alpha J_f x^q)}{x^q} \) in 4, we obtain
\[
\text{Re} \left\{ \frac{L(\alpha J_f x^q)}{x^q} \right\} > \beta
\]
or
\[
\text{Re} \left\{ \frac{L(\alpha J_f x^q)}{x^q} \right\} > \beta
\]

Setting \( \mathcal{V} = r \mathcal{V} \) \( 0 < r < 1 \) and letting \( r \to 1^- \), yields
\[
\text{Re} \left\{ \frac{L(\alpha J_f x^q)}{x^q} \right\} > 0.
\]

By mean value theorem, we have
\[
\sum_{k=n+p}^{\infty} \left( 1 + \alpha - k(p \alpha + \beta) \right) a_k \sigma_k < (1 + \alpha) - p(\beta + \alpha p).
\]

Conversely, for \( f \in \mathcal{M}_{p}^{(\alpha, \beta)} \), it is enough to show that
\[
\left( \frac{L(\alpha J_f x^q)}{x^q} \right) > \alpha \left( \frac{L(\alpha J_f x^q)}{x^q} - p \right) + \beta
\]
\[
\left( \frac{L(\alpha J_f x^q)}{x^q} \right) > \alpha \left( \frac{L(\alpha J_f x^q)}{x^q} - p \right) + \beta
\]
by using the fact \( \text{Re}(w) > \alpha \iff |w - (p + \alpha)| < |w + (p - \beta)| \).

Now let \( M = \frac{L(\alpha J_f x^q)}{x^q} \) and then
\[
M = \frac{1}{\| \mathcal{V} \|} \left| \frac{L(\alpha J_f x^q)}{x^q} \right| - p \mathcal{V} \left( \frac{L(\alpha J_f x^q)}{x^q} \right) - p \mathcal{V} \left( \frac{L(\alpha J_f x^q)}{x^q} \right)
\]
\[
= \frac{1}{\| \mathcal{V} \|} \left| \frac{L(\alpha J_f x^q)}{x^q} \right| - p \mathcal{V} \left( \frac{L(\alpha J_f x^q)}{x^q} \right) - p \mathcal{V} \left( \frac{L(\alpha J_f x^q)}{x^q} \right)
\]
\[
= \sum_{k=n+p}^{\infty} \left( 1 + \alpha - k(p \alpha + \beta) \right) a_k \sigma_k
\]
\[
> (1 + \alpha) - p(\beta + \alpha p) - \sum_{k=n+p}^{\infty} (1 + \alpha - k(\beta + \alpha p) + k(p \alpha + \beta) a_k \sigma_k
\]
3 Extreme Points and Distortion Bounds

Now we obtain the extreme points for the class \( \mathcal{M}_p^{(V)}(\alpha, \beta) \).

**Theorem 3.1:** Let \( f_1(z) = z^p \) and

\[
f_k(z) = z^p - \frac{(1 + \alpha) - p(\alpha p + \beta)}{(1 + \alpha) - k(\alpha p + \beta)} \sigma_k z^k
\]

where, \( k \geq n + p, n, p \in \mathbb{N} \), then \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta) \) if and only if \( f(z) \) can be expressed in the form

\[
f(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^\infty \lambda_k f_k(z)
\]

where

\[
\lambda_1 + \sum_{k=n+p}^\infty \lambda_k = 1 \quad (\lambda_1 \geq 0, \lambda_k \geq 0).
\]

**Proof:** Let \( f \) can be expressed by the form 13, then

\[
f(z) = \lambda_1 z^p + \sum_{k=n+p}^\infty \left[ \lambda_k z^p - \frac{(1 + \alpha) - p(\alpha p + \beta) \lambda_k z^k}{(1 + \alpha) - k(\alpha p + \beta) \sigma_k} \right]
\]

\[
= z^p (\lambda_1 + \sum_{k=n+p}^\infty \lambda_k) - \sum_{k=n+p}^\infty t_k z^k = z^p - \sum_{k=n+p}^\infty t_k z^k
\]

where

\[
t_k = \frac{(1 + \alpha) - p(\alpha p + \beta) \lambda_k}{(1 + \alpha) - k(\alpha p + \beta) \sigma_k}
\]

Since

\[
\sum_{k=n+p}^\infty \frac{(1 + \alpha) - k(\alpha p + \beta) \lambda_k}{(1 + \alpha) - p(\alpha p + \beta)} t_k = \sum_{k=n+p}^\infty \lambda_k = 1 - \lambda_1 < 1
\]

then we conclude that \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta) \).

Conversely, let \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta) \), then by 11

\[
a_k < \frac{(1 + \alpha) - p(\alpha p + \beta)}{(1 + \alpha) - k(\alpha p + \beta) \sigma_k}, \quad k \geq n + p, n \in \mathbb{N},
\]

so, if we set

\[
\lambda_k = \frac{[(1 + \alpha) - k(\alpha p + \beta)] \sigma_k}{(1 + \alpha) - p(\alpha p + \beta)} a_k < 1,
\]

and \( \lambda_1 = 1 - \sum_{k=n+p}^\infty \lambda_k \). Then

\[
f(z) = z^p - \sum_{k=n+p}^\infty \lambda_k z^k
\]

Next, we derive the distribution bound for \( L[\alpha_1] f(V) \).

**Theorem 3.2:** Let \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta) \). Then

\[
r^p - r^{p+n} \frac{(1 + \alpha) - p(\alpha p + \beta)}{(1 + \alpha) - (n + p)(\alpha p + \beta)} \]

\[
< ||L[\alpha_1] f(V)|| < r^p + r^{p+n} \frac{(1 + \alpha) - p(\alpha p + \beta)}{(1 + \alpha) - (n + p)(\alpha p + \beta)}
\]

**Proof:** Since \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta) \), then it follows from 13 that

\[
\sum_{k=n+p}^\infty a_k \sigma_k < \frac{(1 + \alpha) - p(\alpha p + \beta)}{(1 + \alpha) - (n + p)(\alpha p + \beta)}.
\]

Therefore,

\[
||L[\alpha_1] f(V)|| = \left| r^p - \sum_{k=n+p}^\infty a_k \sigma_k r^k \right|
\]

\[
\leq r^p + r^{p+n} \sum_{k=n+p}^\infty a_k \sigma_k
\]

and

\[
||L[\alpha_1] f(V)|| \geq r^p - \sum_{k=n+p}^\infty a_k \sigma_k r^k
\]

\[
\geq r^p - r^{p+n} \sum_{k=n+p}^\infty a_k \sigma_k
\]

\[
> r^p - \frac{(1 + \alpha) - p(\alpha p + \beta)}{(1 + \alpha) - (n + p)(\alpha p + \beta)} r^{p+n}.
\]

4 Radii of Starlikeness and Convexity

Now we obtain the radius of starlikeness and convexity for the class \( \mathcal{M}_p^{(V)}(\alpha, \beta) \).
Theorem 4.1: The radius of starlikeness for the class $f \in \mathcal{M}_p^{(V)}(\alpha, \beta)$ is given by,

$$r_1(\alpha, \beta, \lambda, \mu, p, \gamma) = \inf_k \left[ \frac{[p-\gamma][1+\gamma -k(\alpha p + \beta)]}{1+\gamma} \sigma_k \right]^{1/r}.$$  

Proof: For $0 \leq \gamma < p$, we want to show that

$$\left| \frac{zf'}{f} - p \right| < p - \gamma$$

or equivalently,

$$\frac{\sum_{k=n+p}^{\infty} (k-p)ak^{k-p}}{1 - \sum_{k=n+p}^{\infty} ak^{k-p}} < p - \gamma \Rightarrow \sum_{k=n+p}^{\infty} \frac{k-\gamma}{k-\gamma} ak^{k-p} < 1.$$  

By 11 it is easy to see that the above inequality holds if

$$r^{k-p} < \left[ \frac{(p-\gamma)[(1+\gamma)-k(\alpha p + \beta)]}{(1+\gamma)-p(\beta + \alpha p)} \right] \sigma_k.$$  

Now, since $f$ is convex if and only if $zf'$ is starlike, then we have:

Theorem 4.2: The radius of convexity for the class $f \in \mathcal{M}_p^{(V)}(\alpha, \beta)$ is given by,

$$r_2(\alpha, \beta, \lambda, \mu, p, \gamma) = \inf_k \left[ \frac{(1+\gamma -k(\alpha p + \beta))}{(1+\gamma - p(\beta + \alpha p))} \right]^{1/r}.$$  

5 Neighbourhoods

Now we extend the concept of neighbourhoods of analytic function for the class $\mathcal{M}_p^{(V)}(\alpha, \beta)$. Goodman [6] introduced this concept and, then generalized by Ruscheweyh [10].

Let $\alpha \geq 0, 0 \leq p, \lambda > -1, \delta \geq 0$, we define the $\delta$-neighbourhood of a function $f(z) = z^p - \sum_{k=n+p}^{\infty} ak z^k$ and denote by $N_{\delta, p}^{(\alpha, \mu)}(f)$ consisting of all functions

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$$

satisfying

$$\sum_{k=n+p}^{\infty} \frac{(1+\gamma -k(\alpha p + \beta))}{(1+\gamma - p(\beta + \alpha p))} \sigma_k |a_k - b_k| \leq \delta.$$  

Theorem 5.1: Let $f \in \mathcal{M}_p^{(V)}(\alpha, \beta)$, then $N_{\delta, p}^{(\alpha, \mu)}(f) \subset \mathcal{M}_p^{(V)}(\alpha, \beta)$.  

Before proving this theorem we need the following two lemmas can be found in [11].

Lemma 5.1: If for every complex number $\xi$ with $|\xi| < \delta(0 \leq \delta)$ and $f \in \mathcal{M}_p^{(V)}(\alpha, \beta)$ then $\frac{f(z) + z^p}{1+\xi} \in \mathcal{M}_p^{(V)}(\alpha, \beta)$.  

Lemma 5.2: $f \in \mathcal{M}_p^{(W)}(\alpha, \beta) \iff \frac{f(z) + z^p}{1+\xi} \neq z^p - \sum_{k=n+p}^{\infty} b_k z^k$ and

$$|b_k| \leq \frac{(1+\gamma -k(\alpha p + \beta))}{(1+\gamma - p(\beta + \alpha p))} \sigma_k.$$  

Proof of Theorem 5.1: Since $f \in \mathcal{M}_p^{(W)}(\alpha, \beta)$, then by Lemma 5.1, we have

$$\frac{f(z) + z^p}{1+\xi} \in \mathcal{M}_p^{(V)}(\alpha, \beta),$$

therefore,

$$\left( \frac{f(z) + z^p}{1+\xi} * \psi(z) \right) \neq 0$$

then

$$\left( z^p - \frac{z^p}{1+\xi} * \psi(z) \right) \neq 0,$$

and so

$$\frac{(f * \psi(z))}{(1+\xi)z^p} + \frac{\xi}{1+\xi} \neq 0.$$  

Let $\left| \frac{f(z)}{z^p} \right| < \delta$, then we must have

$$\left| \frac{f(z)}{z^p} \right| \geq \frac{\xi}{1+\xi} - \frac{1}{1+\xi} \left| \frac{f(z)}{z^p} \right| > \frac{\xi}{1+\xi} \geq 0,$$

which is a contradiction with $|\xi| < \delta$, however we have

$$\left| \frac{f(z)}{z^p} \right| > \delta.$$  

Let

$$h(z) = z^p - \sum_{k=n+p}^{\infty} e_k z^k \in N_{\delta, p}^{(\alpha, \mu)}(f),$$

then

$$\delta \geq \left| \frac{f(z)}{z^p} \right| \leq \left| \frac{f(z)}{z^p} \right| - \left| \frac{f(z)}{z^p} \right| \leq \sum_{k=n+p}^{\infty} \left| e_k \right| \left| b_k \right| \left| z^k \right|$$

$$\leq \frac{(1+\gamma -k(\alpha p + \beta))}{(1+\gamma - p(\beta + \alpha p))} \left| a_k - e_k \right| \sigma_k \leq \delta.$$  

So we obtain $\frac{f(z)}{z^p} \neq 0$, and by Lemma 5.2 we have $h \in \mathcal{M}_p^{(V)}(\alpha, \beta)$.

6 Some Properties of Class $\mathcal{M}_p^{(V)}(\alpha, \beta)$

Theorem 6.1: Let $f_1(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ belongs to $\mathcal{M}_p^{(W)}(\alpha, \beta)$ and $0 < \lambda_i < 1$ such that $\sum_{i=1}^{m} \lambda_i = 1$, then the function $G(z) = \sum_{i=1}^{m} \lambda_i f_1(z)$ is also in $\mathcal{M}_p^{(V)}(\alpha, \beta)$.  

Proof: Since $f_1(z) \in \mathcal{M}_p^{(V)}(\alpha, \beta)$, then by 11 we have

$$\sum_{k=n+p}^{\infty} \frac{(1+\gamma -k(\alpha p + \beta))}{(1+\gamma - p(\beta + \alpha p))} a_{k,i} \sigma_k < 1 \quad (i = 1, \cdots, m)$$
\[ G(z) = \sum_{i=1}^{m} \lambda_i f_i(z) = \sum_{i=1}^{m} \lambda_i \left( z^p - \sum_{k=n+p}^{\infty} a_{k,i} z^k \right) \]

\[ = z^p \sum_{i=1}^{m} \lambda_i - \sum_{k=n+p}^{\infty} \left( \sum_{i=1}^{m} \lambda_i a_{k,i} \right) z^k. \]

Now

\[ \sum_{k=n+p}^{\infty} \frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)} \left( \sum_{i=1}^{m} \lambda_i a_{k,i} \right) \sigma_k \]

\[ = \sum_{i=1}^{m} \lambda_i \left[ \sum_{k=n+p}^{\infty} \frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)} a_{k,i} \sigma_k \right] < \sum_{i=1}^{m} \lambda_i = 1, \]

then \( G(z) \in \mathcal{M}_p^{(V)}(\alpha, \beta). \)

Here we introduce an integral operator due to Bernardi [1]

\[ L_\nu[f] = \frac{p+e}{e^\alpha} \int_0^\infty t^p e^{-1} dt \quad (e = -p). \]

Theorem 6.2: If \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta). \) Then \( L_\nu[f] \) also belongs to \( \mathcal{M}_p^{(V)}(\alpha, \beta). \)

Proof: Let \( f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \) then

\[ L_\nu[f] = \frac{p+e}{e^\alpha} \int_0^\infty t^p e^{-1} dt \left( \sum_{k=n+p}^{\infty} a_k z^k \right) \int_0^1 t^e dt \]

\[ = \frac{p+e}{e^\alpha} \left[ \left( \frac{1}{p+e} - \sum_{k=n+p}^{\infty} \frac{1}{k+e} a_k t^k \right) \right] \pmb{1}_0 \]

\[ = z^p - \sum_{k=n+p}^{\infty} \frac{p+e}{k+e} a_k z^k. \]

Since \( e = -p, k \geq n + p > p, \) then \( \frac{p+e}{k+e} \leq 1. \) So we have

\[ \sum_{k=n+p}^{\infty} \frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)} \left( \frac{p+e}{k+e} \right) \sigma_k a_k \]

\[ \leq \sum_{k=n+p}^{\infty} \frac{(1+\alpha)-k(\alpha p+\beta)}{(1+\alpha)-p(\alpha p+\beta)} \sigma_k a_k < 1. \]

By assumption \( f \in \mathcal{M}_p^{(V)}(\alpha, \beta). \) Thus \( L_\nu[f] \in \mathcal{M}_p^{(V)}(\alpha, \beta). \)

References


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