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Generalizations of Derivations in *BCI*-Algebras

G. Muhiuddin[1,*] *and Abdullah M. Al-roqi*[2]

[1] Department of Mathematics, University of Tabuk, P. O. Box 741, Tabuk 71491, Saudi Arabia
[2] Department of Mathematics, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

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Abstract: In the present paper we introduced the notion of (θ,ϕ)-derivations of a *BCI*-algebra X. Some interesting results on inside (or outside) (θ,ϕ)-derivations in BCI-algebras are discussed. It is shown that for any commutative *BCI*-algebra X, every inside (θ,ϕ)-derivation of X is isotone. Furthermore it is also proved that for any outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a *BCI*-algebra X, $d_{(\theta,\phi)}(x) = \theta(x)\wedge d_{(\theta,\phi)}(x)$ if and only if $d_{(\theta,\phi)}(0) = 0$ for all $x\in X$.

Keywords: BCI-algebras, inside (θ,ϕ)-derivation, outside (θ,ϕ)-derivation, (θ,ϕ)-derivation

1 Introduction

Several authors have studied derivations in rings and near-rings (see for example, [1], [2], [12] and [14], where further references can be found). Jun and Xin [9] applied the notion of derivation in ring and near-ring theory to *BCI*-algebras, and as a result they introduced a new concept, called a (regular) derivation, in *BCI*-algebras. Using this concept as defined they investigated some of its properties. As in [9], a self map $d : X \to X$ is called a left-right derivation (briefly (l,r)-derivation)of X if $d(x*y) = d(x)*y\wedge x*d(y)$ holds for all $x,y\in X$. Similarly, a self map $d : X\to X$ is called a right-left derivation (briefly (r,l)-derivation) of X if $d(x*y) = x*d(y)\wedge d(x)*y$ holds for all $x,y\in X$. Moreover, if d is both $(l,r)-$ and $(r,l)-$derivation, it is a derivation on X. Later on, Zhan and Lui [16] introduced the notion of left-right (or right-left) f-derivation of a *BCI*-algebra, and investigated some related properties. Using the idea of regular f-derivation, they gave characterizations of a p-semisimple *BCI*-algebra. Following [16], a self map $d_f : X\to X$ is said to be a left-right f-derivation or (l,r)-f-derivation of X if it satisfies the identity $d_f(x*y) = d_f(x)*f(y)\wedge f(x)*d_f(y)$ for all $x,y\in X$. Similarly, a self map $d_f : X\to X$ is said to be a right-left f-derivation or (r,l)-f-derivation of X if it satisfies the identity $d_f(x*y) = f(x)*d_f(y)\wedge d_f(x)*f(y)$ for all $x,y\in X$. Moreover, if d_f is both (l,r) and (r,l)-f-derivation, it is said that d_f is an $f-$derivation, where f is an endomorphism. Recently, a number of research papers have been devoted to the study of various kinds of derivations in BCI-algebras (see for example, [4], [5], [6], [7], and [8], where further references can be found).

In this paper, we introduce the notion of (θ,ϕ)-derivations of a *BCI*-algebra X and discuss some interesting results on inside (or outside) (θ,ϕ)-derivations in a BCI-algebra X. In the sequel, we obtain that every inside (θ,ϕ)-derivation of X is isotone if X is commutative *BCI*-algebra. Furthermore, it is also prove that for any outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a *BCI*-algebra X, $d_{(\theta,\phi)}(x) = \theta(x)\wedge d_{(\theta,\phi)}(x)$ if and only if $d_{(\theta,\phi)}(0) = 0$ for all $x\in X$.

2 Preliminaries

A nonempty set X with a constant 0 and a binary operation $*$ is called a *BCI-algebra* if for all $x,y,z\in X$ the following conditions hold:

(I)$((x*y)*(x*z))*(z*y) = 0$,
(II)$(x*(x*y))*y = 0$,
(III)$x*x = 0$,
(IV)$x*y = 0$ and $y*x = 0$ imply $x = y$.

A *BCI*-algebra X has the following properties: for all $x,y,z\in X$

(a1)$x*0 = x$.

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* Corresponding author e-mail: chishtygm@gmail.com

(a2) $(x*y)*z=(x*z)*y$.
(a3) $x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$.
(a4) $(x*z)*(y*z)\leq x*y$.
(a5) $x*(x*(x*y))=x*y$.
(a6) $0*(x*y)=(0*x)*(0*y)$.
(a7) $x*0=0$ implies $x=0$.

For a *BCI*-algebra X, denote by X_+ (resp. $G(X)$) the *BCK*-part (resp. the *BCI*-G part) of X, i.e., X_+ is the set of all $x\in X$ such that $0\leq x$ (resp. $G(X):=\{x\in X \mid 0*x=x\}$). Note that $G(X)\cap X_+=\{0\}$ (see [11]). If $X_+=\{0\}$, then X is called a *p-semisimple BCI-algebra*. In a *p*-semisimple *BCI*-algebra X, the following hold:

(a8) $(x*z)*(y*z)=x*y$.
(a9) $0*(0*x)=x$ for all $x\in X$.
(a10) $x*(0*y)=y*(0*x)$.
(a11) $x*y=0$ implies $x=y$.
(a12) $x*a=x*b$ implies $a=b$.
(a13) $a*x=b*x$ implies $a=b$.
(a14) $a*(a*x)=x$.

Let X be a *p*-semisimple *BCI*-algebra. We define addition "+" as $x+y=x*(0*y)$ for all $x,y\in X$. Then $(X,+)$ is an abelian group with identity 0 and $x-y=x*y$. Conversely let $(X,+)$ be an abelian group with identity 0 and let $x*y=x-y$. Then X is a *p*-semisimple *BCI*-algebra and $x+y=x*(0*y)$ for all $x,y\in X$ (see [13]).

For a *BCI*-algebra X we denote $x\wedge y=y*(y*x)$, in particular $0*(0*x)=a_x$, and $L_p(X):=\{a\in X \mid x*a=0 \Rightarrow x=a, \forall x\in X\}$. We call the elements of $L_p(X)$ the *p-atoms* of X. For any $a\in X$, let $V(a):=\{x\in X \mid a*x=0\}$, which is called the *branch* of X with respect to a. It follows that $x*y\in V(a*b)$ whenever $x\in V(a)$ and $y\in V(a)$ for all $x,y\in X$ and all $a,b\in L_p(X)$. Note that $L_p(X)=\{x\in X \mid a_x=x\}$, which is the *p*-semisimple part of X, and X is a *p*-semisimple *BCI*-algebra if and only if $L_p(X)=X$ (see [10, Proposition 3.2]). Note also that $a_x\in L_p(X)$, i.e., $0*(0*a_x)=a_x$, which implies that $a_x*y\in L_p(X)$ for all $y\in X$. It is clear that $G(X)\subset L_p(X)$, and $x*(x*a)=a$ and $a*x\in L_p(X)$ for all $a\in L_p(X)$ and all $x\in X$. For more details, refer to [3], [10], [11], [13].

3 Generalizations of derivations in *BCI*-algebras

In what follows, θ and ϕ are endomorphisms of a *BCI*-algebra X unless otherwise specified.

Definition 1. *A self map $d_{(\theta,\phi)}$ of a BCK/BCI-algebra X is called*

(1) *an inside (θ,ϕ)-derivation of X if it satisfies:* $\left(d_{(\theta,\phi)}(x*y)=\left(d_{(\theta,\phi)}(x)*\theta(y)\right)\wedge\left(\phi(x)*d_{(\theta,\phi)}(y)\right)\right)$ *for all $x,y\in X$.*
(2) *an outside (θ,ϕ)-derivation of X if it satisfies:* $\left(d_{(\theta,\phi)}(x*y)=\left(\theta(x)*d_{(\theta,\phi)}(y)\right)\wedge\left(d_{(\theta,\phi)}(x)*\phi(y)\right)\right)$ *for all $x,y\in X$.*
(3) *a (θ,ϕ)-derivation of X if it is both an inside (θ,ϕ)-derivation and an outside (θ,ϕ)-derivation.*

Note that if $\theta=\phi=f$, then the inside (θ,ϕ)-derivation of a *BCK/BCI*-algebra X is an (l,r)-f-derivation of a *BCK/BCI*-algebra X and the outside (θ,ϕ)-derivation of a *BCK/BCI*-algebra X is an (r,l)-f-derivation of a *BCK/BCI*-algebra X. In this case, $d_{(\theta,\phi)}$ is denoted by d_f.

Example 1. Consider a *BCI*-algebra $X=\{0,a,b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define a map

$$d_{(\theta,\phi)}:X\to X,\ x\mapsto\begin{cases}b \text{ if } x\in\{0,a\},\\ 0 \text{ if } x=b,\end{cases}$$

and define two endomorphisms

$$\theta:X\to X,\ x\mapsto\begin{cases}0 \text{ if } x\in\{0,a\},\\ b \text{ if } x=b,\end{cases}$$

and

$$\phi:X\to X \text{ such that } \phi(x)=x \text{ for all } x\in X.$$

It is routine to verify that $d_{(\theta,\phi)}$ is both an inside (θ,ϕ)-derivation and an outside (θ,ϕ)-derivation of X.

Lemma 1([3]). *Let X be a BCI-algebra. For any $x,y\in X$, if $x\leq y$, then x and y are contained in the same branch of X.*

Lemma 2([3]). *Let X be a BCI-algebra. For any $x,y\in X$, if x and y are contained in the same branch of X, then $x*y,\ y*x\in X_+$.*

Proposition 1. *Let X be a commutative BCI-algebra. Then every inside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of X satisfies the following assertion:*

$$(\forall x,y\in X)\left(x\leq y\ \Rightarrow\ d_{(\theta,\phi)}(x)\leq d_{(\theta,\phi)}(y)\right), \quad (1)$$

that is, every inside (θ,ϕ)-derivation of X is isotone.

Proof. Let $x,y\in X$ be such that $x\leq y$. Since X is commutative, we have $x=x\wedge y$. Hence

$$\begin{aligned}d_{(\theta,\phi)}(x)&=d_{(\theta,\phi)}(x\wedge y)\\&=\left(d_{(\theta,\phi)}(y)*\theta(y*x)\right)\wedge\left(\phi(y)*d_{(\theta,\phi)}(y*x)\right)\\&\leq\left(d_{(\theta,\phi)}(y)*\theta(y*x)\right)\end{aligned} \quad (2)$$

Since every endomorphism of X is isotone, we have $\theta(x) \leq \theta(y)$. It follows from Lemmas 1 and 2 that $0 = \theta(x) * \theta(y) \in X_+$ and $\theta(y) * \theta(x) \in X_+$ so that there exists $a(\neq 0) \in X_+$ such that $\theta(y * x) = \theta(y) * \theta(x) = a$. Hence (2) implies that $d_{(\theta,\phi)}(x) \leq d_{(\theta,\phi)}(y) * a$. Using (a3), (a2) and (III), we have

$$\begin{aligned} d_{(\theta,\phi)}(x) * d_{(\theta,\phi)}(y) &\leq \left(d_{(\theta,\phi)}(y) * a\right) * d_{(\theta,\phi)}(y) \\ &= \left(d_{(\theta,\phi)}(y) * d_{(\theta,\phi)}(y)\right) * a = 0 * a = 0, \end{aligned}$$

and so $d_{(\theta,\phi)}(x) * d_{(\theta,\phi)}(y) = 0$ by (a7), that is, $d_{(\theta,\phi)}(x) \leq d_{(\theta,\phi)}(y)$.

Proposition 2. *Every inside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X)\left(d_{(\theta,\phi)}(x) = d_{(\theta,\phi)}(x) \wedge \phi(x)\right). \tag{3}$$

Proof. Let $d_{(\theta,\phi)}$ be an inside (θ,ϕ)-derivation of X. Using (a2) and (a4), we have

$$\begin{aligned} d_{(\theta,\phi)}(x) &= d_{(\theta,\phi)}(x * 0) \\ &= \left(d_{(\theta,\phi)}(x) * \theta(0)\right) \wedge \left(\phi(x) * d_{(\theta,\phi)}(0)\right) \\ &= \left(d_{(\theta,\phi)}(x) * 0\right) \wedge \left(\phi(x) * d_{(\theta,\phi)}(0)\right) \\ &= d_{(\theta,\phi)}(x) \wedge \left(\phi(x) * d_{(\theta,\phi)}(0)\right) \\ &= \left(\phi(x) * d_{(\theta,\phi)}(0)\right) * \left(\left(\phi(x) * d_{(\theta,\phi)}(0)\right) * d_{(\theta,\phi)}(x)\right) \\ &= \left(\phi(x) * d_{(\theta,\phi)}(0)\right) * \left(\left(\phi(x) * d_{(\theta,\phi)}(x)\right) * d_{(\theta,\phi)}(0)\right) \\ &\leq \phi(x) * \left(\phi(x) * d_{(\theta,\phi)}(x)\right) \\ &= d_{(\theta,\phi)}(x) \wedge \phi(x) \end{aligned}$$

Obviously $d_{(\theta,\phi)}(x) \wedge \phi(x) \leq d_{(\theta,\phi)}(x)$ by (II). Therefore the equality (3) is valid.

If we take $\theta = \phi = 1_X$ in Proposition 2 where 1_X is the identity map, then we have the following corollary.

Corollary 1([9]). *Every (l,r)-derivation d of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X)\left(d(x) = d(x) \wedge x\right).$$

If we take $\theta = \phi = f$, then we have the following corollary.

Corollary 2([16]). *Every (l,r)-f-derivation d_f of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X)\left(d_f(x) = d_f(x) \wedge f(x)\right).$$

Proposition 3. *For any outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCI-algebra X, the following are equivalent:*

(1) $(\forall x \in X)\left(d_{(\theta,\phi)}(x) = \theta(x) \wedge d_{(\theta,\phi)}(x)\right)$.
(2) $d_{(\theta,\phi)}(0) = 0$.

Proof. (1) $\Rightarrow$ (2) is straightforward by taking $x = 0$.

Assume that (2) is valid. Then

$$\begin{aligned} d_{(\theta,\phi)}(x) &= d_{(\theta,\phi)}(x * 0) \\ &= \left(\theta(x) * d_{(\theta,\phi)}(0)\right) \wedge \left(d_{(\theta,\phi)}(x) * \phi(0)\right) \\ &= \left(\theta(x) * 0\right) \wedge \left(d_{(\theta,\phi)}(x) * 0\right) \\ &= \theta(x) \wedge d_{(\theta,\phi)}(x). \end{aligned}$$

This completes the proof.

If we take $\theta = \phi = 1_X$ in Proposition 3 where 1_X is the identity map, then we have the following corollary.

Corollary 3([9]). *For any (r,l)-derivation d of a BCI-algebra X, the following are equivalent:*

(1) $(\forall x \in X)\left(d(x) = x \wedge d(x)\right)$.
(2) $d(0) = 0$.

If we take $\theta = \phi = f$, then we have the following corollary.

Corollary 4([16]). *For any (r,l)-f-derivation d_f of a BCI-algebra X, the following are equivalent:*

(1) $(\forall x \in X)\left(d_f(x) = f(x) \wedge d_f(x)\right)$.
(2) $d_f(0) = 0$.

Proposition 4. *Let $d_{(\theta,\phi)}$ be an inside (θ,ϕ)-derivation of a BCI-algebra X. Then*

(1) *$d_{(\theta,\phi)}(0)$ is a p-atom of X.*
(2) $(\forall a \in X)\left(a \in L_p(X) \Rightarrow \theta(a),\ \phi(a) \in L_p(X)\right)$.
(3) $(\forall a \in L_p(X))\left(d_{(\theta,\phi)}(a) = d_{(\theta,\phi)}(0) + \theta(a)\right)$.
(4) $(\forall a \in X)\left(a \in L_p(X) \Rightarrow d_{(\theta,\phi)}(a) \in L_p(X)\right)$.
(5) $\left(d_{(\theta,\phi)}(a+b) = d_{(\theta,\phi)}(a) + d_{(\theta,\phi)}(b) - d_{(\theta,\phi)}(0)\right)$ *for all $a, b \in L_p(X)$.*

Proof. (1) follows from (3) by taking $x = 0$.

(2) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$, and so $\theta(a) = \theta(0 * (0 * a)) = 0 * (0 * \theta(a))$. Thus $\theta(a) \in L_p(X)$. Similarly, $\phi(a) \in L_p(X)$.

(3) Let $a \in L_p(X)$. Using (2), (a2) and (a8), we have

$$\begin{aligned} d_{(\theta,\phi)}(a) &= d_{(\theta,\phi)}(0 * (0 * a)) \\ &= \left(d_{(\theta,\phi)}(0) * \theta(0 * a)\right) \wedge \left(\phi(0) * d_{(\theta,\phi)}(0 * a)\right) \\ &= \left(d_{(\theta,\phi)}(0) * \theta(0 * a)\right) \wedge \left(0 * d_{(\theta,\phi)}(0 * a)\right) \\ &= \left(0 * d_{(\theta,\phi)}(0 * a)\right) * \left(\left(0 * d_{(\theta,\phi)}(0 * a)\right) * \right. \\ &\qquad \left. \left(d_{(\theta,\phi)}(0) * \theta(0 * a)\right)\right) \\ &= \left(0 * d_{(\theta,\phi)}(0 * a)\right) * \left(\left(0 * \left(d_{(\theta,\phi)}(0) * \theta(0 * a)\right)\right) * \right. \\ &\qquad \left. d_{(\theta,\phi)}(0 * a)\right) \\ &= 0 * \left(0 * \left(d_{(\theta,\phi)}(0) * \theta(0 * a)\right)\right) \\ &= 0 * \left(0 * \left(d_{(\theta,\phi)}(0) * \left(\theta(0) * \theta(a)\right)\right)\right) \\ &= 0 * \left(0 * \left(d_{(\theta,\phi)}(0) * \left(0 * \theta(a)\right)\right)\right) \\ &= d_{(\theta,\phi)}(0) * \left(0 * \theta(a)\right) \\ &= d_{(\theta,\phi)}(0) + \theta(a). \end{aligned}$$

(4) It follows directly from (1) and (3).

(5) Let $a,b \in L_p(X)$. Then $a+b \in L_p(X)$. Using (3), we have

$$\begin{aligned} d_{(\theta,\phi)}(a+b) &= d_{(\theta,\phi)}(0)+\theta(a+b) \\ &= d_{(\theta,\phi)}(0)+\theta(a)+\theta(b) \\ &= d_{(\theta,\phi)}(0)+\theta(a)+d_{(\theta,\phi)}(0)+\theta(b)-d_{(\theta,\phi)}(0) \\ &= d_{(\theta,\phi)}(a)+d_{(\theta,\phi)}(b)-d_{(\theta,\phi)}(0). \end{aligned}$$

This completes the proof.

If we take $\theta=\phi=1_X$ in Proposition 4 where 1_X is the identity map, then we have the following corollary.

Corollary 5([9]). *Let d be an (l,r)-derivation of a BCI-algebra X. Then*

(1)$d(0) \in L_p(X)$, *i.e.,* $d(0)=0*(0*d(0))$.
(2)$(\forall a \in L_p(X))\,(d(a)=d(0)*(0*a)=d(0)+a)$.
(3)$(\forall a \in X)\,(a \in L_p(X) \Rightarrow d(a) \in L_p(X))$.
(4)$(\forall a,b \in L_p(X))\,(d(a+b)=d(a)+d(b)-d(0))$.

If we take $\theta=\phi=f$ in Proposition 4, then we have the following corollary.

Corollary 6([16]). *Let d_f be an (l,r)-f-derivation of a BCI-algebra X. Then*

(1)$d_f(0)$ *is a p-atom of* X.
(2)$(\forall a \in L_p(X))\left(d_f(a)=d_f(0)+f(a)\right)$.
(3)$(\forall a \in X)\left(a \in L_p(X) \Rightarrow d_f(a) \in L_p(X)\right)$.
(4)$(\forall a,b \in L_p(X))\left(d_f(a+b)=d_f(a)+d_f(b)-d_f(0)\right)$.

Proposition 5. *For any outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCI-algebra X, we have the following assertions:*

(1)$(\forall a \in X)\left(a \in G(X) \Rightarrow d_{(\theta,\phi)}(a) \in G(X)\right)$.
(2)$(\forall a \in X)\left(a \in L_p(X) \Rightarrow d_{(\theta,\phi)}(a) \in L_p(X)\right)$.
(3)$a \in L_p(X)$ *implies*

$$d_{(\theta,\phi)}(a)=\theta(a)*d_{(\theta,\phi)}(0)=\theta(a)+d_{(\theta,\phi)}(0)$$

for all $a \in X$.
(4)$d_{(\theta,\phi)}(a+b)=d_{(\theta,\phi)}(a)+d_{(\theta,\phi)}(b)-d_{(\theta,\phi)}(0)$ *for all* $a,b \in L_p(X)$.
(5)*If θ is the identity map on X, then $d_{(\theta,\phi)}$ is identity on $L_p(X)$ if and only if $d_{(\theta,\phi)}(0)=0$.*

Proof.(1) Let $a \in G(X)$. Then $0*a=a$, and so

$$\begin{aligned} d_{(\theta,\phi)}(a) &= d_{(\theta,\phi)}(0*a) \\ &= \left(\theta(0)*d_{(\theta,\phi)}(a)\right) \wedge \left(d_{(\theta,\phi)}(0)*\phi(a)\right) \\ &= \left(d_{(\theta,\phi)}(0)*\phi(a)\right)*\Big(\left(d_{(\theta,\phi)}(0)*\phi(a)\right)* \\ &\qquad \left(\theta(0)*d_{(\theta,\phi)}(a)\right)\Big) \\ &= \left(d_{(\theta,\phi)}(0)*\phi(a)\right)*\Big(\left(d_{(\theta,\phi)}(0)*\phi(a)\right)* \\ &\qquad \left(0*d_{(\theta,\phi)}(a)\right)\Big) \\ &= 0*d_{(\theta,\phi)}(a) \end{aligned}$$

since $0*d_{(\theta,\phi)}(a) \in L_p(X)$. Hence $d_{(\theta,\phi)}(a) \in G(X)$.

(2) For any $a \in L_p(X)$, we have

$$\begin{aligned} d_{(\theta,\phi)}(a) &= d_{(\theta,\phi)}(0*(0*a)) \\ &= \left(\theta(0)*d_{(\theta,\phi)}(0*a)\right) \wedge \left(d_{(\theta,\phi)}(0)*\phi(0*a)\right) \\ &= \left(0*d_{(\theta,\phi)}(0*a)\right) \wedge \left(d_{(\theta,\phi)}(0)*\phi(0*a)\right) \\ &= \left(d_{(\theta,\phi)}(0)*\phi(0*a)\right)*\Big(\left(d_{(\theta,\phi)}(0)*\phi(0*a)\right)* \\ &\qquad \left(0*d_{(\theta,\phi)}(0*a)\right)\Big) \\ &= 0*d_{(\theta,\phi)}(0*a) \in L_p(X). \end{aligned}$$

(3) For any $a \in L_p(X)$, we have

$$\begin{aligned} d_{(\theta,\phi)}(a) &= d_{(\theta,\phi)}(a*0) \\ &= \left(\theta(a)*d_{(\theta,\phi)}(0)\right) \wedge \left(d_{(\theta,\phi)}(a)*\phi(0)\right) \\ &= \left(\theta(a)*d_{(\theta,\phi)}(0)\right) \wedge \left(d_{(\theta,\phi)}(a)*0\right) \\ &= \left(d_{(\theta,\phi)}(a)*0\right)*\left(\left(d_{(\theta,\phi)}(a)*0\right)*\left(\theta(a)*d_{(\theta,\phi)}(0)\right)\right) \\ &= d_{(\theta,\phi)}(a)*\left(d_{(\theta,\phi)}(a)*\left(\theta(a)*d_{(\theta,\phi)}(0)\right)\right) \\ &= \theta(a)*d_{(\theta,\phi)}(0)=\theta(a)*\left(0*d_{(\theta,\phi)}(0)\right) \\ &= \theta(a)+d_{(\theta,\phi)}(0) \end{aligned}$$

since $\theta(a)*d_{(\theta,\phi)}(0) \in L_p(X)$ and $d_{(\theta,\phi)}(0) \in G(X)$.

(4) If $a,b \in L_p(X)$, then $a+b \in L_p(X)$. Using (3), we have

$$\begin{aligned} d_{(\theta,\phi)}(a+b) &= \theta(a+b)+d_{(\theta,\phi)}(0) \\ &= \theta(a)+\theta(b)+d_{(\theta,\phi)}(0) \\ &= \theta(a)+d_{(\theta,\phi)}(0)+\theta(b)+d_{(\theta,\phi)}(0)-d_{(\theta,\phi)}(0) \\ &= d_{(\theta,\phi)}(a)+d_{(\theta,\phi)}(b)-d_{(\theta,\phi)}(0). \end{aligned}$$

(5) It follows from (3).

If we take $\theta=\phi=1_X$ in Proposition 5 where 1_X is the identity map, then we have the following corollary.

Corollary 7([9]). *For any (r,l)-derivation d of a BCI-algebra X, we have the following assertions:*

(1)$(\forall a \in X)\,(a \in G(X) \Rightarrow d(a) \in G(X))$.
(2)$(\forall a \in X)\,(a \in L_p(X) \Rightarrow d(a) \in L_p(X))$.
(3)$a \in L_p(X) \Rightarrow d(a)=a*d(0)=a+d(0)$ *for all* $a \in X$.
(4)$(\forall a,b \in L_p(X))\,(d(a+b)=d(a)+d(b)-d(0))$.
(5)*d is identity on $L_p(X)$ if and only if $d(0)=0$.*

If we take $\theta=\phi=f$ in Proposition 5, then we have the following corollary.

Corollary 8([16]). *For any (r,l)-f-derivation d_f of a BCI-algebra X, we have the following assertions:*

(1)$(\forall a \in X)\left(a \in G(X) \Rightarrow d_f(a) \in G(X)\right)$.
(2)$(\forall a \in X)\left(a \in L_p(X) \Rightarrow d_f(a) \in L_p(X)\right)$.
(3)$a \in L_p(X) \Rightarrow d_f(a)=f(a)*d_f(0)=f(a)+d_f(0)$ *for all* $a \in X$.
(4)$(\forall a,b \in L_p(X))\left(d_f(a+b)=d_f(a)+d_f(b)-d_f(0)\right)$.
(5)*If f is the identity map on X, then d_f is identity on $L_p(X)$ if and only if $d_f(0)=0$.*

4 Applications

Over the past decade, the theory of derivation in algebraic structure more specifically in logical algebras becomes much more interesting research area in mathematics. Our present research emphasizes on derivations of BCK/BCI-algebras. BCK-algebras and BCI-algebras are algebraic formulation of BCK-system and BCI-system in combinatory logic. As it is well know that the notion of BCI-algebra is a generalization of a BCK-algebra. Therefore, most of the algebras related to t-norm base login such as MTL-algebras, BL-algebras, Hoop-algebras, MV-algebras and Boolean algebras etc. are extensions of BCK-algebras i.e. they are subclasses of BCK-algebras which have lot of applications in computer science(see [15]).

5 Conclusion

In the present paper, we have considered the notion of (θ,ϕ)-derivations in BCI-algebras and investigated its useful properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras etc. In future we can study the notion of (θ,ϕ)-derivations on various algebraic structures which may have a lot of applications in different branches of theoretical physics, engineering and computer science. It is our hope that this work would serve as a foundation for the further study in the theory of derivations of BCK/BCI-algebras.

In our future study of (θ,ϕ)-derivations in BCI-algebras, may be the following topics should be considered:

1. To introduce the concepts of a $d_{(\theta,\phi)}$-invariant inside (or outside) (θ,ϕ)-derivation, a regular inside (or outside) (θ,ϕ)-derivation, and θ-ideal.
2. To provide conditions for an inside (or outside) (θ,ϕ)-derivation to be regular.
3. To find the generalized (θ,ϕ)-derivations of BCI-algebras.
4. To consider conditions for an outside (θ,ϕ)-derivation to be regular.
5. To find more results in (θ,ϕ)-derivations of BCI-algebras and its applications.
6. To find the (θ,ϕ)-derivations of B-algebras, Q-algebras, subtraction algebras, d-algebras etc.

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G. Muhiuddin is working at Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia as an Assistant Professor. He received a Ph.D. degree in Pure Mathematics. His mathematical research areas include Algebras related to logic (BCK, BCI, BCC-algebras, Hilbert algebras, implication algebras), Fuzzy logical algebras and Category theory. He has published a number of research papers in internationally reputed mathematical journals. He has served as an expert referee for several research papers in these areas of his expertise.

Abdullah M. Al-roqi is working at Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia as an Associate Professor. He has received his Ph.D. degree in Mathematics from School Of Mathematics and Statistics, University of Birmingham, United Kingdom. His mathematical research areas are Algebras related to logic, Fuzzy logical algebras, Finite Group Theory, Soluble groups, Classification of finite simple groups and Representation Theory.