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# A New Generalized Lindley Distribution

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**Abstract:** In the present paper, we propose a new generalized Lindley distribution, which generalizes the Lindley distribution and many extensions and generalization of it as proposed by various authors and researchers in recent years. Various properties of the generalized Lindley distribution are provided. We have also investigated the characterization of the generalized Lindley distribution by first truncated moment. The characterizations by order statistics and upper record values have also been investigated.

**Keywords:** Characterization, Generalized Lindley distribution, truncated moments

## 1 Introduction

Modeling and analyzing of real lifetime data play very important roles in many fields of research such as biology, computer science, control theory, economics, engineering, genetics, hydrology, medicine, number theory, statistics, physics, psychology, reliability, risk management, etc., among others. The Lindley distribution is one of the most important continuous probability distributions, and has been extensively studied and widely used by many researchers in the fields of probability, statistics and other applied sciences since it was introduced as a mixture of exponential and gamma distributions by Lindley in 1958 in the context of Bayesian statistics, as a counter example to fiducial statistics and because of its superiority over exponential distribution; see [1,2]. In the present paper, we propose a new generalized Lindley distribution (GLD), which generalizes the Lindley 1958 distribution and many extension and generalization of Lindley distribution as proposed by various authors and researchers in recent years, among them, [3,4,5,6,7,8,9,10,11,12,13] and [14,15] are notable.

The organization of the paper is as follows. In Section 2, we introduce the proposed generalized Lindley distribution (GLD), and discuss its various distributional properties. In Section 3, we provide the characterizations of our proposed generalized Lindley distribution. Some concluding remarks are given in Section 4.

## 2 The Proposed Generalized Lindley Distribution

In this section, we introduce our proposed generalized Lindley distribution, and discuss its various distributional properties, which are described below.

### *2.1 Probability Density, Cumulative Distribution and Hazard Rate Functions:*

For a positive continuous random variable $X$ with scale parameter $\theta > 0$ and shape parameters $\alpha, \beta, \gamma, \lambda > 0$ , we define its probability density function (pdf) by

$$f(x, \alpha, \beta, \gamma, \lambda, \theta) = c\theta^{\alpha+1} x^{\alpha-1} (\beta + \gamma x^{\lambda}) e^{-\theta x}, \; x, \alpha, \beta, \gamma, \lambda, \theta > 0, \tag{2.1}$$

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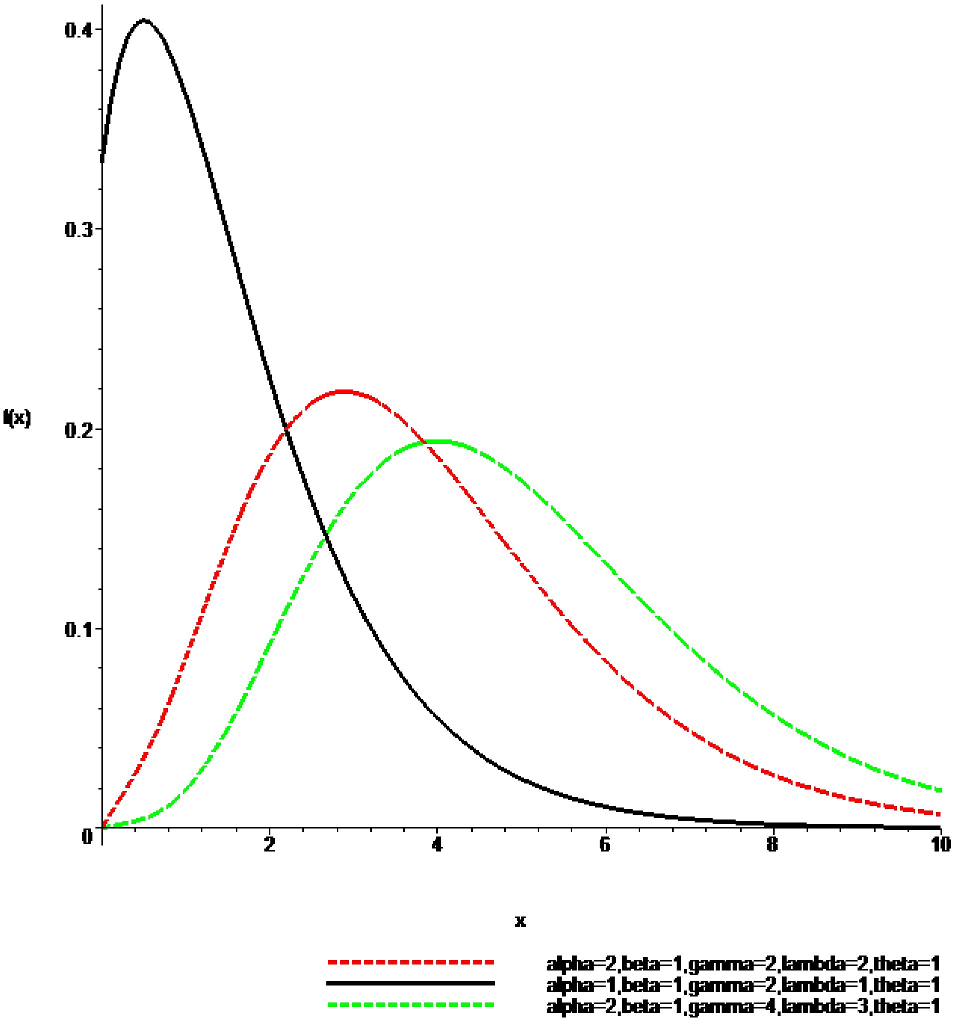


**Fig. 1:** PDF's: Behavior of the pdf of GLD for various values of the parameters

where $c$ denotes the normalizing constant given by

$$\frac{1}{c} = \theta\left(\beta\Gamma(\alpha)+\gamma\frac{\Gamma(\alpha+\lambda)}{\theta^{\lambda}}\right), \tag{2.2}$$

where $\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx,(\alpha>0)$ denotes the gamma function. We denote the random variable having the pdf as given in (2.1) as $X(\alpha,\beta,\gamma,\lambda,\theta)$ . By integrating (2.1) and simplifying, it is easily seen that $\int_0^{\infty}f_X(x)\,d\,x=1$ . So, (2.1) defines a valid pdf. For example, when $\alpha=1,\beta=1,\gamma=1,\lambda=1$ and $\theta=1$, it is easily seen, after integration, that the $f(x)$ given by (2.1) is the pdf of the generalized Lindley distribution. The graphs of the pdf $f(x,\alpha,\beta,\gamma,\lambda,\theta)$ for some selected values of the parameters are given in Figure 2.1. The effects of the parameters can easily be seen from these graphs. For example, it is clear from the plotted Figure 2.1, for selected values of the parameters, that the distributions of the $X(\alpha,\beta,\gamma,\lambda,\theta)$ are unimodal and positively right skewed with longer and heavier right tails.

The corresponding cumulative distribution function (CDF) $f(x,\alpha,\beta,\gamma,\lambda,\theta)$ is

$$\begin{aligned}F(x) &= c\theta\left[\beta\gamma(\alpha,\theta x)+\frac{\gamma}{\theta^{\lambda}}\gamma(\alpha+\lambda,\theta x)\right]\\ &= c\theta\left[\beta\left(\Gamma(\alpha)-\Gamma(\alpha,\theta x)\right)+\frac{\gamma}{\theta^{\lambda}}\left(\Gamma(\alpha+\lambda)-\Gamma(\alpha+\lambda,\theta x)\right)\right]\end{aligned}$$

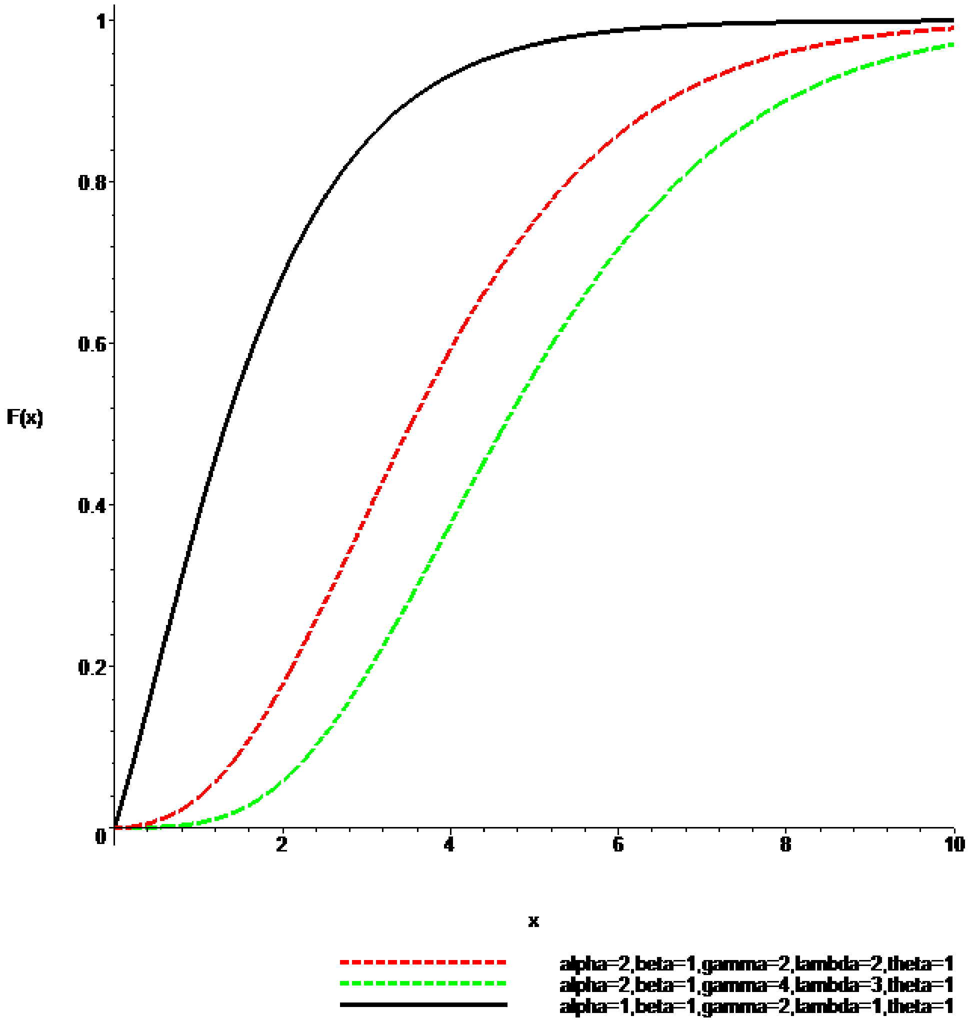


**Fig. 2:** CDF's of GLD for some selected values of the parameters

where $c$ denotes the normalizing constant given by equation (2.2), and $\gamma(\alpha,x) = \int_0^x x^{\alpha-1}e^{-x}dx$ and $\Gamma_\alpha(x) = \int_x^\infty x^{\alpha-1}e^{-x}dx$ denote the lower and upper incomplete gamma functions, respectively, with $\gamma(\alpha,x)+\Gamma(\alpha,x)=\Gamma(\alpha)$ ; see, for example, [16]. The graphs of the cdf $f(x,\alpha,\beta,\gamma,\lambda,\theta)$ are given in Figure 2.2 for some selected values of the parameters.

The hazard rate $h(x,\alpha,\beta,\gamma,\lambda,\theta)$ of $X(\alpha,\beta,\gamma,\lambda,\theta)$ is given by

$$h(x,\alpha,\beta,\gamma,\lambda,\theta) = \frac{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^\lambda)e^{-\theta x}}{1-c\left[\theta\beta\left(\Gamma(\alpha)-\Gamma_\alpha(\theta x)\right)+\theta^{1-\lambda}\gamma\left(\Gamma(\alpha+\lambda)-\Gamma_{(\alpha+\lambda)}(\theta x)\right)\right]}$$

The hazard rates are given in Fig. 2.3 for some particular values for $\alpha,\beta,\gamma,\lambda,\theta$.

## 2.2 Moments

The moment generating function $M_X(t)$ of $X(\alpha,\beta,\gamma,\lambda,\theta)$ is given by

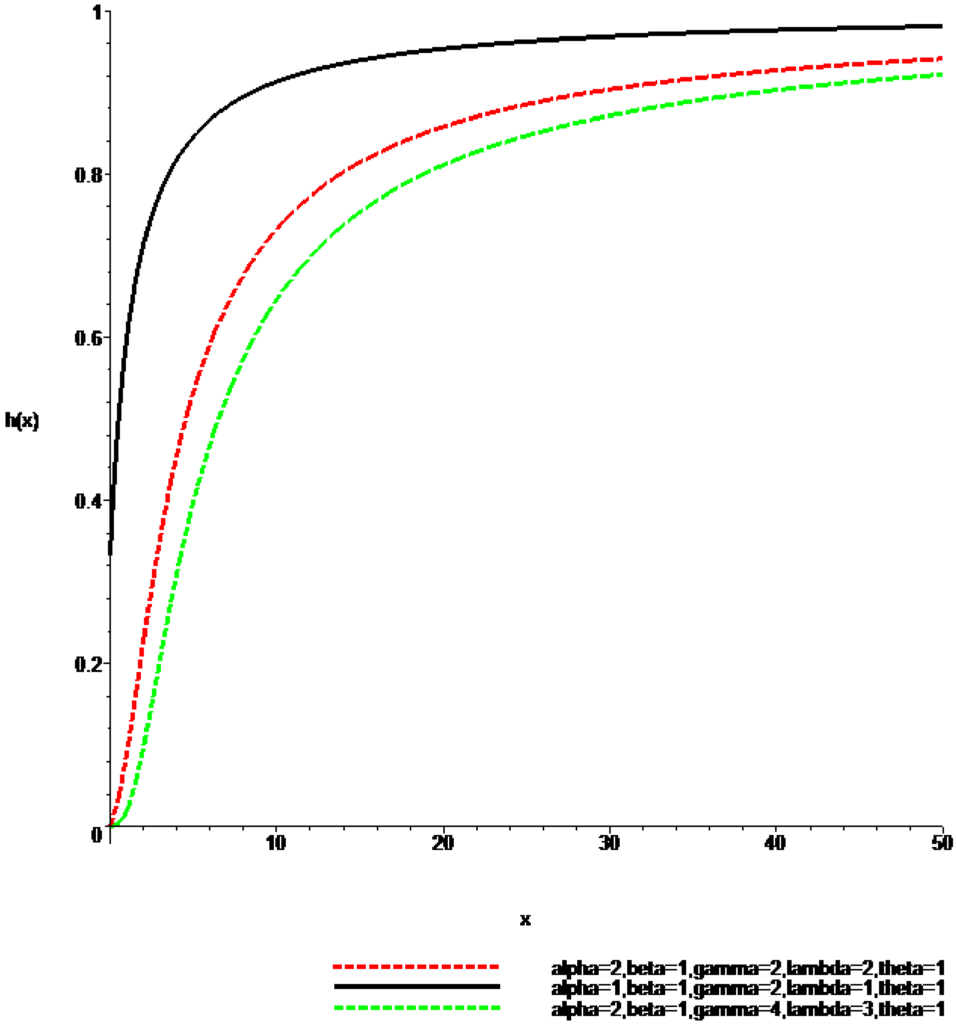


**Fig. 3:** hf's: Behavior of the hf of GLD for various values of the parameters

$$\begin{aligned} M_x(t) &= \int_0^\infty e^{tx} c\theta^{\alpha+1} x^{\alpha-1}(\beta+\gamma x^\lambda)e^{-\theta x}dx \\ &= c\theta\left(\frac{\beta\Gamma(\alpha)}{(1-\frac{t}{\theta})^\alpha}+\frac{\gamma}{\theta^\lambda}\frac{\Gamma(\alpha+\lambda)}{(1-\frac{t}{\theta})^{\alpha+\lambda}}\right) \end{aligned}$$

from which we easily get the various moments as follows:

$$\mu_1 = E(X) = c\theta\left(\frac{\alpha}{\theta}+\frac{\gamma}{\theta^\lambda}\frac{\alpha+\lambda}{\theta}\right) = \frac{1}{\theta^\lambda}\left(c\theta^\lambda\alpha+c\alpha\gamma+c\lambda\gamma\right) \tag{2.3}$$

$$\begin{aligned} \mu_2 = E(X^2) &= c\theta\left(\frac{\alpha(\alpha+1)}{\theta^2}+\frac{\gamma}{\theta^\lambda}\frac{(\alpha+\lambda)(\alpha+\lambda+1)}{\theta^2}\right) \\ &= \frac{c}{\theta^{\lambda+1}}\left(\theta^\lambda\alpha+\alpha\gamma+\lambda\gamma+\theta^\lambda\alpha^2+\alpha^2\gamma+\lambda^2\gamma+2\alpha\lambda\gamma\right) \end{aligned}$$

$$\mu_3 = E(X^3) = c\theta(\frac{\alpha(\alpha+1)(+2)}{\theta^3} + \frac{\gamma}{\theta^\lambda}\frac{(\alpha+\lambda)(\alpha+\lambda+1)(\alpha+\lambda+2)}{\theta^3})$$
$$= \frac{c}{\theta^{\lambda+2}}2\theta^\lambda\alpha + 2\alpha\gamma + 2\lambda\gamma + 2\theta^\lambda\alpha^2 + 3\alpha^2\gamma$$
$$+\alpha^3\gamma + 3\lambda^2\gamma + \lambda^3\gamma + 3\alpha\lambda^2\gamma + 3\alpha^2\lambda\gamma + 6\alpha\lambda\gamma)$$

$$\mu_4 = E(X^4) = c\theta(\frac{\alpha(\alpha+1)(+2)(\alpha+3)}{\theta^4} + \frac{\gamma}{\theta^\lambda}\frac{(\alpha+\lambda)(\alpha+\lambda+1)(\alpha+\lambda+2)(\alpha+\lambda+3)}{\theta^4})$$
$$= \frac{c}{\theta^{\lambda+3}}(6\theta^\lambda\alpha + 6\alpha\gamma + 6\lambda\gamma + 8\theta^\lambda\alpha^2 + 2\theta^\lambda\alpha^3 + 11\alpha^2\gamma + 6\alpha^3\gamma + \lambda^4\gamma$$
$$+18\alpha\lambda^2\gamma + 18\alpha^2\lambda\gamma + 4\alpha\lambda^3\gamma + 4\alpha^3\lambda\gamma + 6\alpha^2\lambda^2\gamma + 22\alpha\lambda\gamma)$$

## 3 Characterizations

We shall now present some characterizations of our proposed GLD by truncated first moment, order statistics and upper record values. We need the following two Lemmas 3.1 and 3.2. The proofs of the lemmas are given in [17]. For completeness of our paper we will give proof of Lemma 3.1 and the proof of the Lemma 3.2 is similar.

**Assumption A.** Assume that the random variable X is absolutely continuous with pdf f(x) and the corresponding cdf F(x).Further we assume that E(X) exists, the f(x) is differentiable and $\alpha = supx|F(x) > 0$ and $\beta = infx|F(x) < 1$ .

**Lemma 3.1.** Under the assumption of Assumption A, if $E(X|X \le x) = g(x)\tau(x)$, where g(x) a continuous differentiable function in $\alpha \le x \le \beta$ and $\tau(x) = \frac{f(x)}{F(x)}$ , if and only if $f(x) = c\exp(\int \frac{x-g'(x)}{g(x)}dx)$, provided $\int_\alpha^x \frac{x-g'(x)}{g(x)}dx$ is finite for all x, $\alpha \le x \le \beta$, where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$.

**Proof.** We know that

$$E(X|X \le x) = \frac{\int_\alpha^x uf(u)du}{F(x)}.$$

Thus

$$\int_\alpha^x uf(u)du = g(x)f(x).$$

Differentiating both sides of the equation, we obtain

$$xf(x) = f(x)g'(x) + f'(x)g(x).$$

Rearranging the above equation, we have

$$\frac{f'(x)}{f(x)} = \frac{x-g'(x)}{g(x)}$$

On integrating both sides of the above equation, we obtain

$$f(x) = c\exp(\int \frac{x-g'(x)}{g(x)}dx),$$

where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$

**Lemma 3.2.** Under the assumption of Assumption A, if $E(X|X \ge x) = h(x)r(x)$ where h(x) is a continuous differentiable function in $\alpha \le x \le \beta$ and $r(x) = \frac{f(x)}{1-F(x)}$ , if and only if $f(x) = c\exp(\int -\frac{x+h'(x)}{h(x)}dx)$ provided $-\int_\alpha^x \frac{x+h'(x)}{h(x)}dx$ is finite for all x, $\alpha \le x \le \beta$, where c is determined by the condition $\int_\alpha^\beta f(x)dx = 1$.

**Proof.** The proof is similar to Lemma 3.1.

### *3.1 Characterizations by Truncated First Moment:*

The following two theorems give the characterization of $X(\alpha,\beta,\gamma,\lambda,\theta)$ by truncated first moment.

**Theorem 3.1.**Theorem 3.1. Suppose the random variable X satisfies the Assumption A and $\alpha = 0$ and $\beta = \infty$ . Then $E(X|X \leq x) = g(x)\tau(x)$,

$$g(x) = \frac{(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}, \tag{3.1}$$

and

$$\tau(x) = \frac{f(x)}{F(x)},$$

if and only if

$$f(x) = c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma\, x^{\lambda})e^{-\theta x}, \alpha,\beta,\gamma,\lambda > 0.$$

**Proof.** Suppose the random variable X has pdf as given in 2.1, then

$$\begin{aligned} f(x)g(x) &= \int_0^x c\theta^{\alpha+1}u^{\alpha}(\beta+\gamma u^{\lambda})e^{-\theta u}du \\ &= c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta\, x) + \Gamma_{(\alpha+\lambda)}(\theta\, x) \end{aligned}$$

Thus we have

$$g(x) = \frac{(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}},$$

Suppose

$$g(x) = \frac{(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}.$$

Then, we have

$$\begin{aligned} g^{/}(x) &= x - \frac{(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}\left(\frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta\right) \\ &= x - g(x)\left(\frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta\right) \end{aligned}$$

Thus

$$\frac{x-g^{/}(x)}{g(x)} = \frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta$$

By Lemma 3.1, and noting that $\frac{d}{dx}ln(x^{\alpha-1}(\beta+\gamma\, x^{\lambda})e^{-\theta x}) = \frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta$ and $\frac{d}{dx}ln(e^{-\theta x}) = -\theta$ , we will have

$$\frac{f'(x)}{f(x)} = \frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta.$$

Integrating both sides of the above equation with respect to x, we obtain

$$f(x) = c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma\, x^{\lambda})e^{-\theta x},$$

where c is a constant.
Using the condition $\int_0^{\infty} f(x)dx = 1$ , we will have

$$f(x) = c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}.$$

This completes the proof of Theorem 3.1.

**Theorem 3.2.** Suppose the random variable X satisfies the Assumption A and $\alpha = 0$ and $\beta = \infty$ . Then

$$E(X|X \geq x) = h(x)r(x),$$

where

$$\begin{aligned} h(x) &= \frac{E(x) - \int_0^x c\theta^{\alpha+1}u^{\alpha}(\beta+\gamma u^{\lambda})e^{-\theta u}du}{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}} \\ &= \frac{E(x) - c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x))}{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}} \end{aligned}$$

and $r(x) = \frac{f(x)}{1-F(x)}$, if and only if

$$f(x) = c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}, \alpha, \beta, \gamma, \lambda > 0.$$

**Proof.** Suppose the random variable X has pdf as given in 2.1, then

$$\begin{aligned} f(x)h(x) &= \int_x^{\infty} c\theta^{\alpha+1}u^{\alpha}(\beta+\gamma u^{\lambda})e^{-\theta u}du \\ &= E(x) - \int_o^x c\theta^{\alpha+1}u^{\alpha}(\beta+\gamma u^{\lambda})e^{-\theta u}du \\ &= E(x) - c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta\, x) + \Gamma_{(\alpha+\lambda)}(\theta\, x) \end{aligned}$$

Thus

$$h(x) = \frac{E(x) - c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x))}{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}.$$

Suppose

$$h(x) = \frac{E(x) - c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}.$$

Then

$$h\prime(x) = -x - \frac{E(x) - c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}\left(\frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta\right)$$

Thus

$$\begin{aligned} h\prime(x) &= -x - \frac{E(x) - c\theta^{\alpha+1}(\beta\Gamma_{\alpha}(\theta x) + \Gamma_{(\alpha+\lambda)}(\theta x)}{c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}}\left(\frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta\right) \\ &= -x - h(x)\left(\frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta\right). \end{aligned}$$

Thus

$$-\frac{x+h(x)}{h(x))} = \frac{\alpha-1}{x} + \frac{\lambda\gamma x^{\lambda-1}}{\beta+x^{\lambda}\gamma} - \theta$$

By Lemma 3.2. we will have

$$\frac{f'(x)}{f(x)} = \frac{\alpha - 1}{x} + \frac{\lambda \gamma x^{\lambda - 1}}{\beta + x^{\lambda}\gamma} - \theta.$$

Integrating both sides of the above equation with respect to x, we obtain

$$f(x) = c\theta^{\alpha+1} x^{\alpha-1} (\beta + \gamma x^{\lambda}) e^{-\theta x}$$

where c is a constant.
Using the condition $\int_0^{\infty} f(x)dx = 1$, we will have,

$$f(x) = c\theta^{\alpha+1} x^{\alpha-1} (\beta + \gamma x^{\lambda}) e^{-\theta x}.$$

This completes the proof of Theorem 3.2.

### *3.2 Characterizations by Order Statistics*

If $X_1, X_2, ..., X_n$ be the $n$ independent copies of the random variable $X$ with absolutely continuous distribution function $F(x)$ and pdf $f(x)$ , and if $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$ be the corresponding order statistics, then it is known from [18], chapter 5, or [19], chapter 2, that $X_{j,n}|X_{k,n} = x$ , for $1 \leq k < j \leq n$ , is distributed as the (j-k)th order statistics from (n-k) independent observations from the random variable $V$ having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1-F(x)}$ , $0 \leq v < x$ , and $X_{i.,n}|X_{k,n} = x, 1 \leq i < k \leq n$ , is distributed as $ith$ order statistics from $k$ independent observations from the random variable $W$ having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x)}, w < x$ . Let $S_{k-1} = \frac{1}{k-1}\left(X_{1,n} + X_{2,n} + ... + X_{k-1,n}\right)$ , and $T_{k,n} = \frac{1}{n-k}\left(X_{k+1,n} + X_{k+2,n} + ... + X_{n.n}\right)$ .

**Theorem 3.3:** Suppose the random variable $X$ satisfies the Assumption A with $\alpha = 0$ and $\beta = \infty$ , then $E(S_{k-1}|X_{k,n} = x) = g(x)\tau(x)$ , where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ being given by 3.1, if and only if $X$ has the pdf

$$f(x) = c\theta^{\alpha+1} x^{\alpha-1} (\beta + \gamma x^{\lambda}) e^{-\theta x}, \alpha, \beta, \gamma, \lambda > 0.$$

**Proof:** It is known that $E(S_{k-1}|X_{k,n} = x) = E(X|X \leq x)$ ; see [13], and [20]. Hence, by Theorem 3.1, the result follows.

**Theorem 3.4:** Suppose the random variable $X$ satisfies the Assumption A with $\alpha = 0$ and $\beta = \infty$ , then $E(T_{k,n}|X_{k,n} = x) = \tilde{g}(x)\frac{f(x)}{1-F(x)}$, where

$$\tilde{g}(x) = \frac{(E(X) - g(x) f(x))}{f(x)},$$

$g(x)$ being given by 3.1 and $E(X)$ being given by 2.3, if and only if $X$ has the pdf

$$f(x) = c\theta^{\alpha+1} x^{\alpha-1} (\beta + \gamma x^{\lambda}) e^{-\theta x}, \alpha, \beta, \gamma, \lambda > 0.$$

**Proof:** Since $E(T_{k,n}|X_{k,n} = x) = E(X|X \geq x)$ , see [13], and [20], the result follows from Theorem 3.2.

### *3.3 Characterization by Upper Record Values*

For details on record values, see [21]. Let $X_1, X_2, ...$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$ . If $Y_n = max(X_1, X_2, ..., X_n)$ for $n \geq 1$ and $Y_j > Y_{j-1}, j > 1$ , then $X_j$ is called an upper record value of $\{X_n, n \geq 1\}$ . The indices at which the upper records occur are given by the record times $\left\{U(n) > min(j|j > U(n+1), X_j > X_{U(n-1)}, n > 1)\right\}$ and $U(1) = 1$ . Let the $nth$ upper record value be denoted by $X(n) = X_{U(n)}$ .

**Theorem 3.5:** Suppose the random variable $X$ satisfies the Assumption A with $\alpha = 0$ and $\beta = \infty$ , then $E(X(n+1)|X(n) = x) = \tilde{g}(x)\frac{f(x)}{1-F(x)}$ , where

$$\tilde{g}(x) = \frac{(E(X) - g(x) f(x))}{f(x)},$$

$g(x)$ being given by 3.1 and $E(X)$ being given by 2.3, if and only if $X$ has the pdf

$$f(x) = c\theta^{\alpha+1}x^{\alpha-1}(\beta+\gamma x^{\lambda})e^{-\theta x}, \alpha, \beta, \gamma, \lambda > 0.$$

Proof: It is known from [18], and [22] that $E(X(n+1)|X(n)=x) = E(X|X \geq x)$ . Then, the result follows from Theorem 3.2.

## 4 Concluding Remarks

The Lindley distribution is one of the most important continuous probability distributions, and has been extensively studied and widely used by many researchers in the fields of probability, statistics and other applied sciences since it was introduced as a mixture of exponential and gamma distributions by Lindley in 1958 in the context of Bayesian statistics, as a counter example to fiducial statistics and because of its superiority over exponential distribution. In the present paper, we have proposed a new generalized Lindley distribution (GLD), which generalizes the Lindley 1958 distribution and many extension and generalization of Lindley distribution as proposed by various authors and researchers in recent years. Various properties of the generalized Lindley distribution have been investigated. We have also investigated the characterization of the generalized Lindley distribution by truncated first moment where we have considered a product of reverse hazard rate and another function of the truncated point. The characterizations order statistics and upper record values have also been investigated. We believe that the findings of this paper would be useful for the practitioners in various fields of studies, and further enhancement of research in distribution theory and its applications.

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